

STEENROD SQUARES, WU CLASSES, AND STIEFEL-WHITNEY CLASSES

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ABSTRACT. There is a list of formulae for the first five Wu classes in terms of Stiefel-Whitney classes on nLab. There was no reference given for the necessary computations, so I tried to do them myself. In doing so, I realised I misunderstood what little I thought I knew about Steenrod squares and Wu classes. In this note I explain what is needed to compute, and hopefully understand, the formulae given on nLab.

For an element $x \in H^*(X; \mathbb{Z}_2)$, we write $|x| = k$ to indicate that $x \in H^k(X; \mathbb{Z}_2)$. For $x \in H^k(X; \mathbb{Z}_2)$ and $\alpha \in H_k(X; \mathbb{Z}_2)$, we write $\langle x, \alpha \rangle_k$ to denote the natural pairing.

STEENROD SQUARES

For any topological space X and integer $m \geq 0$, there is a graded linear map $\text{Sq}^m: H^*(X; \mathbb{Z}_2) \rightarrow H^*(X; \mathbb{Z}_2)$ of degree m called the m^{th} *Steenrod square* (or *Steenrod operation*), such that:

- (1) for any topological space Y and continuous map $f: Y \rightarrow X$, $\text{Sq}^m(f^*x) = f^* \text{Sq}^m(x)$,
- (2) $\text{Sq}^0(x) = x$,
- (3) $\text{Sq}^m(x) = 0$ for $|x| < m$, and $\text{Sq}^m(x) = x \cup x$ for $|x| = m$, and
- (4) $\text{Sq}^k(x \cup y) = \sum_{i+j=k} \text{Sq}^i(x) \cup \text{Sq}^j(y)$.

The fourth condition is called Cartan's formula. We also have the *total Steenrod square* $\text{Sq} := \text{Sq}^0 + \text{Sq}^1 + \text{Sq}^2 + \dots$. Cartan's formula can now be written as $\text{Sq}(x \cup y) = \text{Sq}(x) \cup \text{Sq}(y)$ so $\text{Sq}: H^*(X; \mathbb{Z}_2) \rightarrow H^*(X; \mathbb{Z}_2)$ is an algebra homomorphism.

One can use Steenrod squares together with the Thom isomorphism to define Stiefel-Whitney classes. See Chapter 8 of [1].

WU CLASSES

Now suppose X is a closed, connected n -manifold. Restricting Sq^m to $H^{n-m}(X; \mathbb{Z}_2)$, we obtain a linear map $\text{Sq}^m: H^{n-m}(X; \mathbb{Z}_2) \rightarrow H^n(X; \mathbb{Z}_2)$, and therefore an element of

$$\begin{aligned}
 & \text{Hom}(H^{n-m}(X; \mathbb{Z}_2), H^n(X; \mathbb{Z}_2)) \\
 \cong & \text{Hom}(H^{n-m}(X; \mathbb{Z}_2), \mathbb{Z}_2) && (X \text{ is a closed, connected } n\text{-manifold}) \\
 \cong & \text{Hom}(\text{Hom}(H_{n-m}(X; \mathbb{Z}_2), \mathbb{Z}_2), \mathbb{Z}_2) && (\text{Universal Coefficient Theorem}) \\
 \cong & H_{n-m}(X; \mathbb{Z}_2) && (H_{n-m}(X; \mathbb{Z}_2) \text{ is a finite-dimensional} \\
 & && \text{vector space over } \mathbb{Z}_2) \\
 \cong & H^m(X; \mathbb{Z}_2) && (\text{Poincaré Duality}).
 \end{aligned}$$

Let ν_m denote the element of $H^m(X; \mathbb{Z}_2)$ corresponding to Sq^m under these natural isomorphisms. Unwinding the isomorphisms, we can determine exactly how Sq^m and ν_m are related.

- The first isomorphism sends $x \mapsto \text{Sq}^m(x)$ to $x \mapsto \langle \text{Sq}(x), [X] \rangle_n$ where $[X]$ is the \mathbb{Z}_2 fundamental homology class of X (i.e. the non-zero element of $H_n(X; \mathbb{Z}_2)$).
- The second isomorphism sends $x \mapsto \langle \text{Sq}^m(x), [X] \rangle_n$ to the map $(\alpha \mapsto \langle x, \alpha \rangle_{n-m}) \mapsto \langle \text{Sq}^m(x), [X] \rangle_n$.
- The third isomorphism sends $(\alpha \mapsto \langle x, \alpha \rangle_{n-m}) \mapsto \langle \text{Sq}^m(x), [X] \rangle_n$ to $\alpha \in H_{n-m}(X; \mathbb{Z}_2)$ such that $\langle x, \alpha \rangle_{n-m} = \langle \text{Sq}^m(x), [X] \rangle_n$.
- The final isomorphism sends α to $\nu_m \in H^m(X; \mathbb{Z}_2)$ such that $\alpha = [X] \cap \nu_m$.

So we see that

$$\langle \text{Sq}^m(x), [X] \rangle_n = \langle x, \alpha \rangle_{n-m} = \langle x, [X] \cap \nu_m \rangle_{n-m} = \langle \nu_m \cup x, [X] \rangle_n.$$

Therefore, $\text{Sq}^m(x) = \nu_m \cup x$ for all $x \in H^{n-m}(X; \mathbb{Z}_2)$. We call ν_m the m^{th} Wu class of X , and define $\nu := 1 + \nu_1 + \cdots + \nu_n \in H^*(X; \mathbb{Z}_2)$ to be the total Wu class of X .

Note, if $|x| = k < \frac{n}{2}$, in order to have $\text{Sq}^m(x) \in H^n(X; \mathbb{Z}_2)$, we need $m = n - k > \frac{n}{2}$, but then $\text{Sq}^m(x) = 0$. As we also have $\text{Sq}^m(x) = \nu_m \cup x$, we see that $\nu_m \cup x = 0$ for every $x \in H^k(X; \mathbb{Z}_2)$. It follows from Poincaré duality that $\nu_m = 0$. So for $m > \frac{n}{2}$, we have $\nu_m = 0$ and therefore $\nu = 1 + \nu_1 + \cdots + \nu_{\lfloor \frac{n}{2} \rfloor}$.

WU'S THEOREM

Now suppose X is a smooth, closed n -manifold. Wu's theorem states that w , the total Stiefel-Whitney class of the tangent bundle of X , is related to Steenrod squares and Wu classes by the equation $w = \text{Sq}(\nu)$; see Theorem 11.14 of [1]. Comparing degrees, we see that

$$w_k = \sum_{i+j=k} \text{Sq}^i(\nu_j).$$

Note, $\text{Sq}^i(\nu_j)$ is not simply $\nu_i \cup \nu_j$ unless $i + j = n$. As $\text{Sq}^i(w_j) = 0$ for $i > j$ we can simplify this to

$$w_k = \sum_{\substack{i+j=k \\ i \leq j}} \text{Sq}^i(\nu_j) = \sum_{i=0}^{\lfloor k/2 \rfloor} \text{Sq}^i(\nu_{k-i}).$$

One of the most important applications of this relationship is that the expression $\text{Sq}(\nu)$ depends only on the cohomology ring of X which is an invariant under homotopy equivalence. This is somewhat surprising because the tangent bundle depends on the smooth structure. More precisely, we see that if $f: X \rightarrow Y$ is a homotopy equivalence of smooth, closed n -manifolds, $f^*w(TY) = w(TX)$. In particular, the total Stiefel-Whitney class of X , which is defined as the total Stiefel-Whitney class of the tangent bundle, does not depend on the smooth structure on X . Furthermore, two homotopy equivalent closed manifolds are cobordant as they have the same Stiefel-Whitney numbers.

The right hand side of the equation $w = \text{Sq}(\nu)$ does not rely on a smooth structure, so one could define the total Stiefel-Whitney class of a closed topological manifold to be the expression $\text{Sq}(\nu)$, despite the fact that there is no natural bundle involved.

WU'S FORMULA

The theorem of Wu leads to a computation of the action of the Steenrod squares on the Stiefel-Whitney classes, again due to Wu [3]. Namely, for $0 \leq i \leq j$ we have

$$\text{Sq}^i(w_j) = \sum_{t=0}^i \binom{j-i+t-1}{t} w_{i-t} \cup w_{j+t}$$

where $\binom{p}{q} = 0$ if $p < q$, except when $p = -1$ and $q = 0$ in which case we use the convention $\binom{-1}{0} = 1$. Note, this convention is made in order to deal with the case $i = j$. One could also avoid such a convention by using the formula for $0 \leq i < j$ as we already know $\text{Sq}^j(w_j) = w_j \cup w_j$. Either way, we call the displayed equation Wu's formula.

WU CLASSES IN TERMS OF STIEFEL-WHITNEY CLASSES

In this section the cup product symbol will be omitted and replaced by multiplicative notation.

Wu's theorem allows us to compute the Stiefel-Whitney classes in terms of Wu classes and their Steenrod squares. We can rearrange these relationships to express the Wu classes in terms of Stiefel-Whitney classes. Below we determine the first five as these are the ones listed on the nLab page [2].

1. As $w_1 = \text{Sq}^0(\nu_1) = \nu_1$, we see that $\nu_1 = w_1$.

2. Now we have $w_2 = \text{Sq}^0(\nu_2) + \text{Sq}^1(\nu_1) = \nu_2 + \nu_1^2$ so $\nu_2 = w_2 + \nu_1^2 = w_2 + w_1^2$.

Unlike the first two cases, from this point on we will need to evaluate intermediate Steenrod squares of Wu classes, so the computations become more cumbersome.

3. From Wu's theorem, we have

$$w_3 = \text{Sq}^0(\nu_3) + \text{Sq}^1(\nu_2) = \nu_3 + \text{Sq}^1(\nu_2) = \nu_3 + \text{Sq}^1(w_2 + w_1^2) = \nu_3 + \text{Sq}^1(w_2) + \text{Sq}^1(w_1^2).$$

By Wu's formula, we have

$$\text{Sq}^1(w_2) = \binom{2-1+0-1}{0} w_1 w_2 + \binom{2-1+1-1}{1} w_0 w_3 = \binom{0}{0} w_1 w_2 + \binom{1}{1} w_3 = w_1 w_2 + w_3,$$

and by Cartan's formula we have

$$\text{Sq}^1(w_1^2) = \text{Sq}^0(w_1) \text{Sq}^1(w_1) + \text{Sq}^1(w_1) \text{Sq}^0(w_1) = 0.$$

More generally, for any class x , $\text{Sq}^1(x^2) = 0$ again by Cartan's formula.

Therefore, $\nu_3 = w_3 + \text{Sq}^1(w_2) + \text{Sq}^1(w_1^2) = w_3 + w_1 w_2 + w_3 = w_1 w_2$.

4. This time Wu's theorem gives us three terms,

$$\begin{aligned} w_4 &= \text{Sq}^0(\nu_4) + \text{Sq}^1(\nu_3) + \text{Sq}^2(\nu_2) \\ &= \nu_4 + \text{Sq}^1(w_1 w_2) + \nu_2^2 \\ &= \nu_4 + \text{Sq}^1(w_1 w_2) + (w_2 + w_1^2)^2 \\ &= \nu_4 + \text{Sq}^1(w_1 w_2) + w_2^2 + w_1^4. \end{aligned}$$

By Cartan's formula we have

$$\begin{aligned} \text{Sq}^1(w_1 w_2) &= \text{Sq}^0(w_1) \text{Sq}^1(w_2) + \text{Sq}^1(w_1) \text{Sq}^0(w_2) \\ &= w_1(w_1 w_2 + w_3) + w_1^2 w_2 \\ &= w_1^2 w_2 + w_1 w_3 + w_1^2 w_2 \\ &= w_1 w_3. \end{aligned}$$

Therefore, $\nu_4 = w_4 + \text{Sq}^1(w_1 w_2) + w_2^2 + w_1^4 = w_4 + w_1 w_3 + w_2^2 + w_1^4$.

5. Finally, Wu's theorem gives

$$\begin{aligned}
w_5 &= \text{Sq}^0(\nu_5) + \text{Sq}^1(\nu_4) + \text{Sq}^2(\nu_3) \\
&= \nu_5 + \text{Sq}^1(w_4 + w_1w_3 + w_2^2 + w_1^4) + \text{Sq}^2(w_1w_2) \\
&= \nu_5 + \text{Sq}^1(w_4) + \text{Sq}^1(w_1w_3) + \text{Sq}^1(w_2^2) + \text{Sq}^1(w_1^4) + \text{Sq}^2(w_1w_2) \\
&= \nu_5 + \text{Sq}^1(w_4) + \text{Sq}^1(w_1w_3) + \text{Sq}^2(w_1w_2).
\end{aligned}$$

By Wu's formula we have

$$\text{Sq}^1(w_4) = \binom{4-1+0-1}{0} w_1w_4 + \binom{4-1+1-1}{1} w_0w_5 = \binom{2}{0} w_1w_4 + \binom{3}{1} w_5 = w_1w_4 + w_5.$$

Now by Cartan's formula, we have

$$\begin{aligned}
\text{Sq}^1(w_1w_3) &= \text{Sq}^0(w_1) \text{Sq}^1(w_3) + \text{Sq}^1(w_1) \text{Sq}^0(w_3) \\
&= w_1 \text{Sq}^1(w_3) + w_1^2w_3.
\end{aligned}$$

Again by Wu's formula,

$$\text{Sq}^1(w_3) = \binom{3-1+0-1}{0} w_1w_3 + \binom{3-1+1-1}{1} w_0w_4 = \binom{1}{0} w_1w_3 + \binom{2}{1} w_4 = w_1w_3,$$

so $\text{Sq}^1(w_1w_3) = w_1 \text{Sq}^1(w_3) + w_1^2w_3 = w_1^2w_3 + w_1^2w_3 = 0$.

Returning to Cartan's formula, we obtain

$$\begin{aligned}
\text{Sq}^2(w_1w_2) &= \text{Sq}^0(w_1) \text{Sq}^2(w_2) + \text{Sq}^1(w_1) \text{Sq}^1(w_2) + \text{Sq}^2(w_1) \text{Sq}^0(w_2) \\
&= w_1w_2^2 + w_1^2(w_1w_2 + w_3) + 0w_2 \\
&= w_1w_2^2 + w_1^3w_2 + w_1^2w_3.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\nu_5 &= w_5 + \text{Sq}^1(w_4) + \text{Sq}^1(w_1w_3) + \text{Sq}^2(w_1w_2) \\
&= w_5 + w_1w_4 + w_5 + w_1w_2^2 + w_1^3w_2 + w_1^2w_3 \\
&= w_1w_4 + w_1w_2^2 + w_1^2w_3 + w_1^3w_2.
\end{aligned}$$

In summary, we have

$$\begin{aligned}
\nu_1 &= w_1 \\
\nu_2 &= w_2 + w_1^2 \\
\nu_3 &= w_1w_2 \\
\nu_4 &= w_4 + w_1w_3 + w_2^2 + w_1^4 \\
\nu_5 &= w_1w_4 + w_1w_2^2 + w_1^2w_3 + w_1^3w_2
\end{aligned}$$

which agree with the identities listed on nLab.

SOME APPLICATIONS

Let X be an smooth, closed, orientable, three-manifold. As X is orientable, $w_1 = 0$, and as X is three-dimensional and $2 > \lfloor \frac{3}{2} \rfloor$, we see that $\nu_2 = 0$. By the above computation we have $\nu_2 = w_2 + w_1^2 = w_2$, so $w_2 = 0$ and therefore X is spin. This fact is used in the proof of the following theorem: every smooth, closed, orientable three-manifold is parallelisable. Similarly, by looking at the fourth Wu class, one can show that a smooth, closed, spin four-manifold has even Euler characteristic. Furthermore, it is null-cobordant.

On a smooth, closed n -manifold, the fact that $\nu_n = 0$ imposes a restriction on its Stiefel-Whitney numbers. For example, when $n = 2$, we see that $w_2 + w_1^2 = 0$ so the two Stiefel-Whitney numbers are equal. Hence, two closed surfaces are cobordant if and only if their Euler characteristics have the same parity. Therefore, $\Omega_2^O \cong \mathbb{Z}_2$ where Ω_n^O denotes the group of cobordism classes of smooth, closed n -manifolds and the isomorphism is given by evaluating the Euler characteristic of a representative mod 2. For example, we see that the Klein bottle, $\mathbb{R}P^2 \# \mathbb{R}P^2$, is null-cobordant and therefore serves as an example of a manifold with non-trivial Stiefel-Whitney classes, but trivial Stiefel-Whitney numbers.

REFERENCES

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