HOMOTOPY GROUPS OF A WEDGE SUM OF SPHERES

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Abstract. There is a trick for computing the first few homotopy groups of a wedge sum of spheres which uses cellular approximation. But how do you compute the remaining homotopy groups? The answer is given by Hilton's Theorem. After introducing the trick, I explain Hilton's theorem and how to implement it to calculate the homotopy groups of a wedge sum of spheres in terms of the homotopy groups of spheres.

Consider the space $S^{m_1} \vee \cdots \vee S^{m_k}$. As $A \vee B$ and $B \vee A$ are homotopy equivalent, we can (and will) assume $m_1 \leq \cdots \leq m_k$.

If $m_1 = 0$, then $\pi_0(S^{m_1} \vee \cdots \vee S^{m_k}) = \pi_0(S^{m_2} \vee \cdots \vee S^{m_k}) \oplus \pi_0(S^{m_2} \vee \cdots \vee S^{m_k})$ and for $n > 0$, $\pi_n(S^{m_1} \vee \cdots \vee S^{m_k}) = \pi_n(S^{m_2} \vee \cdots \vee S^{m_k})$. From now on we will assume that $m_1 > 0$ (and hence $m_i > 0$ for all i).

Cellular Approximation

If X is an p-dimensional CW complex and Y is a CW complex, then by cellular approximation $[X, Y] = [X, Y^{(p+1)}]$ where $Y^{(p+1)}$ denotes the $(p+1)$-skeleton of Y. If X and Y are also pointed, then the same is true of the pointed homotopy classes; in particular, $\pi_p(Y) = \pi_k(Y^{(p+1)})$.

Returning to the problem at hand, note that $S^{m_1} \vee \cdots \vee S^{m_k}$ is a subcomplex of $S^{m_1} \times \cdots \times S^{m_k}$ – the latter is obtained from the former by attaching cells of dimension at least $m_1 + m_2$. However, $S^{m_1} \vee \cdots \vee S^{m_k}$ need not be the $(m_1 + m_2 - 1)$-skeleton of $S^{m_1} \times \cdots \times S^{m_k}$ as it may contain cells of dimension greater than $m_1 + m_2 - 1$ (e.g. the 4-skeleton of $S^2 \times S^3 \times S^5$ is $S^2 \vee S^3$, not $S^2 \vee S^3 \vee S^5$). However, $S^{m_1} \vee \cdots \vee S^{m_k}$ and $S^{m_1} \times \cdots \times S^{m_k}$ have the same $(m_1 + m_2 - 1)$-skeleton, namely $S^{m_1} \vee \cdots \vee S^{m_a}$ where a is such that $m_a \leq m_1 + m_2 - 1 < m_{a+1}$. Therefore, for any $n < m_1 + m_2 - 1$ we have

$$\begin{aligned}
\pi_n(S^{m_1} \vee \cdots \vee S^{m_k}) &= \pi_n(S^{m_1} \vee \cdots \vee S^{m_a}) \\
&= \pi_n(S^{m_1} \times \cdots \times S^{m_k}) \\
&= \pi_n(S^{m_1}) \oplus \cdots \oplus \pi_n(S^{m_k}).
\end{aligned}$$

As $m_1 + m_2 - 1 < m_{a+1} \leq \cdots \leq m_k$, $\pi_n(S^{m_{a+1}}) = \cdots = \pi_n(S^{m_k}) = 0$ so we can also express the above result as

$$\pi_n(S^{m_1} \vee \cdots \vee S^{m_k}) = \pi_n(S^{m_1}) \oplus \cdots \oplus \pi_n(S^{m_a}).$$

Hilton's Theorem

Before introducing Hilton's Theorem, we need to make one further reduction.

If $m_1 = \cdots = m_b = 1$ and $m_{b+1}, \dots, m_k > 1$, the Seifert-van Kampen Theorem shows that $\pi_1(S^{m_1} \vee \cdots \vee S^{m_k}) \cong F_b$, the free group on b generators. The higher homotopy groups are isomorphic to the higher homotopy groups of the universal cover which is homotopy equivalent to the wedge sum of countably many copies of $S^{m_{b+1}} \vee \dots \vee S^{m_k}$. With this in mind, we will assume from now on that $m_1 > 1$ (and hence $m_i > 1$ for all i) and set $m_i = r_i + 1$; note $r_i \geq 1$.

In order to state Hilton's Theorem, we need to introduce what he calls *basic products*.

Let α_j be the positive generator of $\pi_{m_j}(S^{m_j})$ (i.e. the homotopy class of the identity map)[1]. We call $\alpha_1, \dots, \alpha_k$ basic products of weight one, and we order them as follows: $\alpha_1 < \dots < \alpha_k$.

Now assume the basic products of weight less than w have been defined and ordered. Basic products of weight w are Whitehead products $[a, b]$ where a, b are basic products of weights u and v respectively, $u+v = w$, $a < b$ (in the ordering), and if $b = [c, d]$ where c and d are basic products, then $c \leq a$. Order the weight w elements arbitrarily among themselves and greater than all lower weight basic products. It follows that $u \leq v$.

Example: Suppose $k = 3$. Then there are three weight one basic products, namely $\alpha_1, \alpha_2, \alpha_3$, which are ordered as follows: $\alpha_1 < \alpha_2 < \alpha_3$.

The weight two basic products are $[\alpha_1, \alpha_2]$, $[\alpha_1, \alpha_3]$, $[\alpha_2, \alpha_3]$. We choose to extend the order as follows: $\alpha_1 < \alpha_2 < \alpha_3 < [\alpha_1, \alpha_2] < [\alpha_1, \alpha_3] < [\alpha_2, \alpha_3]$.

The weight three basic products are $[\alpha_1, [\alpha_1, \alpha_2]]$, $[\alpha_1, [\alpha_1, \alpha_2]]$, $[\alpha_2, [\alpha_1, \alpha_2]]$, $[\alpha_2, [\alpha_1, \alpha_3]]$, $[\alpha_2, [\alpha_2, \alpha_3]]$, $[\alpha_3, [\alpha_1, \alpha_2]]$, $[\alpha_3, [\alpha_1, \alpha_3]]$, $[\alpha_3, [\alpha_2, \alpha_3]]$. One possible ordering is

$$\alpha_1 < \alpha_2 < \alpha_3 < [\alpha_1, \alpha_2] < [\alpha_1, \alpha_3] < [\alpha_2, \alpha_3] < [\alpha_1, [\alpha_1, \alpha_2]] < [\alpha_1, [\alpha_1, \alpha_2]] < [\alpha_2, [\alpha_1, \alpha_2]]$$
$$< [\alpha_2, [\alpha_1, \alpha_3]] < [\alpha_2, [\alpha_2, \alpha_3]] < [\alpha_3, [\alpha_1, \alpha_2]] < [\alpha_3, [\alpha_1, \alpha_3]] < [\alpha_3, [\alpha_2, \alpha_3]].$$

Note, the ordering on the weight two basic products played no role in determining the basic products of weight three. However, they do now play a role in determining the basic products of weight four. For example, $[[\alpha_1, \alpha_2], [\alpha_1, \alpha_3]]$ is a basic product of weight four but $[[\alpha_1, \alpha_3], [\alpha_1, \alpha_2]]$ is not; this is because we chose an order in which $[\alpha_1, \alpha_2] < [\alpha_1, \alpha_3]$. Had we chosen to extend the order to basic products of weight two in such a way that $[\alpha_1, \alpha_3] < [\alpha_1, \alpha_2]$, then $[[\alpha_1, \alpha_3], [\alpha_1, \alpha_2]]$ would be a basic product of weight four but $[[\alpha_1, \alpha_2], [\alpha_1, \alpha_3]]$ would not be. In general, the ordering of the elements of weight k only affects the basic products of weight $2k$ and above.

Any basic product p of weight w is a string of symbols $\alpha_{j_1}, \dots, \alpha_{j_w}$ suitably bracketed. Let w_j be the number of occurences of α_j in the string representing p. The *height* of p is defined to be $q = \sum_{i=1}^{k} r_i w_i$.

Let $\{p_s\}$ be the sequence of basic products written in increasing order, and denote the height of p_s by q_s. Then Hilton's Theorem [1] states that there is an isomorphism

$$\pi_n(S^{m_1} \vee \dots \vee S^{m_k}) \cong \bigoplus_{i=1}^{\infty} \pi_n(S^{q_i+1}).$$

There were choices involved in the definition of basic products (namely the orderings of the weight w basic products for $w \geq 2$). It turns out that had we made different choices, the only difference is that the direct summands are reordered. More precisely, by a theorem of Witt [2], the number of basic products involving w_j copies of α_j, which necessarily have weight $w = w_1 + \dots + w_k$, is given by

$$A(w_1, \dots, w_k) = \frac{1}{w} \sum_{d | w_j} \frac{\mu(d)(w/d)!}{(w_1/d)! \dots (w_k/d)!}$$

where μ denotes the Möbius function defined on the positive integers by

$$\mu(d) = \begin{cases} 1 & d \text{ is square-free with an even number of prime factors} \\ -1 & d \text{ is square-free with an odd number of prime factors} \\ 0 & d \text{ has a squared prime factor.} \end{cases}$$

[1]Note that Hilton uses the notation ι_j instead of α_j.

It follows that for any q, the number of direct summands of the form $\pi_n(S^{q+1})$ is independent of the choices made. For the purposes of calculation, it is useful to note that $A(w_{\sigma(1)},\dots,w_{\sigma(k)}) = A(w_1,\dots,w_k)$ for all $\sigma \in S_k$.

We can calculate $\pi_n(S^{m_1}\vee\cdots\vee S^{m_k})$ as follows: for each q, find the sum of all $A(w_1,\dots,w_k)$ for which $\sum_{i=1}^k r_i w_i = q$; call it c_{q+1} (this is the number of direct summands of the form $\pi_n(S^{q+1})$). Therefore we have

$$\pi_n(S^{m_1}\vee\cdots\vee S^{m_k}) \cong \bigoplus_{q=1}^{\infty} \pi_n(S^{q+1})^{\oplus c_{q+1}} = \bigoplus_{q=2}^{\infty} \pi_n(S^q)^{c_q}.$$

Furthermore, $\pi_n(S^q) = 0$ for $q > n$, so we only need to consider $q \le n$ and hence

$$\pi_n(S^{m_1}\vee\cdots\vee S^{m_k}) \cong \bigoplus_{q=2}^{n} \pi_n(S^q)^{c_q}.$$

If $n < m_1 + m_2 - 1$, this agrees with the expression we found earlier. To see this, note that for $2 \le q < m_1 + m_2 - 1$, any solution of the equation $(m_1-1)w_1 + \cdots + (m_k-1)w_k = q-1$ must be of the form $w_i = 1$ for some i with $m_i = q$ and zero for all other m_j. This solution corresponds to a unique basic product of weight one, namely α_i. So we see that c_q is equal to the number of spheres in the wedge product of dimension q and hence recover the previous result.

Example: Suppose we want to calculate the homotopy groups of $S^3\vee S^4\vee S^5 = S^{2+1}\vee S^{3+1}\vee S^{4+1}$. For $n < 3+4-1 = 6$ we have $\pi_n(S^3\vee S^4\vee S^5) = \pi_n(S^3)\oplus\pi_n(S^4)\oplus\pi_n(S^5)$.

For $n = 6$ we want to find the solutions of the equation $2w_1 + 3w_2 + 4w_3 = 5$. The only solution is $(1,1,0)$ and the only basic product involving α_1 once and α_2 once is $[\alpha_1,\alpha_2]$, so $c_6 = A(1,1,0) = 1$ and therefore

$$\pi_6(S^3\vee S^4\vee S^5) \cong \pi_6(S^3)\oplus\pi_6(S^4)\oplus\pi_6(S^5)\oplus\pi_6(S^6).$$

For $n = 7$ the equation of interest is $2w_1 + 3w_2 + 4w_3 = 6$. The solutions are (3, 0, 0), (0, 2, 0), and (1, 0, 1). There are no basic products involving α_1 three times but no α_2 and α_3. Let's check with the formula

$$A(3,0,0) = \frac{1}{3}\sum_{d|w_i}\frac{\mu(d)(3/d)!}{(3/d)!0!0!} = \frac{1}{3}\sum_{d|w_i}\mu(d) = \frac{1}{3}(\mu(1)+\mu(3)) = \frac{1}{3}(1-1) = 0$$

Likewise, there are no basic products with α_2 twice, but no α_1 or α_3 (note, $[\alpha_2,\alpha_2]$ is not a basic product), so $A(0,2,0) = 0$. Finally, there is only one basic product with $(w_1,w_2,w_3) = (1,0,1)$, namely $[\alpha_1,\alpha_3]$. Therefore, $c_7 = A(3,0,0) + A(0,2,0) + A(1,0,1) = 1$, so

$$\pi_7(S^3\vee S^4\vee S^5) \cong \pi_7(S^3)\oplus\pi_7(S^4)\oplus\pi_7(S^5)\oplus\pi_7(S^6)\oplus\pi_7(S^7).$$

Here is a table of the relevant information for the next few values of n:

n	solutions of $2w_1 + 3w_2 + 4w_3 = n - 1$	A	c_n
8	(2, 1, 0)	$A(2,1,0) = 1$	
	(0, 1, 1)	$A(0,1,1) = 1$	$1 + 1 = 2$
9	(4, 0 ,0)	$A(4,0,0) = 0$	
	(2, 0, 1)	$A(2,0,1) = 1$	
	(1, 2, 0)	$A(1,2,0) = 1$	
	(0, 0, 2)	$A(0,0,2) = 0$	$0 + 1 + 1 + 0 = 2$
10	(3, 1, 0)	$A(3,1,0) = 1$	
	(1, 1, 1)	$A(1,1,1) = 2$	
	(0, 3, 0)	$A(0,3,0) = 0$	$1 + 2 + 0 = 3$
11	(5, 0, 0)	$A(5,0,0) = 0$	
	(3, 0, 1)	$A(3,0,1) = 1$	
	(2, 2, 0)	$A(2,2,0) = 1$	
	(1, 0, 2)	$A(1,0,2) = 1$	
	(0, 2, 1)	$A(0,2,1) = 1$	$0 + 1 + 1 + 1 + 1 = 4$
12	(4, 1, 0)	$A(4,1,0) = 1$	
	(2, 1, 1)	$A(2,1,1) = 3$	
	(1, 2, 0)	$A(1,2,0) = 1$	
	(0, 1, 2)	$A(0,1,2) = 1$	$1 + 3 + 1 + 1 = 6$

So, using our knowledge of the homotopy groups of spheres, we see for example that

$$\begin{aligned}
&\pi_{12}(S^3 \vee S^4 \vee S^5) \\
&\cong \pi_{12}(S^3) \oplus \pi_{12}(S^4) \oplus \pi_{12}(S^5) \oplus \pi_{12}(S^6) \oplus \pi_{12}(S^7) \oplus \pi_{12}(S^8)^2 \oplus \pi_{12}(S^9)^2 \oplus \pi_{12}(S^{10})^3 \\
&\quad \oplus \pi_{12}(S^{11})^4 \oplus \pi_{12}(S^{12})^6 \\
&\cong \mathbb{Z}_2^2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{30} \oplus \mathbb{Z}_2 \oplus 0 \oplus 0^2 \oplus \mathbb{Z}_{24}^2 \oplus \mathbb{Z}_2^3 \oplus \mathbb{Z}_2^4 \oplus \mathbb{Z}^6 \\
&\cong \mathbb{Z}^6 \oplus \mathbb{Z}_{30} \oplus \mathbb{Z}_{24}^2 \oplus \mathbb{Z}_2^{11} \\
&\cong \mathbb{Z}^6 \oplus (\mathbb{Z}_5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2) \oplus (\mathbb{Z}_8 \oplus \mathbb{Z}_3)^2 \oplus \mathbb{Z}_2^{11} \\
&\cong \mathbb{Z}^6 \oplus \mathbb{Z}_8^2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3^3 \oplus \mathbb{Z}_2^{12}
\end{aligned}$$

In this example, every sphere of dimension greater than $m_1 + m_2 - 1$ occurs at least once. This is not always the case. For example, consider $S^3 \vee S^5$. As the expression $2w_1 + 4w_2$ is never odd, $c_q = 0$ for q even, i.e. only homotopy groups of odd-dimensional spheres appear as direct summands. More generally, if the greatest common divisor of $r_1, \dots, r_k$ is r, then $c_q = 0$ if $r \nmid q$. Even if the relevant equation has solutions for a given q, there may not be any corresponding basic products. For example, $S^3 \vee S^4$. The equation $2w_1 + 3w_2 = 4$ has a unique solution, namely $(2, 0)$, but there are no basic products with two α_1 and no α_2, so $\pi_n(S^5)$ does not appear as a direct summand of $\pi_n(S^3 \vee S^4)$.

Here is some pseudocode for calculating the values of c_q

- Enter n.
- Enter $m_1, \dots, m_k$.
- Reorder m_i in increasing order to get m_i'
- Set $r_i = m_i' - 1$.
- For $q = 2, \dots, \min(r_1 + r_2, n)$, set c_q to be the number of elements of $[r_i]$ equal to $q - 1$.
- For $q = r_1 + r_2 + 1, \dots, n$
  - Calculate non-negative integer solutions of $[r_i]^T \mathbf{w} = n - 1$
  - For each solution $\mathbf{w}$, calculate (using Witt's Theorem) $A(\mathbf{w})$.

  - Set c_q = sum of $A(\mathbf{w})$
- Output $[c_q]$

References

[1] P. Hilton, *On the homotopy groups of the union of spheres*, Journal of the London Mathematical Society, 30 (1955), pp. 154–172.

[2] E. Witt, *Treue Darstellung Liescher Ringe*, Journal für die reine und angewandte Mathematik, 177 (1937), pp. 152–160.