

THE HIRZEBRUCH χ_y GENUS AND A THEOREM OF HIRZEBRUCH ON ALMOST COMPLEX MANIFOLDS

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ABSTRACT. The purpose of this note is to give an introduction to the Hirzebruch χ_y genus and to give a proof of a theorem of Hirzebruch which states that on a closed almost complex manifold M of dimension $4m$ we have $\chi(M) \equiv (-1)^m \sigma(M) \pmod{4}$.

Let (M, g) be an $2n$ -dimensional closed Riemannian manifold. Given a spin^c structure, one can form the complex spin^c bundles $\mathbb{S}_\mathbb{C}^+$ and $\mathbb{S}_\mathbb{C}^-$. Then there is a spin^c Dirac operator $\not{D}^c : \Gamma(\mathbb{S}_\mathbb{C}^+) \rightarrow \Gamma(\mathbb{S}_\mathbb{C}^-)$ which has index

$$\text{ind}(\not{D}^c) = \int_M \exp(c_1(L)/2) \hat{A}(TM)$$

where L is the complex line bundle associated to the spin^c structure; see Theorem D.15 of [3].

If $E \rightarrow M$ is a hermitian vector bundle, then there is a twisted spin^c Dirac operator $\not{D}_E^c : \Gamma(\mathbb{S}_\mathbb{C}^+ \otimes E) \rightarrow \Gamma(\mathbb{S}_\mathbb{C}^- \otimes E)$ which has index

$$\text{ind}(\not{D}_E^c) = \int_M \exp(c_1(L)/2) \text{ch}(E) \hat{A}(TM).$$

I don't know a reference for this precise statement (if you do, please let me know), but the fact that this quantity is an integer is Theorem 26.1.1 of [1].

Suppose now that M admits an almost complex structure and g is hermitian. Then there is a canonical spin^c structure which has associated line bundle $L = \det_{\mathbb{C}}(TM)$, so $c_1(L) = c_1(M)$; see Example D.6 of [3]. Using the fact that $\exp(c_1(M)/2) \hat{A}(TM) = \text{Td}(TM)$, the index becomes

$$\text{ind}(\not{D}_E^c) = \int_M \text{ch}(E) \text{Td}(TM). \tag{1}$$

In addition, the complex spin^c bundles take the form $\mathbb{S}_\mathbb{C}^+ \cong \bigwedge^{0,\text{even}} M$ and $\mathbb{S}_\mathbb{C}^- = \bigwedge^{0,\text{odd}} M$; see corollary 3.4.6 of [4]. If $E = \bigwedge^{p,0} M$, then we have a twisted spin^c Dirac operator $\not{D}_{\bigwedge^{p,0} M}^c : \Gamma(\bigwedge^{p,\text{even}} M) \rightarrow \Gamma(\bigwedge^{p,\text{odd}} M)$; for notational convenience, we will instead write \not{D}_p^c for this operator. We define $\chi^p(M) := \text{ind}(\not{D}_p^c)$; if $p = 0$, this is just the Todd genus. The Hirzebruch χ_y genus is defined to be

$$\chi_y(M) := \sum_{p=0}^n \chi^p(M) y^p.$$

INTEGRABLE CASE

Suppose now that J is integrable, in which case $n = \dim_{\mathbb{C}} M$. Then $\not{D}^c = \sqrt{2}(\bar{\partial} + \bar{\partial}^*)$; see chapter 3, section 4 of [4]. In addition, if E is holomorphic, then $\not{D}_E^c = \sqrt{2}(\bar{\partial}_E + \bar{\partial}_E^*)$ and (1) becomes the statement of the Hirzebruch-Riemann-Roch theorem.

Furthermore, $\Lambda^{p,0} M$ is holomorphic and $\bar{\partial}_p^c : \Gamma(\Lambda^{p,\text{even}} M) \rightarrow \Gamma(\Lambda^{p,\text{odd}} M)$ is just $\sqrt{2}(\bar{\partial} + \bar{\partial}^*)$. If $\mathcal{H}_{\bar{\partial}}^{p,q}(M)$ denotes the $\bar{\partial}$ -harmonic (p,q) -forms on M , then

$$\begin{aligned} \text{ind}(\bar{\partial}_p^c) &= \dim \left(\bigoplus_{q \text{ even}} \mathcal{H}_{\bar{\partial}}^{p,q}(M) \right) - \dim \left(\bigoplus_{q \text{ odd}} \mathcal{H}_{\bar{\partial}}^{p,q}(M) \right) \\ &= \sum_{q \text{ even}} \dim \mathcal{H}_{\bar{\partial}}^{p,q}(M) - \sum_{q \text{ odd}} \dim \mathcal{H}_{\bar{\partial}}^{p,q}(M) \\ &= \sum_{q \text{ even}} h^{p,q}(M) - \sum_{q \text{ odd}} h^{p,q}(M) \\ &= \sum_{q=0}^n (-1)^q h^{p,q}(M) \\ &= \chi(M, \Omega^p). \end{aligned}$$

Using the penultimate equality above, we have

$$\chi_y(M) = \sum_{p=0}^n \sum_{q=0}^n (-1)^q h^{p,q}(M) y^p.$$

We now list some properties of $\chi_y(M)$ in the integrable case.

Property 1. $\chi_y(M) = (-y)^n \chi_{y^{-1}}(M)$.

Proof. As

$$(-y)^n \chi_{y^{-1}}(M) = (-y)^n \sum_{p=0}^n \chi^p(M) y^{-p} = \sum_{p=0}^n (-1)^n \chi^p(M) y^{n-p},$$

this property is equivalent to $\chi^p(M) = (-1)^n \chi^{n-p}(M)$.

By Serre duality we have $h^{p,q}(M) = h^{n-p,n-q}(M)$, so

$$\begin{aligned} \chi^p(M) &= \sum_{q=0}^n (-1)^q h^{p,q}(M) \\ &= \sum_{q=0}^n (-1)^q h^{n-p,n-q}(M) \\ &= (-1)^n \sum_{q=0}^n (-1)^{n-q} h^{n-p,n-q}(M) \\ &= (-1)^n \chi^{n-p}(M). \end{aligned}$$

□

Property 2. If M admits a Kähler metric, then $\chi_{-1}(M) = \chi(M)$.

Proof. Note that in the Kähler case

$$\chi_{-1}(M) = \sum_{p=0}^n \sum_{q=0}^n (-1)^{p+q} h^{p,q}(M) = \sum_{k=0}^{2n} (-1)^k \sum_{p+q=k} h^{p,q}(M) = \sum_{k=0}^{2n} (-1)^k b_k(M) = \chi(M).$$

□

Property 3. Suppose that n is even and M admits a Kähler metric. Then $\chi_1(M) = \sigma(M)$.

Proof. Using the fact that $h^{p,q}(M) = h^{q,p}(M)$, we have

$$\chi_1(M) = \sum_{p=0}^n \sum_{q=0}^n (-1)^q h^{p,q}(M) = \sum_{p=0}^n \sum_{q=0}^n (-1)^q h^{q,p}(M) = \sum_{p=0}^n \sum_{q=0}^n (-1)^p h^{p,q}(M).$$

It follows from the Hard Lefschetz Theorem that the final expression is equal to $\sigma(M)$; see Corollary 3.3.18 of [2]. \square

As we will see in the next section, all three of these properties hold in general.

NON-INTEGRABLE CASE

Suppose now that J is not integrable.

In order to establish the properties mentioned in the previous section, we need the following expression for $\chi_y(M)$.

Theorem. *Let x_i be the Chern roots of TM . Then*

$$\chi_y(M) = \int_M \prod_{i=1}^n \frac{x_i(1 + ye^{-x_i})}{1 - e^{-x_i}}.$$

Proof. By the splitting principle, we can suppose that $TM = \ell_1 \oplus \cdots \oplus \ell_n$, and hence $T^*M = \ell_1^* \oplus \cdots \oplus \ell_n^*$, without any loss of generality. Defining $x_i = c_1(\ell_i)$, we have $-x_i = c_1(\ell_i^*)$. Note that

$$\bigwedge^{p,0} M = \bigwedge^p T^*M = \bigwedge^p (\ell_1^* \oplus \cdots \oplus \ell_n^*) = s_p(\ell_1^*, \dots, \ell_n^*)$$

where s_p is the p^{th} elementary symmetric polynomial (addition and multiplication correspond to direct sum and tensor product respectively). Therefore

$$\text{ch} \left(\bigwedge^{p,0} M \right) = \text{ch}(s_p(\ell_1^*, \dots, \ell_n^*)) = s_p(\text{ch}(\ell_1^*), \dots, \text{ch}(\ell_n^*)) = s_p(e^{-x_1}, \dots, e^{-x_n}).$$

So we have

$$\begin{aligned} \chi_y(M) &= \sum_{p=0}^n \chi^p(M) y^p \\ &= \sum_{p=0}^n \text{ind}(\not\partial_p^c) y^p \\ &= \sum_{p=0}^n \left(\int_M \text{ch} \left(\bigwedge^{p,0} M \right) \text{Td}(M) \right) y^p \\ &= \int_M \left(\sum_{p=0}^n \text{ch} \left(\bigwedge^{p,0} M \right) y^p \right) \text{Td}(M) \\ &= \int_M \left(\sum_{p=0}^n s_p(e^{-x_1}, \dots, e^{-x_n}) y^p \right) \prod_{i=1}^n \frac{x_i}{1 - e^{-x_i}} \\ &= \int_M \prod_{i=1}^n (1 + e^{-x_i} y) \prod_{i=1}^n \frac{x_i}{1 - e^{-x_i}} \\ &= \int_M \prod_{i=1}^n \frac{x_i(1 + e^{-x_i} y)}{1 - e^{-x_i}}. \end{aligned}$$

\square

Setting $y = -1$, we now see that property 2 holds in the non-integrable case:

$$\chi_{-1}(M) = \int_M \prod_{i=1}^n \frac{x_i(1 - e^{-x_i})}{1 - e^{-x_i}} = \int_M \prod_{i=1}^n x_i = \int_M c_n(M) = \int_M e(M) = \chi(M).$$

When J is integrable, property 2 gives us the following result (which also follows from the existence of the Frölicher spectral sequence).

Corollary. *Let M be a compact complex manifold. The Euler characteristic of M is given by*

$$\chi(M) = \sum_{p=0}^n \sum_{q=0}^n (-1)^{p+q} h^{p,q}(M).$$

For the other two properties, we need the following lemma. Thanks to Professor Ping Li for pointing this out to me.

Lemma. *Let t be a parameter. Then*

$$\chi_y(M) = \int_M \prod_{i=1}^n \frac{x_i(1 + ye^{-tx_i})}{1 - e^{-tx_i}}.$$

Proof. The key is to note that $\chi_y(M)$ only depends on the degree $2n$ part of the integrand. As $\deg x_i = 2$, if we replace x_i by tx_i , then we have

$$\int_M \prod_{i=1}^n \frac{tx_i(1 + ye^{-tx_i})}{1 - e^{-tx_i}} = t^n \int_M \prod_{i=1}^n \frac{x_i(1 + ye^{-x_i})}{1 - e^{-x_i}} = t^n \chi_y(M).$$

Dividing through by t^n , we arrive at the result. \square

With this lemma in hand, we have

$$\begin{aligned} \chi_y(M) &= \int_M \prod_{i=1}^n \frac{x_i(1 + ye^{-x_i})}{1 - e^{-x_i}} \\ &= \int_M \prod_{i=1}^n \frac{x_i(1 + ye^{x_i y^{-1}})}{1 - e^{x_i y^{-1}}} \\ &= y^n \int_M \prod_{i=1}^n \frac{x_i(y^{-1} + e^{x_i y^{-1}})}{1 - e^{x_i y^{-1}}} \\ &= (-y)^n \int_M \prod_{i=1}^n \frac{x_i(y^{-1} + e^{x_i y^{-1}})}{e^{x_i y^{-1}} - 1} \\ &= (-y)^n \int_M \prod_{i=1}^n \frac{x_i(y^{-1} e^{-x_i y^{-1}} + 1)}{1 - e^{-x_i y^{-1}}} \\ &= (-y)^n \int_M \prod_{i=1}^n \frac{x_i(1 + y^{-1} e^{-x_i})}{1 - e^{-x_i}} \\ &= (-y)^n \chi_{y^{-1}}(M). \end{aligned}$$

and hence we see that property 1 holds.

For property 3, suppose $n = 2m$. Setting $y = 1$ gives

$$\chi_1(M) = \int_M \prod_{i=1}^n \frac{x_i(1 + e^{-x_i})}{1 - e^{-x_i}} = \int_M \prod_{i=1}^n \frac{x_i(1 + e^{2x_i})}{1 - e^{2x_i}} = \int_M \prod_{i=1}^n \frac{x_i}{\tanh(x_i)}$$

where the lemma was used with $t = -2$ in the second equality. Recall that the power series which generates the L genus (in terms of Pontryagin classes) is

$$Q(x) = \frac{\sqrt{x}}{\tanh(\sqrt{x})}.$$

By lemma 1.3.1 of [1], the corresponding power series which generates the L genus (in terms of Chern classes) is

$$\tilde{Q}(x) = \frac{x}{\tanh(x)}.$$

So the above computation shows that

$$\chi_1(M) = \int_M \prod_{i=1}^n \frac{x_i}{\tanh(x_i)} = \int_M L_m(p_1, \dots, p_m) = \sigma(M).$$

In the integrable case, we immediately obtain the following corollary.

Corollary. *Let M be a $2n$ -dimensional compact complex manifold. The signature of M is given by*

$$\sigma(M) = \sum_{p=0}^n \sum_{q=0}^n (-1)^q h^{p,q}(M).$$

Note, in the proof of property 3 in the Kähler case, the exponent of -1 is p , not q . Because $h^{p,q}(M) = h^{q,p}(M)$, it doesn't make any difference. However, in the non-Kähler case, having exponent p does not compute the signature as can easily be checked for a Hopf surface.

If M has odd dimension, it follows from Serre duality that $\chi_1(M) = 0$.

A THEOREM OF HIRZEBRUCH

The three properties of $\chi_y(M)$ give rise to the following result of Hirzebruch.

Theorem. *Suppose M is a closed $4m$ -dimensional manifold which admits an almost complex structure, then $\chi(M) \equiv (-1)^m \sigma(M) \pmod{4}$.*

Proof. Suppose m is even, so $4m = 8k$. Then

$$\begin{aligned} \chi(M) &= \sum_{p=0}^{4k} (-1)^p \chi^p(M) \\ &= \sum_{p=0}^{4k} \chi^p(M) - 2 \left[\sum_{p=0}^{2k-1} \chi^{2p+1}(M) \right] \\ &= \sigma(M) - 2 \left[\sum_{p=0}^{k-1} \chi^{2p+1}(M) + \sum_{p=k}^{2k-1} \chi^{2p+1}(M) \right] \\ &= \sigma(M) - 2 \left[\sum_{p=0}^{k-1} \chi^{2p+1}(M) + \sum_{p=k}^{2k-1} (-1)^{4k} \chi^{4k-2p-1}(M) \right] \\ &= \sigma(M) - 2 \left[\sum_{p=0}^{k-1} \chi^{2p+1}(M) + \sum_{p=k}^{2k-1} \chi^{4k-2p-1}(M) \right] \\ &= \sigma(M) - 2 \left[\sum_{p=0}^{k-1} \chi^{2p+1}(M) + \sum_{p=0}^{k-1} \chi^{2p+1}(M) \right] \end{aligned}$$

$$= \sigma(M) - 4 \sum_{p=0}^{k-1} \chi^{2p+1}(M).$$

Therefore $\chi(M) \equiv \sigma(M) \pmod{4}$.

Suppose now that m is odd, so $4m = 8k + 4$. Then

$$\begin{aligned} \chi(M) &= \sum_{p=0}^{4k+2} (-1)^p \chi^p(M) \\ &= - \sum_{p=0}^{4k+2} \chi^p(M) + 2 \left[\sum_{p=0}^{2k+1} \chi^{2p}(M) \right] \\ &= -\sigma(M) + 2 \left[\sum_{p=0}^k \chi^{2p}(M) + \sum_{p=k+1}^{2k+1} \chi^{2p}(M) \right] \\ &= -\sigma(M) + 2 \left[\sum_{p=0}^k \chi^{2p}(M) + \sum_{p=k+1}^{2k+1} (-1)^{4k+2} \chi^{4k+2-2p}(M) \right] \\ &= -\sigma(M) + 2 \left[\sum_{p=0}^k \chi^{2p}(M) + \sum_{p=k+1}^{2k+1} \chi^{4k+2-2p}(M) \right] \\ &= -\sigma(M) + 2 \left[\sum_{p=0}^k \chi^{2p}(M) + \sum_{p=0}^k \chi^{2p}(M) \right] \\ &= -\sigma(M) + 4 \sum_{p=0}^k \chi^{2p}(M). \end{aligned}$$

Therefore $\chi(M) \equiv -\sigma(M) \pmod{4}$. □

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