WHICH GRASSMANNIANS ARE SPIN MANIFOLDS?

MICHAEL ALBANESE

Abstract. The purpose of this note is to determine which (unoriented, oriented, and complex) grassmannians are spin manifolds. In order to achieve this goal, formulae for the first and second Stiefel-Whitney class of a tensor product are derived.

Let $\mathrm{Gr}(a, b)$ denote the grassmanian of a-dimensional subspaces of a real b-dimensional vector space, and denote the tautological bundle over it by γ. Recall that $T\,\mathrm{Gr}(a, b) \cong \mathrm{Hom}(\gamma, \gamma^\perp) \cong \gamma^* \otimes \gamma^\perp \cong \gamma \otimes \gamma^\perp$ where $\gamma^\perp$ is the orthogonal complement of $\gamma \subset \varepsilon^b$ with respect to a fixed Riemannian metric on ε^b. As a smooth manifold M is spin if and only if $w_1(M) = 0$ and $w_2(M) = 0$, we need to determine formulae for $w_1(E \otimes F)$ and $w_2(E \otimes F)$.

Stiefel-Whitney Classes of a Tensor Product

Lemma. *Let L_1 and L_2 be real line bundles over a paracompact space B. Then $w_1(L_1 \otimes L_2) = w_1(L_1) + w_1(L_2)$.*

Proof. Let $\pi_i : \mathbb{RP}^\infty \times \mathbb{RP}^\infty$ denote projection onto the i^{th} factor and let $\mu : \mathbb{RP}^\infty \times \mathbb{RP}^\infty \to \mathbb{RP}^\infty$ be a classifying map for $\pi_1^*\gamma \otimes \pi_2^*\gamma$. By the Künneth theorem, $\pi_1^* w_1(\gamma)$ and $\pi_2^* w_1(\gamma)$ form a basis for $H^1(\mathbb{RP}^\infty \times \mathbb{RP}^\infty; \mathbb{Z}_2)$, so $w_1(\pi_1^*\gamma \otimes \pi_2^*\gamma) = a\pi_1^* w_1(\gamma) + b\pi_2^* w_1(\gamma)$ for some $a, b \in \mathbb{Z}_2$.

If $\sigma : \mathbb{RP}^\infty \times \mathbb{RP}^\infty \to \mathbb{RP}^\infty \times \mathbb{RP}^\infty$ is the map which interchanges factors, then $\pi_1 \circ \sigma = \pi_2$ and $\pi_2 \circ \sigma = \pi_1$, so $\sigma^*\mu^* w_1(\gamma) = a\pi_2^* w_1(\gamma) + b\pi_1^* w_1(\gamma)$, but $\sigma \circ \mu$ classifies $\pi_2^*\gamma \otimes \pi_1^*\gamma \cong \pi_1^*\gamma \otimes \pi_2^*\gamma$ so $\sigma \circ \mu$ is homotopic to μ. Therefore

$$a\pi_2^* w_1(\gamma) + b\pi_1^* w_1(\gamma) = (\sigma \circ \mu)^* w_1(\gamma) = \mu^* w_1(\gamma) = a\pi_1^* w_1(\gamma) + b\pi_2^* w_1(\gamma),$$

which implies $a = b$. So either $w_1(\pi_1^*\gamma \otimes \pi_2^*\gamma) = \pi_1^* w_1(\gamma) + \pi_2^* w_1(\gamma)$, or $w_1(\pi_1^*\gamma \otimes \pi_2^*\gamma) = 0$.

Now let $f_i : B \to \mathbb{RP}^\infty$ be a classifying map for L_i. Then

$$\begin{aligned}(f_1, f_2)^*(\pi_1^*\gamma \otimes \pi_2^*\gamma) &\cong ((f_1, f_2)^*\pi_1^*\gamma) \otimes ((f_1, f_2)^*\pi_2^*\gamma)\\ &\cong (\pi_1 \circ (f_1, f_2))^*\gamma \otimes (\pi_2 \circ (f_1, f_2)^*)\gamma\\ &\cong f_1^*\gamma \otimes f_2^*\gamma\\ &\cong L_1 \otimes L_2.\end{aligned}$$

As $w_1(L_1 \otimes L_2) = w_1((f_1, f_2)^*(\pi_1^*\gamma \otimes \pi_2^*\gamma) = (f_1, f_2)^* w_1(\pi_1^*\gamma \otimes \pi_2^*\gamma)$, if $w_1(\pi_1^*\gamma \otimes \pi_2^*\gamma) = 0$, then $w_1(L_1 \otimes L_2) = 0$. This is clearly false, just take L_1 to be non-trivial and L_2 to be trivial. Therefore $w_1(\pi_1^*\gamma \otimes \pi_2^*\gamma) = \pi_1^* w_1(\gamma) + \pi_2^* w_1(\gamma)$ and so

$$\begin{aligned}w_1(L_1 \otimes L_2) &= (f_1, f_2)^* w_1(\pi_1^*\gamma \otimes \pi_2^*\gamma)\\ &= (f_1, f_2)^*(\pi_1^* w_1(\gamma) + \pi_2^* w_1(\gamma))\\ &= (f_1, f_2)^*\pi_1^* w_1(\gamma) + (f_1, f_2)^*\pi_2^* w_1(\gamma)\\ &= (\pi_1 \circ (f_1, f_2))^* w_1(\gamma) + (\pi_2 \circ (f_1, f_2))^* w_1(\gamma)\\ &= f_1^* w_1(\gamma) + f_2^* w_1(\gamma)\end{aligned}$$

$$
\begin{aligned}
&= w_1(f_1^*\gamma) + w_1(f_2^*\gamma) \\
&= w_1(L_1) + w_1(L_2).
\end{aligned}
$$

□

With this lemma in hand, we can move on to the general case thanks to the splitting principle.

Theorem. *Let E and F be real vector bundles over a paracompact space B. Let $m = \operatorname{rank} E$ and $n = \operatorname{rank} F$. Then $w(E \otimes F) = p_{m,n}(w_1(E), \dots, w_m(E), w_1(F), \dots, w_n(F))$ where $p_{m,n}$ is the unique polynomial which satisfies*

$$p_{m,n}(\sigma_1, \dots, \sigma_m, \tau_1, \dots, \tau_n) = \prod_{i=1}^{m}\prod_{j=1}^{n}(1 + x_i + y_j)$$

where $\sigma_k = \sigma_k(x_1, \dots, x_m)$ and $\tau_k = \tau_k(y_1, \dots, y_n)$ are the k^{th} elementary symmetric polynomials in m and n variables respectively.

Proof. By the splitting principle, there is a paracompact space Y and a map $g : Y \to B$ such that $g^*E \cong \ell_1' \oplus \cdots \oplus \ell_m'$ and $g^* : H^*(Y; \mathbb{Z}_2) \to H^*(B; \mathbb{Z}_2)$ is injective. Again by the splitting principle, there is a paracompact space X and a map $f : X \to Y$ such that $f^*g^*F \cong \eta_1 \oplus \cdots \oplus \eta_n$, and $f^* : H^*(X; \mathbb{Z}_2) \to H^*(Y; \mathbb{Z}_2)$ is injective. Letting $\ell_i = f^*\ell_i'$, we have $f^*g^*E \cong \ell_1 \oplus \cdots \oplus \ell_m$. So

$$f^*g^*(E \otimes F) \cong (f^*g^*E) \otimes (f^*g^*F) \cong (\ell_1 \oplus \cdots \oplus \ell_m) \otimes (\eta_1 \oplus \cdots \oplus \eta_n) \cong \bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n} \ell_i \otimes \eta_j.$$

Therefore,

$$
\begin{aligned}
w(f^*g^*(E \otimes F)) &= w\left(\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n} \ell_i \otimes \eta_j\right) \\
&= \prod_{i=1}^{m}\prod_{j=1}^{n} w(\ell_i \otimes \eta_j) \\
&= \prod_{i=1}^{m}\prod_{j=1}^{n}(1 + w_1(\ell_i \otimes \eta_j)) \\
&= \prod_{i=1}^{m}\prod_{j=1}^{n}(1 + w_1(\ell_i) + w_1(\eta_j)) \\
&= \prod_{i=1}^{m}\prod_{j=1}^{n}(1 + x_i + y_j)
\end{aligned}
$$

where the penultimate equality uses the lemma and $x_i := w_1(\ell_i)$, $y_j := w_1(\eta_j)$.

Denote the final expression above by $q(x_1, \dots, x_m, y_1, \dots, y_n)$. Note that q is a polynomial which is symmetric in the x_i and the y_j separately, so by the fundamental theorem of symmetric polynomials, there is a unique polynomial $p_{m,n}$ such that

$$q(x_1, \dots, x_m, y_1, \dots, y_m) = p_{m,n}(\sigma_1, \dots, \sigma_m, \tau_1, \dots, \tau_n).$$

Now note that $\sigma_i(x_1, \dots, x_m) = w_i(\ell_1 \oplus \cdots \oplus \ell_m) = w_i(f^*g^*E) = f^*g^*w_i(E)$ and likewise $\tau_j(y_1, \dots, y_n) = f^*g^*w_j(F)$, so

$$
\begin{aligned}
f^*g^*w(E \otimes F) &= w(f^*g^*(E \otimes F)) \\
&= q(x_1, \dots, x_m, y_1, \dots, y_n) \\
&= p_{m,n}(\sigma_1, \dots, \sigma_m, \tau_1, \dots, \tau_n)
\end{aligned}
$$

$$= p_{m,n}(f^*g^*w_1(E), \dots, f^*g^*w_m(E), f^*g^*w_1(F), \dots, f^*g^*w_n(F))$$
$$= f^*g^*p_{m,n}(w_1(E), \dots, w_m(E), w_1(F), \dots, w_n(F)).$$

By the injectivity of f^* and g^*, we have $w(E \otimes F) = p_{m,n}(w_1(E), \dots, w_m(E), w_1(F), \dots, w_n(F))$. $\square$

The two proofs above constitute a solution to Problem 7-C from [3].

As in the proof, we will use $q(x_1, \dots, x_m, y_1, \dots, y_n)$ to denote the right hand side of the equation in the theorem.

If we can identify the degree k part of $p_{m,n}$, then we can obtain an explicit formula for $w_k(E \otimes F)$ in terms of $w_1(E), \dots, w_k(E), w_1(F), \dots, w_k(F)$. In particular, we need to express the degree k part of q as a polynomial in elementary symmetric polynomials. To achieve our main goal, we only need to do this for $k = 1$ and 2.

The degree one part of q is

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(x_i + y_j) = n\sum_{i=1}^{m} x_i + m\sum_{j=1}^{n} y_j = n\sigma_1(x_1, \dots, x_m) + m\tau_1(y_1, \dots, y_n).$$

Therefore,

$$\boxed{w_1(E \otimes F) = nw_1(E) + mw_1(F).}$$

Now we need to identify the degree two part of q; this is more difficult. First note that q is the product of mn factors, and any two factors gives rise to four degree two terms, so there should be a total of $4\binom{mn}{2}$ terms in the degree two part of q. There are five distinct types of terms that can appear: x_i^2, y_j^2, $x_ix_{i'}$ with $i \neq i'$, $y_jy_{j'}$ with $j \neq j'$, and x_iy_j.

The x_i^2 terms only arise from the subproduct $(1 + x_i + y_1) \dots (1 + x_i + y_n)$, and each choice of two factors gives rise to one such term, so in total there are $\binom{n}{2}$ copies of x_i^2.

The y_j^2 terms only arise from the subproduct $(1 + x_1 + y_j) \dots (1 + x_m + y_j)$, and each choice of two factors gives rise to one such term, so in total there are $\binom{m}{2}$ copies of y_j^2.

The $x_ix_{i'}$ terms with $i \neq i'$ only arise from the subproduct $(1 + x_i + y_1) \dots (1 + x_i + y_n)(1 + x_{i'} + y_1) \dots (1 + x_{i'} + y_n)$, and each choice of a factor from the first n and a factor from the second n gives rise to one such term, so in total there are n^2 copies of $x_ix_{i'}$.

The $y_jy_{j'}$ terms with $j \neq j'$ only arise from the subproduct $(1 + x_1 + y_j) \dots (1 + x_m + y_j)(1 + x_1 + y_{j'}) \dots (1 + x_m + y_{j'})$, and each choice of a factor from the first m and a factor from the second m gives rise to one such term, so in total there are m^2 copies of $y_jy_{j'}$.

Now consider terms of the form x_iy_j. They can only arise from products of factors of the form $(1 + x_{i'} + y_{j'})$ where $i = i'$ or $j = j'$. Given one of the $n - 1$ factors of the form $(1 + x_i + y_{j'})$ with $j' \neq j$, there are precisely m factors which contain y_j, namely $(1 + x_1 + y_j), \dots, (1 + x_m + y_j)$, which can combine with $(1 + x_i + y_{j'})$ to produce one x_iy_j term. Likewise, given one of the $m - 1$ factors of the form $(1 + x_{i'} + y_j)$ with $i' \neq i$, there are precisely n factors which contain x_i, namely $(1+x_i+y_1), \dots, (1+x_m+y_j)$, which can combine with $(1+x_{i'}+y_j)$ to produce one x_iy_j term. Finally, the unique factor $(1 + x_i + y_j)$ can combine with $(m - 1) + (n - 1)$ factors to produce one x_iy_j term, namely factors of the form $(1 + x_{i'} + y_{j'})$ where $i = i'$ or $j = j'$, but not both. Note, we have double counted each appearance of x_iy_j, so in total there are $\frac{1}{2}[m(n-1)+n(m-1)+(m-1)+(n-1)] = mn-1$ copies of x_iy_j.

We should check that we haven't missed any terms. There are m terms of the form x_i^2, n terms of the form y_j^2, $\binom{m}{2}$ terms of the form $x_ix_{i'}$ with $i \neq i'$, $\binom{n}{2}$ terms of the form $y_jy_{j'}$ with $j \neq j'$, and mn

terms of the form x_iy_j. Therefore, there are a total of

$$\begin{aligned}
&m\binom{n}{2}+n\binom{m}{2}+\binom{m}{2}n^2+\binom{n}{2}m^2+mn(mn-1)\\
&=\frac{1}{2}mn(n-1)+\frac{1}{2}mn(m-1)+\frac{1}{2}m^2n(n-1)+\frac{1}{2}mn^2(m-1)+mn(mn-1)\\
&=\frac{1}{2}mn[(n-1)+(m-1)+m(n-1)+n(m-1)+2(mn-1)]\\
&=\frac{1}{2}mn[n-1+m-1+mn-m+mn-n+2mn-2]\\
&=\frac{1}{2}mn[4mn-4]\\
&=4\frac{mn(mn-1)}{2}\\
&=4\binom{mn}{2}
\end{aligned}$$

terms in the degree two part of q as predicted.

So the degree two part of q is

$$\begin{aligned}
&\binom{n}{2}\sum_{i=1}^m x_i^2+\binom{m}{2}\sum_{j=1}^n y_j^2+n^2\sum_{1\le i<i'\le m}x_ix_{i'}+m^2\sum_{1\le j<j'\le n}y_jy_{j'}+(mn-1)\sum_{i=1}^n\sum_{j=1}^n x_iy_j\\
=&\binom{n}{2}\left(\sum_{i=1}^m x_i\right)^2+\binom{m}{2}\left(\sum_{j=1}^n y_j\right)^2+n^2\sigma_2(x_1,\dots,x_m)+m^2\tau_2(y_1,\dots,y_n)\\
&+(mn-1)\left(\sum_{i=1}^m x_i\right)\left(\sum_{j=1}^n y_j\right)\\
=&\binom{n}{2}\sigma_1(x_1,\dots,x_m)^2+\binom{m}{2}\tau_1(y_1,\dots,y_n)^2+n^2\sigma_2(x_1,\dots,x_n)+m^2\tau_2(y_1,\dots,y_n)\\
&+(mn-1)\sigma_1(x_1,\dots,x_m)\tau_1(y_1,\dots,y_n).
\end{aligned}$$

Therefore,

$$\boxed{w_2(E\otimes F)=\binom{n}{2}w_1(E)^2+\binom{m}{2}w_1(F)^2+n^2w_2(E)+m^2w_2(F)+(mn-1)w_1(E)w_1(F).}$$

Which Unoriented Grassmannians are spin manifolds?

Write the grassmannian $\mathrm{Gr}(a,b)$ as $\mathrm{Gr}(m,m+n)$ where $m=a$ and $n=b-a$. Then $\gamma^\perp$ has rank n. As $T\,\mathrm{Gr}(m,m+n)=\gamma\otimes\gamma^\perp$, we have

$$w_1(\mathrm{Gr}(m,m+n))=nw_1(\gamma)+mw_1(\gamma^\perp).$$

Using the fact that $\gamma\oplus\gamma^\perp\cong\varepsilon^{m+n}$, we see that $w_1(\gamma^\perp)=w_1(\gamma)$ and therefore

$$w_1(\mathrm{Gr}(m,m+n))=nw_1(\gamma)+mw_1(\gamma^\perp)=nw_1(\gamma)+mw_1(\gamma)=(m+n)w_1(\gamma).$$

Proceeding in a similar way, we have

$$w_2(\mathrm{Gr}(m,m+n))=\binom{n}{2}w_1(\gamma)^2+\binom{m}{2}w_1(\gamma^\perp)^2+n^2w_2(\gamma)+m^2w_2(\gamma^\perp)+(mn-1)w_1(\gamma)w_1(\gamma^\perp).$$

Again, as $\gamma\oplus\gamma^\perp\cong\varepsilon^{m+n}$, we see that $w_2(\gamma^\perp)=w_2(\gamma)+w_1(\gamma)w_1(\gamma^\perp)=w_2(\gamma)+w_1(\gamma)^2$, so

$$w_2(\mathrm{Gr}(m,m+n))$$

$$\begin{aligned}
&= \binom{n}{2} w_1(\gamma)^2 + \binom{m}{2} w_1(\gamma^\perp)^2 + n^2 w_2(\gamma) + m^2 w_2(\gamma^\perp) + (mn-1) w_1(\gamma) w_1(\gamma^\perp)\\
&= \binom{n}{2} w_1(\gamma)^2 + \binom{m}{2} w_1(\gamma)^2 + n^2 w_2(\gamma) + m^2 (w_2(\gamma) + w_1(\gamma)^2) + (mn-1) w_1(\gamma) w_1(\gamma)\\
&= \left[\binom{m}{2} + \binom{n}{2} + m^2 + mn - 1\right] w_1(\gamma)^2 + (m^2+n^2) w_2(\gamma).
\end{aligned}$$

As $\binom{d}{2} = \frac{1}{2}d(d-1)$, its parity is determined by $d \bmod 4$. More precisely, $\binom{d}{2}$ is even if $d \equiv 0, 1 \bmod 4$ and odd if $d \equiv 2, 3 \bmod 4$. So the parity of the first two terms is determined by the values of m and n modulo 4, while the parity of remaining terms is determined by the values of m and n modulo 2. So we see that

$$w_2(\mathrm{Gr}(m, m+n)) = \begin{cases} 0 & (m,n) \equiv (0,2), (1,3), (2,0), (3,1) \bmod 4\\ w_2(\gamma) & (m,n) \equiv (0,3), (1,0), (2,1), (3,2) \bmod 4\\ w_1(\gamma)^2 & (m,n) \equiv (0,0), (1,1), (2,2), (3,3) \bmod 4\\ w_2(\gamma) + w_1(\gamma)^2 & (m,n) \equiv (0,1), (1,2), (2,3), (3,0) \bmod 4. \end{cases}$$

Note that the difference $m - n$ is constant in each row, so we can more succinctly express the above as

$$w_2(\mathrm{Gr}(m, m+n)) = \begin{cases} 0 & m-n \equiv 2 \bmod 4\\ w_2(\gamma) & m-n \equiv 1 \bmod 4\\ w_1(\gamma)^2 & m-n \equiv 0 \bmod 4\\ w_2(\gamma) + w_1(\gamma)^2 & m-n \equiv 3 \bmod 4. \end{cases}$$

Upon first glance, the above description seems to contradict the fact that $\mathrm{Gr}(m, m+n)$ and $\mathrm{Gr}(n, m+n)$ are diffeomorphic, at least in the case where $m - n$ is odd. Why does interchanging m and n give a different expression for w_2? In order to understand this disparity, denote the tautological bundles over $\mathrm{Gr}(m, m+n)$ and $\mathrm{Gr}(n, m+n)$ by γ_m and γ_n respectively.

Recall that there is a diffeomorphism $\varphi : \mathrm{Gr}(m, m+n) \to \mathrm{Gr}(n, m+n)$ given by $P \mapsto P^\perp$; note, this requires an inner product on the ambient vector space. It follows that $\varphi^*\gamma_n \cong \gamma_m^\perp$. So, if $m - n \equiv 3 \bmod 4$, we have $w_2(\mathrm{Gr}(m, m+n)) = w_2(\gamma_m) + w_1(\gamma_m)^2 \in H^2(\mathrm{Gr}(m, m+n); \mathbb{Z}_2)$ and $w_2(\mathrm{Gr}(n, m+n)) = w_2(\gamma_n) \in H^2(\mathrm{Gr}(n, m+n); \mathbb{Z}_2)$. The cohomology rings are not equal, so we cannot compare these two elements, but the diffeomorphism φ gives rise to an isomorphism between them, namely φ^*. Under this isomorphism,

$$\varphi^* w_2(\gamma_n) = w_2(\varphi^*\gamma_n) = w_2(\gamma_m^\perp) = w_2(\gamma_m) + w_1(\gamma_m)^2.$$

The case $m - n \equiv 1 \bmod 4$ is similar.

Now that we have expressions for $w_1(\mathrm{Gr}(m, m+n))$ and $w_2(\mathrm{Gr}(m, m+n))$, we can finally determine for which m and n the manifold $\mathrm{Gr}(m, m+n)$ is spin.

Recall that $H^*(\mathrm{Gr}(m, m+n); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1(\gamma), \dots, w_m(\gamma)]/(\bar{w}_{n+1}, \dots, \bar{w}_{m+n})$ where $\deg \bar{w}_i = i$ and $\bar{w} = 1 + \bar{w}_1 + \dots + \bar{w}_{m+n}$ satisfies $w(\gamma)\bar{w} = 1$; see Proposition 11.1 of [1]. In particular, if $m, n \geq 2$, then $w_1(\gamma)$, $w_2(\gamma)$, $w_1(\gamma)^2$, and $w_2(\gamma) + w_1(\gamma)^2$ are all non-zero. If $m = 1$ or $n = 1$, then the grassmannian is a projective space, in which case it is easy to check whether $w_1(\gamma)$, $w_2(\gamma)$, $w_1(\gamma)^2$, and $w_2(\gamma) + w_1(\gamma)^2$ are non-zero or not.

We can also ask about the non-orientable anologues of spin, namely pin$^+$ and pin$^-$. Recall that the obstruction to a smooth manifold M admitting a pin$^+$ structure is $w_2(M)$, and the obstruction to admitting a pin$^-$ structure is $w_2(M) + w_1(M)^2$.

Theorem. *The grassmannian* $\mathrm{Gr}(m, m+n)$ *is:*

- *orientable if and only if* $m + n$ *is even.*

- *spin if and only if* $m - n \equiv 2 \bmod 4$*, or* $m = n = 1$*, i.e.* $\mathrm{Gr}(1,2) = \mathbb{RP}^1 = S^1$.
- *pin*$^+$ *if and only if it is spin or is a projective space of dimension* $4k$.
- *pin*$^-$ *if and only if it is spin or is a projective space of dimension* $4k+2$.

Which Oriented Grassmannians are spin?

Let $\mathrm{Gr}^+(a,b)$ denote the grassmanian of oriented a-dimensional subspaces of a real b-dimensional vector space, and denote the tautological bundle over it by γ_+. Similar to the unoriented case, we have $T\,\mathrm{Gr}^+(a,b) \cong \gamma_+ \otimes \gamma_+^\perp$ where $\gamma_+^\perp$ is the orthogonal complement of $\gamma_+ \subset \varepsilon^b$ with respect to a fixed Riemannian metric on ε^b.

There is a double covering $\pi : \mathrm{Gr}^+(a,b) \to \mathrm{Gr}(a,b)$ which forgets the orientation of the subspace. It follows that $\pi^*\gamma \cong \gamma_+$, and hence $w_i(\gamma_+) = w_i(\pi^*\gamma) = \pi^* w_i(\gamma)$. The Gysin sequence associated to π is

$$\dots \to H^i(\mathrm{Gr}(a,b);\mathbb{Z}_2) \xrightarrow{\cup e} H^{i+1}(\mathrm{Gr}(a,b);\mathbb{Z}_2) \xrightarrow{\pi^*} H^{i+1}(\mathrm{Gr}^+(a,b);\mathbb{Z}_2) \to H^{i+1}(\mathrm{Gr}(a,b);\mathbb{Z}_2) \to \dots$$

where $e \in H^1(\mathrm{Gr}(a,b);\mathbb{Z}_2) = \{0, w_1(\gamma)\}$ is the Euler class of π; as π is not the trivial double cover, $e = w_1(\gamma)$.

By the exactness of the Gysin sequence, $w_i(\gamma_+)$ is zero if and only if $w_i(\gamma) = w_1(\gamma) \cup \alpha$ for some α; that is, $w_i(\gamma)$ is not in the ideal generated by $w_1(\gamma)$. In particular, $w_1(\gamma_+) = 0$, and hence $w_1(\mathrm{Gr}^+(m, m+n)) = 0$.

It now follows from the computation of $w_2(\mathrm{Gr}(m, m+n))$ in the previous section that

$$w_2(\mathrm{Gr}^+(m,m+n)) = \begin{cases} 0 & m - n \equiv 0 \bmod 2 \\ w_2(\gamma_+) & m - n \equiv 1 \bmod 2. \end{cases}$$

As $H^*(\mathrm{Gr}(k,n);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1(\gamma), \dots, w_k(\gamma)]/(\overline{w}_{n+1}, \dots, \overline{w}_{m+n})$, if $m, n \geq 2$, then $w_2(\gamma)$ is not in the ideal generated by $w_1(\gamma)$ and hence $w_2(\gamma^+) \neq 0$. If $m = 1$ or $n = 1$, then the orientable grassmannian is a sphere and hence $w_2(\mathrm{Gr}^+(m, m+n)) = 0$.

Theorem. *The grassmannian* $\mathrm{Gr}^+(m, m+n)$ *is always orientable. Moreover, the obstructions to spin, pin*$^+$*, and pin*$^-$ *structures coincide and they vanish if and only if* $m - n$ *is even,* $m = 1$*, or* $n = 1$.

This agrees with Theorem 8 of [2].

Which Complex Grassmannians are spin?

Let $\mathrm{Gr}^{\mathbb{C}}(a,b)$ denote the grassmanian of complex a-dimensional subspaces of a complex b-dimensional vector space, and denote the tautological bundle over it by $\gamma_{\mathbb{C}}$. Similar to the previous cases, we have $T\,\mathrm{Gr}^{\mathbb{C}}(a,b) \cong \overline{\gamma_{\mathbb{C}}} \otimes \gamma_{\mathbb{C}}^\perp$ as complex vector bundles, where $\gamma_{\mathbb{C}}^\perp$ is the orthogonal complement of $\gamma_{\mathbb{C}} \subset \varepsilon_{\mathbb{C}}^b$ with respect to some fixed hermitian metric on $\varepsilon_{\mathbb{C}}^b$.

As $\mathrm{Gr}^{\mathbb{C}}(m, m+n)$ is a complex manifold, it is orientable, i.e. $w_1(\mathrm{Gr}^{\mathbb{C}}(m, m+n)) = 0$. Instead of using the formula for $w_2(E \otimes F)$, we have a shortcut in the complex case: we can use the Chern character to compute $c_1(\mathrm{Gr}^{\mathbb{C}}(m, m+n))$ and hence $w_2(\mathrm{Gr}^{\mathbb{C}}(m, m+n))$.

The Chern character is extremely useful as it satisfies $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E)\,\mathrm{ch}(F)$. As $\mathrm{ch}(E) = \mathrm{rank}(E) + c_1(E) + \dots$ this immediately implies

$$c_1(E \otimes F) = \mathrm{rank}(F) c_1(E) + \mathrm{rank}(E) c_1(F).$$

In particular,

$$c_1(\mathrm{Gr}^{\mathbb{C}}(m, m+n)) = c_1(\overline{\gamma_{\mathbb{C}}} \otimes \gamma_{\mathbb{C}}^\perp) = n c_1(\overline{\gamma_{\mathbb{C}}}) + m c_1(\gamma_{\mathbb{C}}^\perp).$$

As $\gamma_{\mathbb{C}} \oplus \gamma_{\mathbb{C}}^{\perp} \cong \varepsilon_{\mathbb{C}}^{m+n}$, we see that $c_1(\gamma_{\mathbb{C}}^{\perp}) = -c_1(\gamma^{\mathbb{C}})$. As for the other term, $c_i(\overline{E}) = (-1)^i c_i(E)$ so we conclude that

$$c_1(\mathrm{Gr}^+(m, m+n)) = nc_1(\overline{\gamma_{\mathbb{C}}}) + mc_1(\gamma_{\mathbb{C}}^{\perp}) = -nc_1(\gamma_{\mathbb{C}}) - mc_1(\gamma_{\mathbb{C}}) = -(m+n)c_1(\gamma_{\mathbb{C}}).$$

As $H^*(\mathrm{Gr}^{\mathbb{C}}(m, m+n); \mathbb{Z}) \cong \mathbb{Z}[c_1(\gamma_{\mathbb{C}}), \ldots, c_m(\gamma_{\mathbb{C}})]/(\bar{c}_{m+1}, \ldots, \bar{c}_{m+n})$ where $\bar{c}_i$ are defined in analogy with the previous cases, we see that $c_1(\gamma_{\mathbb{C}})$ is non-zero and is not divisible by 2. Therefore $w_2(\mathrm{Gr}^{\mathbb{C}}(m, m+n)) = (m+n)w_2(\gamma_{\mathbb{C}})$; as $c_1(\gamma_{\mathbb{C}})$ is not divisible by 2, we see that $w_2(\gamma_{\mathbb{C}}) \neq 0$. Therefore, we arrive at the following result.

Theorem. *The grassmannian* $\mathrm{Gr}^{\mathbb{C}}(m, m+n)$ *is always orientable. Moreover, the obstructions to spin, pin$^+$, and pin$^-$ structures coincide and they vanish if and only if* $m+n$ *is even.*

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