WHICH GRASSMANNIANS ARE SPIN MANIFOLDS?

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Abstract. The purpose of this note is to determine which (unoriented, oriented, and complex) grassmannians are spin manifolds. In order to achieve this goal, formulae for the first and second Stiefel-Whitney class of a tensor product are derived.

Let Gr(a, b) denote the grassmanian of a-dimensional subspaces of a real b-dimensional vector space, and denote the tautological bundle over it by γ. Recall that TGr(a, b) ∼= Hom(γ, γ+) ∼= γ∗ ⊗ γ+ ∼= γ ⊗ γ+ where γ+ is the orthogonal complement of γ C εb with respect to a fixed Riemannian metric on εb. As a smooth manifold M is spin if and only if w1(M) = 0 and w2(M) = 0, we need to determine formulae for w1(E ⊗ F) and w2(E ⊗ F).

Stiefel-Whitney Classes of a Tensor Product

Lemma. Let L1 and L2 be real line bundles over a paracompact space B. Then w1(L1 ⊗ L2) = w1(L1) + w1(L2).

Proof. Let πi : RP∞ × RP∞ denote projection onto the ith factor and let μ : RP∞ × RP∞ → RP∞ be a classifying map for π1∗γ ⊗ π2∗γ. By the Küneth theorem, π1∗w1(γ) and π2∗w1(γ) form a basis for

\[ H^1(RP\infty \times RP\infty; \mathbb{Z}_2) \]

so w1(π1∗γ ⊗ π2∗γ) = aπ1∗w1(γ) + bπ2∗w1(γ) for some a, b ∈ \mathbb{Z}_2.

If σ : RP∞ × RP∞ → RP∞ × RP∞ is the map which interchanges factors, then π1 ∘ σ = π2 and π2 ∘ σ = π1, so σ∗μ∗w1(γ) = aπ2∗w1(γ) + bπ1∗w1(γ), but σ ∘ μ classifies π2∗γ ⊗ π1∗γ ∼= π1∗γ ⊗ π2∗γ so σ ∘ μ is homotopic to μ. Therefore

\[ aπ2∗w1(γ) + bπ1∗w1(γ) = (σ ∘ μ)∗w1(γ) = μ∗w1(γ) = aπ1∗w1(γ) + bπ2∗w1(γ), \]

which implies a = b. So either w1(π1∗γ ⊗ π2∗γ) = π1∗w1(γ) + π2∗w1(γ), or w1(π1∗γ ⊗ π2∗γ) = 0.

Now let f : B → RP∞ be a classifying map for L_i. Then

\[ (f_1, f_2)^∗(π_1∗γ ⊗ π_2∗γ) \cong ((f_1, f_2)^∗π_1∗γ) ⊗ ((f_1, f_2)^∗π_2∗γ) \cong (f_1^∗ ⊗ f_2^∗)γ; f_1^* ⊗ f_2^* \cong L_1 ⊗ L_2. \]

As w1(L1 ⊗ L2) = w1((f_1, f_2)^∗(π_1∗γ ⊗ π_2∗γ)) = (f_1, f_2)^∗w1(π_1∗γ ⊗ π_2∗γ), if w1(π1∗γ ⊗ π2∗γ) = 0, then w1(L1 ⊗ L2) = 0. This is clearly false, just take L1 to be non-trivial and L2 to be trivial. Therefore w1(π1∗γ ⊗ π2∗γ) = π1∗w1(γ) + π2∗w1(γ) and so

\[ w1(L1 ⊗ L2) = (f_1, f_2)^∗w1(π_1∗γ ⊗ π_2∗γ) = (f_1, f_2)^∗(π_1∗w1(γ) + π_2∗w1(γ)) = (f_1, f_2)^∗π_1∗w1(γ) + (f_1, f_2)^∗π_2∗w1(γ) = (π_1 ∘ (f_1, f_2))^∗w1(γ) + (π_2 ∘ (f_1, f_2))^∗w1(γ) = f_1^∗w_1(γ) + f_2^∗w_1(γ) \]
Let Theorem. With this lemma in hand, we can move on to the general case thanks to the splitting principle.

\[ w(F \otimes E) = \sum_{i=1}^{m} \sum_{j=1}^{n} w(\ell_i \otimes \eta_j) \]

where \( x_i := w_1(\ell_i), y_j := w_1(\eta_j) \).

With this lemma in hand, we can move on to the general case thanks to the splitting principle.

**Theorem.** Let \( E \) and \( F \) be real vector bundles over a paracompact space \( B \). Let \( m = \text{rank} E \) and \( n = \text{rank} F \). Then \( w(E \otimes F) = p_{m,n}(w_1(E), \ldots, w_m(E), w_1(F), \ldots, w_n(F)) \) where \( p_{m,n} \) is the unique polynomial which satisfies

\[
p_{m,n}(\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n) = \prod_{i=1}^{m} \prod_{j=1}^{n} (1 + x_i + y_j)
\]

where \( \sigma_k = \sigma_k(x_1, \ldots, x_m) \) and \( \tau_k = \tau_k(y_1, \ldots, y_n) \) are the \( k \)th elementary symmetric polynomials in \( m \) and \( n \) variables respectively.

**Proof.** By the splitting principle, there is a paracompact space \( Y \) and a map \( g : Y \to B \) such that \( g^*E \cong \ell_1' \oplus \cdots \oplus \ell_m' \) and \( g^* : H^*(Y; \mathbb{Z}_2) \to H^*(B; \mathbb{Z}_2) \) is injective. Again by the splitting principle, there is a paracompact space \( X \) and a map \( f : X \to Y \) such that \( f^*g^*F \cong \eta_1 \oplus \cdots \oplus \eta_n \), and \( f^* : H^*(X; \mathbb{Z}_2) \to H^*(Y; \mathbb{Z}_2) \) is injective. Letting \( \ell_i = f^*\ell_i' \), we have \( f^*g^*E \cong \ell_1 \oplus \cdots \oplus \ell_m \). So

\[
f^*g^*(E \otimes F) \cong (f^*g^*E) \otimes (f^*g^*F) \cong (\ell_1 \oplus \cdots \oplus \ell_m) \otimes (\eta_1 \oplus \cdots \oplus \eta_n) \cong \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} \ell_i \otimes \eta_j.
\]

Therefore,

\[
w(f^*g^*(E \otimes F)) = w\left( \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} \ell_i \otimes \eta_j \right) = \prod_{i=1}^{m} \prod_{j=1}^{n} w(\ell_i \otimes \eta_j) = \prod_{i=1}^{m} \prod_{j=1}^{n} (1 + w_1(\ell_i \otimes \eta_j)) = \prod_{i=1}^{m} \prod_{j=1}^{n} (1 + w_1(\ell_i) + w_1(\eta_j)) = \prod_{i=1}^{m} \prod_{j=1}^{n} (1 + x_i + y_j)
\]

where the penultimate equality uses the lemma and \( x_i := w_1(\ell_i), y_j := w_1(\eta_j) \).

Denote the final expression above by \( q(x_1, \ldots, x_m, y_1, \ldots, y_n) \). Note that \( q \) is a polynomial which is symmetric in the \( x_i \) and the \( y_j \) separately, so by the fundamental theorem of symmetric polynomials, there is a unique polynomial \( p_{m,n} \) such that

\[
q(x_1, \ldots, x_m, y_1, \ldots, y_n) = p_{m,n}(\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n).
\]

Now note that \( \sigma_i(x_1, \ldots, x_m) = w_1(\ell_1 \oplus \cdots \oplus \ell_m) = w_1(f^*g^*E) = f^*g^*w_i(E) \) and likewise \( \tau_j(y_1, \ldots, y_n) = f^*g^*w_j(F) \), so

\[
f^*g^*w(E \otimes F) = w(f^*g^*(E \otimes F)) = q(x_1, \ldots, x_m, y_1, \ldots, y_n) = p_{m,n}(\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n)
\]
By the injectivity of $f^*$ and $g^*$, we have $w(E \otimes F) = p_{m,n}(w_1(E), \ldots, w_m(E), w_1(F), \ldots, w_n(F))$. □

The two proofs above constitute a solution to Problem 7-C from [3].

As in the proof, we will use $q(x_1, \ldots, x_m, y_1, \ldots, y_n)$ to denote the right hand side of the equation in the theorem.

If we can identify the degree $k$ part of $p_{m,n}$, then we can obtain an explicit formula for $w_k(E \otimes F)$ in terms of $w_1(E), \ldots, w_k(E), w_1(F), \ldots, w_k(F)$. In particular, we need to express the degree $k$ part of $q$ as a polynomial in elementary symmetric polynomials. To achieve our main goal, we only need to do this for $k = 1$ and 2.

The degree one part of $q$ is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (x_i + y_j) = n \sum_{i=1}^{m} x_i + m \sum_{j=1}^{n} y_j = n\sigma_1(x_1, \ldots, x_m) + m\tau_1(y_1, \ldots, y_n).$$

Therefore,

$$w_1(E \otimes F) = nw_1(E) + mw_1(F).$$

Now we need to identify the degree two part of $q$; this is more difficult. First note that $q$ is the product of $mn$ factors, and any two factors gives rise to four degree two terms, so there should be a total of $4^{\binom{mn}{2}}$ terms in the degree two part of $q$. There are five distinct types of terms that can appear: $x_i^2$, $y_j^2$, $x_i x_{i'}$ with $i \neq i'$, $y_j y_{j'}$ with $j \neq j'$, and $x_i y_j$.

The $x_i^2$ terms only arise from the subproduct $(1 + x_i + y_1) \ldots (1 + x_i + y_n)$, and each choice of two factors gives rise to one such term, so in total there are $\binom{n}{2}$ copies of $x_i^2$.

The $y_j^2$ terms only arise from the subproduct $(1 + x_1 + y_j) \ldots (1 + x_m + y_j)$, and each choice of two factors gives rise to one such term, so in total there are $\binom{m}{2}$ copies of $y_j^2$.

The $x_i x_{i'}$ terms with $i \neq i'$ only arise from the subproduct $(1 + x_i + y_1) \ldots (1 + x_i + y_n)(1 + x_{i'} + y_1) \ldots (1 + x_{i'} + y_n)$, and each choice of a factor from the first $n$ and a factor from the second $n$ gives rise to one such term, so in total there are $n^2$ copies of $x_i x_{i'}$.

The $y_j y_{j'}$ terms with $j \neq j'$ only arise from the subproduct $(1 + x_1 + y_j) \ldots (1 + x_m + y_j)(1 + x_1 + y_{j'}) \ldots (1 + x_m + y_{j'})$, and each choice of a factor from the first $m$ and a factor from the second $m$ gives rise to one such term, so in total there are $m^2$ copies of $y_j y_{j'}$.

Now consider terms of the form $x_i y_j$. They can only arise from products of factors of the form $(1 + x_i + y_j')$ where $i = i'$ or $j = j'$. Given one of the $n - 1$ factors of the form $(1 + x_i + y_j')$ with $j' \neq j$, there are precisely $m$ factors which contain $y_j$, namely $(1 + x_i + y_j), \ldots, (1 + x_m + y_j)$, which can combine with $(1 + x_i + y_j')$ to produce one $x_i y_j$ term. Likewise, given one of the $m - 1$ factors of the form $(1 + x_{i'} + y_j)$ with $i' \neq i$, there are precisely $n$ factors which contain $x_i$, namely $(1 + x_i + y_1), \ldots, (1 + x_m + y_j)$, which can combine with $(1 + x_{i'} + y_j)$ to produce one $x_i y_j$ term. Finally, the unique factor $(1 + x_i + y_j)$ can combine with $(m - 1) + (n - 1)$ factors to produce one $x_i y_j$ term, namely factors of the form $(1 + x_{i'} + y_{j'})$ where $i = i'$ or $j = j'$, but not both. Note, we have double counted each appearance of $x_i y_j$, so in total there are $\frac{1}{2}[m(n-1)+n(m-1)+(m-1)+(n-1)] = mn - 1$ copies of $x_i y_j$.

We should check that we haven’t missed any terms. There are $m$ terms of the form $x_i^2$, $n$ terms of the form $y_j^2$, $\binom{m}{2}$ terms of the form $x_i x_{i'}$ with $i \neq i'$, $\binom{n}{2}$ terms of the form $y_j y_{j'}$ with $j \neq j'$, and $mn$
terms of the form \(x_iy_j\). Therefore, there are a total of
\[
\begin{align*}
&\frac{m}{2} \binom{n}{2} + n \binom{m}{2} + \binom{m}{2} n^2 + \binom{n}{2} m^2 + mn(mn - 1) \\
&= \frac{1}{2} mn(n - 1) + \frac{1}{2} mn(m - 1) + \frac{1}{2} m^2 n(n - 1) + \frac{1}{2} mn^2 (m - 1) + mn(mn - 1) \\
&= \frac{1}{2} mn[(n - 1) + (m - 1) + m(n - 1) + n(m - 1) + 2(mn - 1)] \\
&= \frac{1}{2} mn[n - 1 + m - 1 + mn - m + mn - n + 2mn - 2] \\
&= \frac{1}{2} mn[4mn - 4] \\
&= 4 \frac{mn(mn - 1)}{2} \\
&= 4 \binom{mn}{2}
\end{align*}
\]
terms in the degree two part of \(q\) as predicted.

So the degree two part of \(q\) is
\[
\begin{align*}
&\binom{n}{2} \sum_{i=1}^{m} x_i^2 + \binom{m}{2} \sum_{j=1}^{n} y_j^2 + n^2 \sum_{1 \leq i < j \leq m} x_i x_j + m^2 \sum_{1 \leq j < i \leq n} y_j y_i + (mn - 1) \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j \\
&= \binom{n}{2} \left( \sum_{i=1}^{m} x_i \right)^2 + \binom{m}{2} \left( \sum_{j=1}^{n} y_j \right)^2 + n^2 \sigma_2 (x_1, \ldots, x_m) + m^2 \tau_2 (y_1, \ldots, y_n) \\
&\quad + (mn - 1) \left( \sum_{i=1}^{m} x_i \right) \left( \sum_{j=1}^{n} y_j \right) \\
&= \binom{n}{2} \sigma_1 (x_1, \ldots, x_m)^2 + \binom{m}{2} \tau_1 (y_1, \ldots, y_n)^2 + n^2 \sigma_2 (x_1, \ldots, x_n) + m^2 \tau_2 (y_1, \ldots, y_n) \\
&\quad + (mn - 1) \sigma_1 (x_1, \ldots, x_m) \tau_1 (y_1, \ldots, y_n).
\end{align*}
\]
Therefore,
\[
\begin{align*}
w_2(E \otimes F) = &\binom{n}{2} w_1(E)^2 + \binom{m}{2} w_1(F)^2 + n^2 w_2(E) + m^2 w_2(F) + (mn - 1) w_1(E) w_1(F).
\end{align*}
\]

**Which Unoriented Grassmannians are spin manifolds?**

Write the grassmanian \(\text{Gr}(a, b)\) as \(\text{Gr}(m, m + n)\) where \(m = a\) and \(n = b - a\). Then \(\gamma^+\) has rank \(n\).

As \(T \text{Gr}(m, m + n) = \gamma \otimes \gamma^+\), we have
\[
w_1(\text{Gr}(m, m + n)) = n w_1(\gamma) + m w_1(\gamma^+).
\]

Using the fact that \(\gamma \oplus \gamma^+ \cong \varepsilon^{m+n}\), we see that \(w_1(\gamma^+) = w_1(\gamma)\) and therefore
\[
w_1(\text{Gr}(m, m + n)) = n w_1(\gamma) + m w_1(\gamma^+) = n w_1(\gamma) + m w_1(\gamma) = (m + n) w_1(\gamma).
\]

Proceeding in a similar way, we have
\[
w_2(\text{Gr}(m, m + n)) = \binom{n}{2} w_1(\gamma)^2 + \binom{m}{2} w_1(\gamma^+)^2 + n^2 w_2(\gamma) + m^2 w_2(\gamma^+) + (mn - 1) w_1(\gamma) w_1(\gamma^+).
\]

Again, as \(\gamma \oplus \gamma^+ \cong \varepsilon^{m+n}\), we see that \(w_2(\gamma^+) = w_2(\gamma) + w_1(\gamma) w_1(\gamma^+) = w_2(\gamma) + w_1(\gamma)^2\), so
\[
w_2(\text{Gr}(m, m + n))
\]

which grassmannians are spin manifolds?

\[ (\frac{d}{2}) = \frac{1}{2}d(d - 1), \] its parity is determined by \( d \mod 4 \). More precisely, \( (\frac{d}{2}) \) is even if \( d \equiv 0, 1 \mod 4 \) and odd if \( d \equiv 2, 3 \mod 4 \). So the parity of the first two terms is determined by the values of \( m \) and \( n \) modulo 4, while the parity of remaining terms is determined by the values of \( m \) and \( n \) modulo 2. So we see that

\[
\begin{align*}
    w_2(\text{Gr}(m, m + n)) = & \begin{cases} 
    0 & (m, n) \equiv (0, 2), (1, 3), (2, 0), (3, 1) \mod 4 \\
    w_2(\gamma) & (m, n) \equiv (0, 3), (1, 0), (2, 1), (3, 2) \mod 4 \\
    w_1(\gamma)^2 & (m, n) \equiv (0, 1), (1, 2), (2, 3), (3, 0) \mod 4.
\end{cases}
\end{align*}
\]

Note that the difference \( m - n \) is constant in each row, so we can more succinctly express the above as

\[
\begin{align*}
    w_2(\text{Gr}(m, m + n)) = & \begin{cases} 
    0 & m - n \equiv 2 \mod 4 \\
    w_2(\gamma) & m - n \equiv 1 \mod 4 \\
    w_1(\gamma)^2 & m - n \equiv 0 \mod 4 \\
    w_2(\gamma) + w_1(\gamma)^2 & m - n \equiv 3 \mod 4.
\end{cases}
\end{align*}
\]

Upon first glance, the above description seems to contradict the fact that \( \text{Gr}(m, m + n) \) and \( \text{Gr}(n, n + n) \) are diffeomorphic, at least in the case where \( m - n \) is odd. Why does interchanging \( m \) and \( n \) give a different expression for \( w_2 \)? In order to understand this disparity, denote the tautological bundles over \( \text{Gr}(m, m + n) \) and \( \text{Gr}(n, n + n) \) by \( \gamma_m \) and \( \gamma_n \) respectively.

Recall that there is a diffeomorphism \( \varphi : \text{Gr}(m, m + n) \to \text{Gr}(n, n + n) \) given by \( P \mapsto P^\perp \); note, this requires an inner product on the ambient vector space. It follows that \( \varphi^*\gamma_n \cong \gamma_m^\perp \). So, if \( m - n \equiv 3 \mod 4 \), we have \( w_2(\text{Gr}(m, m + n)) = w_2(\gamma_m) + w_1(\gamma_m)^2 \in H^2(\text{Gr}(m, m + n); \mathbb{Z}_2) \) and \( w_2(\text{Gr}(n, n + n)) = w_2(\gamma_n) \in H^2(\text{Gr}(n, n + n); \mathbb{Z}_2) \). The cohomology rings are not equal, so we cannot compare these two elements, but the diffeomorphism \( \varphi \) gives rise to an isomorphism between them, namely \( \varphi^* \). Under this isomorphism,

\[
\varphi^* w_2(\gamma_n) = w_2(\varphi^* \gamma_n) = w_2(\gamma_n^\perp) = w_2(\gamma_m) + w_1(\gamma_m)^2.
\]

The case \( m - n \equiv 1 \mod 4 \) is similar.

Now that we have expressions for \( w_1(\text{Gr}(m, m + n)) \) and \( w_2(\text{Gr}(m, m + n)) \), we can finally determine for which \( m \) and \( n \) the manifold \( \text{Gr}(m, m + n) \) is spin.

Recall that \( H^*(\text{Gr}(m, m + n); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1(\gamma), \ldots, w_m(\gamma)]/(\bar{w}_{n+1}, \ldots, \bar{w}_{m+n}) \) where \( \bar{w}_i = i \) and \( \bar{w} = 1 + \bar{w}_1 + \cdots + \bar{w}_{m+n} \) satisfies \( \bar{w}(\gamma)\bar{w} = 1 \); see Proposition 11.1 of [1]. In particular, if \( m, n \geq 2 \), then \( w_1(\gamma), w_2(\gamma), w_1(\gamma)^2, \) and \( w_2(\gamma) + w_1(\gamma)^2 \) are all non-zero. If \( m = 1 \) or \( n = 1 \), then the grassmannian is a projective space, in which case it is easy to check whether \( w_1(\gamma), w_2(\gamma), w_1(\gamma)^2, \) and \( w_2(\gamma) + w_1(\gamma)^2 \) are non-zero or not.

We can also ask about the non-orientable analogues of spin, namely \( \text{pin}^+ \) and \( \text{pin}^- \). Recall that the obstruction to a smooth manifold \( M \) admitting a \( \text{pin}^+ \) structure is \( w_2(M) \), and the obstruction to admitting a \( \text{pin}^- \) structure is \( w_2(M) + w_1(M)^2 \).

**Theorem.** The grassmannian \( \text{Gr}(m, m + n) \) is:

- orientable if and only if \( m + n \) is even.
• spin if and only if \( m - n \equiv 2 \mod 4 \), or \( m = n = 1 \), i.e. \( \text{Gr}(1, 2) = \mathbb{R}P^1 = S^1 \).

• \( \text{pin}^+ \) if and only if it is spin or is a projective space of dimension \( 4k \).

• \( \text{pin}^- \) if and only if it is spin or is a projective space of dimension \( 4k + 2 \).

**Which Oriented Grassmannians are spin?**

Let \( \text{Gr}^+(a, b) \) denote the grassmanian of oriented \( a \)-dimensional subspaces of a real \( b \)-dimensional vector space, and denote the tautological bundle over it by \( \gamma_+ \). Similar to the unoriented case, we have \( T \text{Gr}^+(a, b) \cong \gamma_+ \otimes \gamma_+^\perp \) where \( \gamma_+^\perp \) is the orthogonal complement of \( \gamma_+ \subset \mathbb{e}_b^h \) with respect to a fixed Riemannian metric on \( \mathbb{e}_b^h \).

There is a double covering \( \pi : \text{Gr}^+(a, b) \to \text{Gr}(a, b) \) which forgets the orientation of the subspace. It follows that \( \pi^* \gamma \cong \gamma_+ \), and hence \( w_1(\gamma_+) = \pi(w_1(\gamma)) = \pi^* w_1(\gamma) \). The Gysin sequence associated to \( \pi \) is

\[
\cdots \to H^i(\text{Gr}(a, b); \mathbb{Z}_2) \xrightarrow{\iota_*} H^{i+1}(\text{Gr}(a, b); \mathbb{Z}_2) \xrightarrow{\pi^*} H^{i+1}(\text{Gr}^+(a, b); \mathbb{Z}_2) \to H^{i+1}(\text{Gr}(a, b); \mathbb{Z}_2) \to \cdots
\]

where \( e \in H^1(\text{Gr}(a, b); \mathbb{Z}_2) = \{0, w_1(\gamma)\} \) is the Euler class of \( \pi \); as \( \pi \) is not the trivial double cover, \( e = w_1(\gamma) \).

By the exactness of the Gysin sequence, \( w_1(\gamma_+) \) is zero if and only if \( w_1(\gamma) = w_1(\gamma) \cup \alpha \) for some \( \alpha \); that is, \( w_1(\gamma) \) is not in the ideal generated by \( w_1(\gamma) \). In particular, \( w_1(\gamma_+) = 0 \), and hence \( w_1(\text{Gr}^+(m, m + n)) = 0 \).

It now follows from the computation of \( w_2(\text{Gr}(m, m + n)) \) in the previous section that

\[
w_2(\text{Gr}^+(m, m + n)) = \begin{cases} 0 & m - n \equiv 0 \mod 2 \\
w_2(\gamma_+) & m - n \equiv 1 \mod 2. \end{cases}
\]

As \( H^*(\text{Gr}(k, n); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1(\gamma), \ldots, w_k(\gamma)]/(\overline{w}_n, \ldots, \overline{w}_m) \), if \( m, n \geq 2 \), then \( w_2(\gamma) \) is not in the ideal generated by \( w_1(\gamma) \) and hence \( w_2(\gamma^+) \neq 0 \). If \( m = 1 \) or \( n = 1 \), then the orientable grassmannian is a sphere and hence \( w_2(\text{Gr}^+(m, m + n)) = 0 \).

**Theorem.** The grassmannian \( \text{Gr}^+(m, m + n) \) is always orientable. Moreover, the obstructions to spin, \( \text{pin}^+ \), and \( \text{pin}^- \) structures coincide and they vanish if and only if \( m - n \) is even, \( m = 1 \), or \( n = 1 \).

This agrees with Theorem 8 of [2].

**Which Complex Grassmannians are spin?**

Let \( \text{Gr}^C(a, b) \) denote the grassmanian of complex \( a \)-dimensional subspaces of a complex \( b \)-dimensional vector space, and denote the tautological bundle over it by \( \gamma^C \). Similar to the previous cases, we have \( T \text{Gr}^C(a, b) \cong \overline{\gamma} \otimes \gamma^C_+ \) as complex vector bundles, where \( \gamma^C_+ \) is the orthogonal complement of \( \gamma^C \subset \mathbb{e}_b^h \) with respect to some fixed hermitian metric on \( \mathbb{e}_b^h \).

As \( \text{Gr}^C(m, m + n) \) is a complex manifold, it is orientable, i.e. \( w_1(\text{Gr}^C(m, m + n)) = 0 \). Instead of using the formula for \( w_2(E \otimes F) \), we have a shortcut in the complex case: we can use the Chern character to compute \( c_1(\text{Gr}^C(m, m + n)) \) and hence \( w_2(\text{Gr}^C(m, m + n)) \).

The Chern character is extremely useful as it satisfies \( \text{ch}(E \otimes F) = \text{ch}(E) \text{ch}(F) \). As \( \text{ch}(E) = \text{rank}(E) + c_1(E) + \ldots \) this immediately implies

\[
c_1(E \otimes F) = \text{rank}(F)c_1(E) + \text{rank}(E)c_1(F).
\]

In particular,

\[
c_1(\text{Gr}^C(m, m + n)) = c_1(\overline{\gamma} \otimes \gamma^C_+) = nc_1(\overline{\gamma}) + mc_1(\gamma^C_+).
\]
As \( \gamma_C \oplus \gamma_{\perp} \cong \varepsilon_C^{m+n} \), we see that \( c_1(\gamma_{\perp}) = -c_1(\gamma_C) \). As for the other term, \( c_i(E) = (-1)^i c_i(E) \) so we conclude that
\[
c_1(\text{Gr}^+(m, m + n)) = nc_1(\gamma_C) + mc_1(\gamma_{\perp}) = -nc_1(\gamma_C) - mc_1(\gamma_C) = -(m + n)c_1(\gamma_C).
\]
As \( H^*(\text{Gr}^C(m, m + n); \mathbb{Z}) \cong \mathbb{Z}[c_1(\gamma_C), \ldots, c_m(\gamma_C)]/(\bar{\tau}_{m+1}, \ldots, \bar{\tau}_{m+n}) \) where \( \bar{\tau}_i \) are defined in analogy with the previous cases, we see that \( c_1(\gamma_C) \) is non-zero and is not divisible by 2. Therefore \( w_2(\text{Gr}^C(m, m + n)) = (m + n)w_2(\gamma_C) \); as \( c_1(\gamma_C) \) is not divisible by 2, we see that \( w_2(\gamma_C) \neq 0 \). Therefore, we arrive at the following result.

**Theorem.** The grassmannian \( \text{Gr}^C(m, m + n) \) is always orientable. Moreover, the obstructions to spin, \( \text{pin}^+ \), and \( \text{pin}^- \) structures coincide and they vanish if and only if \( m + n \) is even.

**References**

