ALGEBRAIC TOPOLOGY - The squares of a grassmannian.

Note by M. **WU WEN-TSÜN**, presented by M. Élie Cartan.

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1. In what follows, the ring of coefficients of the cohomology ring $H^*(M)$ of a space M will be exclusively the ring of integers mod 2.

For any $i \geq 0$, W^i are the W-classes (characteristic classes of Stiefel-Whitney) of a s. f. s. (spherical fibre structure) with the convention $W^0 = 1$ (the unit class of the base), and $W^i = 0$ for $i > m$, $m-1$ being the dimension of the fibre sphere. We will show the following formula:

$$\text{Sq}^r W^s = \sum\nolimits_t C^t_{s-r+t-1} W^{r-t} W^{s+t} \qquad (s \geq r > 0), \tag{1}$$

where C^q_p = binomial coefficient for $p \geq q > 0$, $= 0$ for $p < q > 0$, and $= 1$ for $p = -1$ and $q = 0$ (all are reduced mod 2).

First note some consequences of the this formula: define, in the base, a system of classes U^p (any $p \geq 0$) by the following equations:

$$W^i = \sum\nolimits_p \text{Sq}^{i-p} U^p, \qquad \text{any } p \geq 0; \tag{2}$$

we call them *canonical classes* of the structure considered. If the s. f. s. is in particular the tangent structure associated to a differentiable manifold of dimension m, we see, in comparing equations (1) and (2) with the previous note ([1]), that the name of canonical classes is justified; moreover, among all the s. f. s. (the fibres S^{m-1}) on the manifold M as base, the tangent structure of M possesses the following remarkable property:

$$U^p = 0 \qquad \text{for} \quad 2p > m, \tag{3}$$

From (1) and (3) we deduce:

a. For an orientable structure we have $U^{2k+1} = 0$, any k, which generalises a theorem of H. Cartan ([1]),

b. For the tangent structure on a differentiable manifold of dimension m, we have $W^1 W^{m-2} = 0$ if $m = 4k$; $W^1 W^{m-3} = 0$; $W^1 W^{m-1} = 0$ if $m = 4k+1$; $W^m = W^1 W^{m-1}$ if $m = 4k+2$; $W^1 W^{m-1} = 0$, $W^{m-1} = W^1 W^{m-2}$ if $m = 4k+3$.

2. Let $G_{n,m}$ be the grassmannian of m linear elements in the Euclidean space $\mathbb{R}^{n+m}$ of dimension $n+m$ passing through the origin in $\mathbb{R}^{n+m}$. We know ([2]) that the ring $H^*(G_{n,m})$ is generated by the classes W^i of the s. f. s. $\mathcal{G}_{n,m}$ (fibres S^{m-1}) with base $G_{n,m}$ canonically associated to $G_{n,m}$. Moreover, as pointed out to me by H. Cartan:

Lemma 1. *Let $\varphi_p(W^i)$ be a polynomial not identically zero in $W^1, \ldots, W^m$ such that for each term $W^{i_1} \ldots W^{i_k}$ of this polynomial we have $i_1 + \cdots + i_k = p \leq n$. Then $\varphi_p(W^i)$ is a non-zero element of $H^*(G_{n,m})$.*

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[1] *Comptes rendus*, **230**, 1950, p. 508-511.

[2] S. CHERN, *Annals of Math.*, **49**, 1948, p. 362-372.

Suppose then that $\mathbb{R}^{n+m}$ is the product of two Euclidean spaces $\mathbb{R}_j^{n_j+m}$ of dimension $n_j + m_j$ $(j = 1, 2)$. Let G_{n_j,m_j} $(j = 1, 2)$ be the grassmannians defined respectively in $\mathbb{R}_j^{n_j+m_j}$. For $X_j \in G_{n_j,m_j}$ let $X \in G_{n,m}$ be the join of X_1 and X_2, we then have a canonical map

$$f\colon G_{n_1,m_1} \times G_{n_2,m_2} \to G_{n,m}$$

defined by $f(X_1 \times X_2) = X$. Denoting by W_j^i $(j = 1, 2)$ the respective W-classes of the structures $\mathcal{G}_{n_j,m_j}$, we have:

Lemma 2. *The mod 2 homotopy type of f is determined by* ([3] *):*

$$f^*W^i = \sum_j W_1^j \otimes W_2^{i-j} \qquad (\textit{for any } i \geq 0).$$

As a consequence of Lemmas 1 and 2, retaining the notations, we have:

Lemma 3. *For $p \leq n_1$ and n_2, $\varphi_p(W^i)$ is a non-zero element of $H^*(G_{n,m})$ if and only if $\varphi_p(W^i)$ is a non-zero element of $H^*(G_{n_1,m_1} \times G_{n_2,m_2})$.*

3. *Proof of* (1). - We set

$$\varphi_{r,s}(W^i) = \operatorname{Sq}^r W^s + \sum_t C^t_{s-r+t-1} W^{r-t} W^{s+t}.$$

The formula (1), or, equivalently, the formula $\varphi_{r,s}(W_j^i) = 0$, being obvious for $m = 1$, we will assume by induction that it is true for the structures which have sphere fibers of dimension $< m - 1$, where $m > 1$. Now let W^i, W_j^i respectively be the classes of the structures $\mathcal{G}_{n,m}$ and $\mathcal{G}_{n_j,m_j}$ $(j = 1, 2)$ where $n = n_1 + n_2$, $n_j \geq r + s$, $m_1 = m - 1$, $m_2 = 1$. From the formula $f^* \operatorname{Sq}^i = \operatorname{Sq}^i f^*$, a theorem of H. Cartan ([4]), and lemma 2 of section 2, we deduce

$$f^*\varphi_{r,s}(W^i) = \varphi_{r,s}(W_1^2) \otimes 1 + \varphi_{r,s-1}(W_1^i) \otimes W_2^1 + \varphi_{r-1,s-1}(W_1^i) \otimes (W_2^1)^2.$$

By the induction hypothesis $f^*\varphi_{r,s}(W^i) = 0$ consequently $\varphi_{r,s}(W^i) = 0$ by lemma 3. Since the structure $\mathcal{G}_{n,m}$ is universal for n large enough, we have $\varphi_{r,s}(W^i) = 0$ for any s.f.s.. Formula (1) is thus proved by induction.

Let, in particular, W^i be the W-classes of the structures $\mathcal{G}_{n,m}$ on the base $G_{n,m}$. The ring $H^*(G_{n,m})$ being generated by the classes W^i, we see that formula (1) completely determines the squares in $G_{n,m}$ by expressing them as polynomials in W^i.

(Excerpt from *Comptes rendus des séances de l'Académie des Sciences*,
t. **230**, p. 918-920, meeting of 6 March 1950.)

[3] We note that the theorem of Whitney on the product of two spherical fiber structures is a consequence of this lemma whose proof is given in my Thesis, Strasbourg, 1949.

[4] *Comptes rendus*, **230**, 1950, p. 425-427.