HW 4

2.3 (6) $w(x,t) = \sum_{n=1}^{\infty} b_n \sin(\lambda_n x)\cdot e^{-\lambda_n^2 Kt}$ with $\lambda_n = \frac{n\pi}{a}$.

$$b_n = \frac{2}{a}\int_0^a g(x)\sin\left(\frac{n\pi x}{a}\right)dx = \frac{2}{a}\cdot\int_0^a \beta x \sin\left(\frac{n\pi x}{a}\right)dx =$$

$$= \frac{2\beta}{a}\cdot\left(-\frac{a^2}{n\pi}\cos(n\pi) - \int_0^a -\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)dx\right) =$$

$$= \frac{2\beta a}{n\pi}\cdot(-1)^{n+1} + 0.$$

(8) $w(x,t)$ as above, with:

$$b_n = \frac{2}{a}\cdot\int_0^{a/2} g(x)\sin\left(\frac{n\pi x}{a}\right)dx + \frac{2}{a}\int_{a/2}^a g(x)\cdot\sin\left(\frac{n\pi x}{a}\right)dx =$$

$$= \frac{2}{a}\int_0^{a/2}\frac{2T_0 x}{a}\sin\left(\frac{n\pi x}{a}\right)dx + \frac{2}{a}\int_{a/2}^a \frac{2T_0(a-x)}{a}\sin\left(\frac{n\pi x}{a}\right)dx$$

$$= \frac{4T_0}{a^2}\left(-\frac{a^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{a^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right) +$$

$$+ \frac{4T_0}{a^2}\left(+\frac{a^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{a^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right) = \frac{8T_0}{n^2\pi^2}\cdot\sin\left(\frac{n\pi}{2}\right)$$

So $w(x,t) = \sum_{n=1}^{\infty}\frac{8T_0\cdot(-1)^{n+1}}{(2n-1)^2\pi^2}\cdot\sin\left(\frac{(2n-1)\pi}{a}x\right)\cdot e^{-\frac{(2n-1)^2\pi^2}{a^2}Kt}$.

<u>Project 2.2.</u>

(a) Let $v(x) = \lim_{t\to\infty} C(x,t)$ be the steady state solution.

$$\begin{cases} v''(x) = 0 \\ v(0) = C_1 \\ v(a) = C_1 \end{cases} \Longrightarrow v(x) = C_1.$$

(b) Let $w(x,t) = C(x,t) - v(x)$ be the transient solution.
The problem is:

$$\begin{cases} D \cdot \dfrac{\partial^2 w}{\partial x^2} = \dfrac{\partial w}{\partial t} & 0 < x < a,\ t > 0 \\ w(0,t) = 0 \quad w(a,t) = 0 & t > 0 \\ w(x,0) = C_0 - C_1 & \end{cases}$$

(c) $w(x,t) = \sum_{n=1}^{\infty} b_n \sin(\lambda_n x) \cdot e^{-\lambda_n^2 Dt}$, $\lambda_n = \dfrac{n\pi}{a}$

$$b_n = \frac{2}{a}\int_0^a (C_0 - C_1) \cdot \sin\left(\frac{n\pi x}{a}\right) dx =$$

$$= \frac{2(C_0 - C_1)}{n\pi} \cdot (1 - \cos(n\pi)) = \frac{2(1-(-1)^n)(C_0 - C_1)}{n\pi}$$

$C(x,t) = C_1 + \sum_{n=1}^{\infty} b_n \sin(\lambda_n x)\, e^{-\lambda_n Dt}$ (as above).

(d) $C(\frac{a}{2}, t) - C_0 = 0.9(C_1 - C_0) \iff$

$$\iff w\left(\frac{a}{2}, t\right) + \underbrace{v\left(\frac{a}{2}\right)}_{= C_1} = 0.9 C_1 + 0.1 C_0 \iff$$

$$\iff w\left(\frac{a}{2}, t\right) = -0.1(C_1 - C_0).$$

The first term for $x = \frac{a}{2}$ is:

$$\frac{4}{\pi}(C_0 - C_1) \underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1} \cdot \exp\left(-\frac{\pi^2}{a^2} Dt\right) \approx 0.1(C_0 - C_1)$$

$$\implies \exp\left(-\frac{\pi^2}{a^2} Dt\right) = \frac{\pi}{40} \implies t = -\frac{a^2}{\pi^2 D} \cdot \ln\left(\frac{\pi}{40}\right).$$

(e) $\left[-\frac{a^2}{\pi^2 D}\ln\left(\frac{\pi}{40}\right)\right] = \frac{[a]^2}{[D]} = \frac{m^2}{\frac{m^2}{s}} = s = [t]$

Plugging in, $t = 6444\,s$.

$\boxed{2.4}$ (2) $u(x,0) = T_0 + T_1 \cdot \frac{x^2}{a^2} = f(x)$

the solution is $u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\lambda_n x) \cdot e^{-\lambda_n^2 K t}$

with $\lambda_n = \frac{n\pi}{a}$,

$$a_0 = \frac{1}{a}\int_0^a f(x)\,dx = \frac{1}{a}\int_0^a T_0\,dx + \frac{1}{a}\int_0^a \frac{T_1}{a^2}\cdot x^2\,dx = T_0 + \frac{T_1}{3}$$

$$a_n = \frac{2}{a}\int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)dx = \frac{2}{a}T_0 \cdot \int_0^a \cos\left(\frac{n\pi x}{a}\right)dx +$$

$$+ \frac{2}{a^3}T_1 \cdot \int_0^a x^2 \cos\left(\frac{n\pi x}{a}\right)dx = \frac{4T_1}{n^2\pi^2}(-1)^n.$$

(4) As above, with $a_0 = \frac{1}{a}\int_{\frac{a}{2}}^{a} T_2\,dx + \frac{1}{a}\int_0^{a/2} T_1\,dx = \frac{T_1+T_2}{2}$

$$a_n = \frac{2}{a}\int_0^{a/2} T_1 \cos\left(\frac{n\pi x}{a}\right)dx + \frac{2}{a}\int_{a/2}^{a} T_2 \cos\left(\frac{n\pi x}{a}\right)dx$$

$$= \frac{2T_1}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{2T_2}{n\pi}\sin\left(\frac{n\pi}{2}\right) = \frac{2(T_1-T_2)\cdot(-1)^{n+1}}{(2n-1)\cdot\pi}$$

(6) For $u_0 = 1$ it's trivial. For $n \geq 1$:

$\frac{\partial^2 u_n}{\partial x^2} = -\lambda_n^2 \cdot u_n(x,t)$ and $\frac{\partial u_n}{\partial t} = -\lambda_n^2 \cdot K \cdot u_n(x,t)$

so $\frac{\partial^2 u_n}{\partial x^2} = \frac{1}{K}\frac{\partial u_n}{\partial t}$.

Also $\frac{\partial u_n}{\partial x} = -\lambda_n \sin(\lambda_n x)\exp(-\lambda_n^2 K t)$ and for

$x=0$, $\sin\left(\frac{n\pi}{a}\cdot 0\right)=0$, for $x=a$, $\sin\left(\frac{n\pi}{a}\cdot a\right)=0$.

$\boxed{2.5}$ (4) $u(x,t)=w(x,t)+v(x)$, where $v(x)=T_0$.

For $\lambda_n=\frac{(2n-1)\pi}{2a}$ and $g(x)=u(x,0)-v(x)=$

$$=\begin{cases} 0 & 0<x<\frac{a}{2} \\ T_1-T_0 & \frac{a}{2}<x<a \end{cases}$$

$$w(x,t)=\sum_{n=1}^{\infty} b_n \sin(\lambda_n x)\exp(-\lambda_n^2 k t), \text{ with}$$

$$b_n=\frac{2}{a}\int_0^a g(x)\sin(\lambda_n x)\,dx=\frac{2}{a}\cdot\int_{a/2}^{a}(T_1-T_0)\sin\left(\frac{(2n-1)\pi}{2a}x\right)dx$$

$$=-\frac{2}{a}(T_1-T_0)\cdot\frac{1}{\lambda_n}\left(\underbrace{\cos(\lambda_n\cdot a)}_{=0}-\cos\left(\lambda_n\cdot\frac{a}{2}\right)\right)=$$

$$=\frac{4(T_1-T_0)}{(2n-1)\pi}\cdot\cos\left(\frac{(2n-1)\pi}{4}\right)=\frac{4(T_1-T_0)(-1)^{n+1}\cdot\sqrt{2}}{(2n-1)\cdot\pi\cdot 2}$$

So $u(x,t)=T_0+\sum_{n=1}^{\infty}\frac{4(T_1-T_0)}{(2n-1)\pi}\cos\left(\frac{(2n-1)\pi}{4}\right)\sin\left(\frac{(2n-1)\pi}{2a}x\right)\exp(-\lambda_n^2 kt)$

(6) The problem is homogeneous.

Assume $u(x,t)=\phi(x)\cdot T(t)$. Then $\frac{\phi''}{\phi}=\frac{1}{k}\cdot\frac{T'}{T}+\gamma^2=-\lambda^2$

$\Rightarrow \phi''+\lambda^2\phi=0$, $T'+(\gamma^2+\lambda^2)kT=0 \Rightarrow$

$\Rightarrow \phi(x)=c_1\cos(\lambda x)+c_2\sin(\lambda x)$, $T(t)=c\cdot e^{-(\gamma^2+\lambda^2)kt}$.

$u(0,t)=0 \Rightarrow \phi(0)=0 \Rightarrow c_1=0$

$\frac{\partial u}{\partial x}(a,t)=0 \Rightarrow \phi'(0)=0 \Rightarrow \cos(\lambda a)=0 \Rightarrow$

$$\Rightarrow \lambda = \lambda_n = \frac{(2n-1)\pi}{2a} \quad \text{for some (any) } n.$$

Then:

$$u(x,t) = w(x,t) = \sum_{n=1}^{\infty} b_n \sin(\lambda_n x) \exp\left(-(\lambda_n^2+\gamma^2)\kappa t\right)$$

with $b_n = \frac{2}{a}\int_0^a T_0 \sin(\lambda_n x)\,dx =$

$$= \frac{2T_0}{a\lambda_n} \cdot \Big[-\cos(\lambda_n x)\Big]_{x=0}^{x=a} = \frac{2T_0}{a\lambda_n}(-0+1) = \frac{4T_0}{(2n-1)\pi}.$$