ALMOST COMPLEX STRUCTURES AND OBSTRUCTION THEORY

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Abstract. These are notes for a lecture I gave in John Morgan's Homotopy Theory course at Stony Brook in Fall 2018.

Let X be a CW complex and Y a simply connected space. Last time we discussed the obstruction to extending a map $f : X^{(n)} \to Y$ to a map $X^{(n+1)} \to Y$; recall that $X^{(k)}$ denotes the k-skeleton of X. There is an obstruction $\mathfrak{o}(f) \in C^{n+1}(X; \pi_n(Y))$ which vanishes if and only if f can be extended to $X^{(n+1)}$. Moreover, $\mathfrak{o}(f)$ is a cocycle and $[\mathfrak{o}(f)] \in H^{n+1}(X; \pi_n(Y))$ vanishes if and only if $f|_{X^{(n-1)}}$ can be extended to $X^{(n+1)}$; that is, f may need to be redefined on the n-cells.

Obstructions to lifting a map

Given a fibration $F \to E \xrightarrow{p} B$ and a map $f : X \to B$, when can f be lifted to a map $g : X \to E$? If $X = B$ and $f = \mathrm{id}_B$, then we are asking when p has a section. For convenience, we will only consider the case where F and B are simply connected, from which it follows that E is simply connected. For a more general statement, see Theorem 7.37 of [2].

Suppose g has been defined on $X^{(n)}$. Let e^{n+1} be an n-cell and $\alpha : S^n \to X^{(n)}$ its attaching map, then $p \circ g \circ \alpha : S^n \to B$ is equal to $f \circ \alpha$ and is nullhomotopic (as f extends over the $(n+1)$-cell). From the long exact sequence of a fibration (here we use simply connected so $[S^n, F] = \pi_n(F)$ etc.), we see that there is a map $\beta : S^n \to F$ such that $g \circ \alpha$ is homotopic to $i \circ \beta$ where $i : F \to E$ is the inclusion. So we obtain $\mathfrak{o}(g) \in C^{n+1}(X; \pi_n(F))$ which vanishes if and only if g extends to $X^{(n+1)}$. As before, $\mathfrak{o}(g)$ is a cocycle and $[\mathfrak{o}(g)] \in H^{n+1}(X; \pi_n(F))$ vanishes if and only if $g|_{X^{(n-1)}}$ extends to $X^{(n+1)}$.

Lots of interesting problems can be analysed using obstructions to lifting a map. For example:

- When does a vector bundle have a nowhere-zero section?
- When is a smooth manifold orientable?
- When is a smooth manifold spin?
- When does a smooth manifold admit an almost complex structure?
- When does a topological manifold admit a PL structure or smooth structure?

We're going to focus on the fourth one.

Almost Complex Structures

A *linear complex structure* on a real vector space V is an endomorphism $J : V \to V$ such that $J \circ J = -\mathrm{id}_V$. If V is endowed with a linear complex structure J, then we can view V as a complex vector space by defining $(a + bi) \cdot v = av + bJ(v)$. In particular, if V is finite-dimensional, then $\dim_{\mathbb{R}} V = 2 \dim_{\mathbb{C}} V$ is even. Moreover, if $\{e_1, \dots, e_n\}$ is a basis for V as a complex vector space, then

$\{e_1, J(e_1), \ldots, e_n, J(e_n)\}$ is a basis for V as a real vector space and $e_1 \wedge J(e_1) \wedge \cdots \wedge e_n \wedge J(e_n)$ defines an orientation; this orientation is independent of the choice of basis $\{e_1, \ldots, e_n\}$.

Let $E \to B$ be a real vector bundle. An *almost complex structure* on E is a bundle endomorphism $J : E \to E$ such that $J \circ J = -\operatorname{id}_E$. It follows that in order for an almost complex structure to exist, E must have even rank and be orientable. Note, given an almost complex structure, one can view E as a complex vector bundle.

Remark: The reason I use the terminology 'linear almost complex structure' on V rather than 'almost complex structure' is that the latter could be interpreted as an almost complex structure on the manifold V, i.e. an almost complex structure on the vector bundle TV.

An almost complex structure on a smooth manifold M is defined to be an almost complex structure on TM. Again, if M admits an almost complex structure then M has even dimension and is orientable. Moreover, TM can be viewed as a complex vector bundle.

Question: Does every even-dimensional orientable smooth manifold admit an almost complex structure?

Answer: No, there are obstructions.

Classifying Spaces

A *topological group* is a group $(G, *)$ such that G is a topological space, and the maps $* : G \times G \to G$ and $i : G \to G$, $g \mapsto g^{-1}$ are continuous. If G is a smooth manifold and the maps $*$ and i are smooth, then $(G, *)$ is called a *Lie group*.

A *fiber bundle* with fiber F is a continuous map $\pi : E \to B$ such that for every $b \in B$, there is an open neighbourhood $U \subseteq B$ of b and a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that $\pi = \operatorname{pr}_1 \circ \varphi$.

Let G be topological group. A *principal G-bundle* is a fiber bundle $\pi : E \to B$ together with a continuous right action $E \times G \to E$ which preserves fibers (i.e. $\pi(e \cdot g) = \pi(e)$), and acts freely and transitively on them. As the action is free and transitive, we can (non-canonically) identify the fibers of π with G.

An isomorphism between principal G-bundles $P \to B$ and $Q \to B$ is a G-equivariant map $\phi : P \to Q$ covering the identity. Denote the isomorphism classes of principal G-bundles on a topological space B by $\operatorname{Prin}_G(B)$.

Fiber bundles, and hence principal bundles, are Serre fibrations; see Proposition 4.48 of [4]. Note however, they are not necessarily Hurewicz fibrations, see [1].

Examples

1. If G a discrete group, a principal G-bundle is a normal covering with group of deck transformations isomorphic to G.

2. If H is a closed subgroup of a Lie group G, then $G \to G/H$ is a principal H-bundle.

3. Main example, frame bundles.

Let $E \to B$ be a real rank n vector bundle. The frame bundle of E is a space $F(E)$ together with a map $\pi : F(E) \to B$ such that $\pi^{-1}(p)$ is the collection of ordered bases, or frames, for E_p. Any two frames are related by a unique element of $GL(n, \mathbb{R})$. This is a principal $GL(n, \mathbb{R})$-bundle. Conversely, given a principal $GL(n, \mathbb{R})$-bundle, one can build a real vector bundle via a process known as the associated bundle construction. This defines a bijection between $\operatorname{Prin}_{GL(n,\mathbb{R})}(B)$ and $\operatorname{Vect}_n(B)$, the collection of isomorphism classes of real rank n vector bundles.

Equipping E with a Riemannian metric, we can take the orthogonal frame bundle which is a principal $O(n)$-bundle. Different Riemannian metrics give isomorphic principal $O(n)$-bundles. Again by the associated bundle construction, there is a bijection between $\mathrm{Prin}_{O(n)}(B)$ and $\mathrm{Vect}_n(B)$.

If E also admits an orientation, we can take the oriented orthonormal frame bundle which is a principal $SO(n)$-bundle. Now we obtain a bijection between $\mathrm{Prin}_{SO(n)}(B)$ and $\mathrm{Vect}_n^+(B)$, the collection of isomorphism classes of oriented real rank n vector bundles.

If E has rank $2n$ and is the underlying real vector bundle of a complex vector bundle, then one can take the bundle of complex frames which is a principal $GL(n,\mathbb{C})$-bundle. If E is equipped with a hermitian metric, we can take the bundle of unitary frames which is a principal $U(n)$-bundle. As in the real case, there is a bijection $\mathrm{Prin}_{GL(n,\mathbb{C})}(B)$ and $\mathrm{Prin}_{U(n)}(B)$, and a bijection $\mathrm{Prin}_{GL(n,\mathbb{C})}(B)$-bundles and $\mathrm{Vect}_n^{\mathbb{C}}(B)$, the collection of isomorphism classes of rank n complex vector bundles.

Theorem. *Let G be a topological group. There is a space BG and a principal G-bundle $G \to EG \to BG$ such that for every paracompact topological space B, isomorphism classes of principal G-bundles on B are in bijection with $[B, BG]$.*

The space BG is unique up to homotopy and is called the *classifying space*. Milnor gave an explicit model for BG using the join construction, see [5]. We call $G \to EG \to BG$ the *universal principal G-bundle*; it is characterised by the fact that EG is weakly contractible; it follows from the long exact sequence in homotopy that $\pi_n(BG) \cong \pi_{n-1}(G)$. Given a map $f : B \to BG$, we can associate to it the principal G-bundle $f^*EG \to B$. If $P \to B$ is a principal G-bundle, a map $f : B \to BG$ such that $f^*EG \cong P$ is called a *classifying map* for P.

The association $G \to BG$ is functorial. In particular, given a continuous group homomorphism $\rho : H \to G$, there is an associated continuous map $B\rho : BH \to BG$. If $i : H \to G$ is inclusion, then the homotopy fiber of $Bi : BH \to BG$ is G/H.

Characteristic Classes

From the theorem, we see that there is a bijection between $\mathrm{Vect}_n^+(B)$ and $[B, BSO(n)]$, as well as a bijection between $\mathrm{Vect}_n^{\mathbb{C}}(B)$ and $[B, BU(n)]$. The grassmannians $\mathrm{Gr}_n^+(\mathbb{R}^\infty)$ and $\mathrm{Gr}_n^{\mathbb{C}}(\mathbb{C}^\infty)$ are explicit models for $BSO(n)$ and $BU(n)$, and the tautological bundles over them are the universal bundles.

One can show that $H^*(BSO(n);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_2, \dots, w_n]$ where $\deg w_i = i$. Given a principal $SO(n)$-bundle $P \to B$, we define $w_i(P) = f^*w_i$ where $f : B \to BSO(n)$ is any classifying map for P – these are the Stiefel-Whitney classes for P. Note, the class $w_i(P)$ doesn't depend on the choice of classifying map as homotopic maps induce the same map on cohomology.

Similarly, we have $H^*(BU(n);\mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_n]$ where $\deg c_i = 2i$. Given a principal $U(n)$-bundle $P \to B$, we define $c_i(P) = f^*c_i$ where $f : B \to BU(n)$ is any classifying map for P – these are the Chern classes of P.

The integral cohomology of $BSO(2n)$ is more complicated than that of $BU(n)$. There are elements $p_i \in H^{4i}(BSO(2n);\mathbb{Z})$ for $i = 1, \dots, n$ and $e \in H^{2n}(BSO(2n);\mathbb{Z})$. Modulo torsion, these classes generate the cohomology, but not freely. More precisely, $H^*(BSO(2n);\mathbb{Q}) \cong \mathbb{Q}[p_1, \dots, p_n, e]/(p_n - e^2)$. Given a principal $SO(2n)$-bundle $P \to B$, we define $p_i(P) = f^*p_i$ where $f : B \to BSO(2n)$ is any classifying map for P – these are the Pontryagin classes of P. We define $e(P) = f^*e$ – this is the Euler class of P.

Obstructions to the Existence of an Almost Complex Structure

Let $p : BU(n) \to BSO(2n)$ be the map induced by the inclusion $i : U(n) \to SO(2n)$; i.e. $p = Bi$. Postcomposition with p gives a map $[B, BU(n)] \to [B, BSO(2n)]$ and hence a map from complex rank n vector bundles to orientable rank $2n$ real vector bundles; this just forgets the almost complex structure. We want to know when a principal $SO(2n)$-bundle comes from a principal $U(n)$-bundle, that is when $f : B \to BSO(2n)$ admits a lift $g : B \to BU(n)$. Suppose g is a lift of f, i.e. then $f = g \circ p$. It follows that if E is a complex rank n vector bundle, $c_i(E) \equiv w_{2i}(E) \bmod 2$ and $w_{2i+1}(E) = 0$.

The obstructions to a such a lift lie in $H^{k+1}(X; \pi_k(F))$ where F is the homotopy fiber of $BU(n) \to BSO(2n)$. As the map $BU(n) \to BSO(2n)$ is induced by inclusion, the homotopy fiber is $SO(2n)/U(n)$ which can be identified with the space of linear complex structures on $\mathbb{R}^{2n}$ which are compatible with a given inner product and orientation. It is a closed manifold of dimension $n(n-1)$. Note, when $n = 1$, this space is a point as $U(1) = SO(2)$ which corresponds to the fact that every orientable rank 2 real vector bundle is a complex line bundle.

In order to do obstruction theory, we need to determine the first non-zero homotopy group of $SO(2n)/U(n)$. From the long exact sequence in homotopy associated to the fibration $U(n) \to SO(2n) \to SO(2n)/U(n)$ together with the fact that $\pi_2(G) = 0$ for Lie groups[1] we see that

$$0 \to \pi_2(SO(2n)/U(n)) \to \mathbb{Z} \to \mathbb{Z}_2 \to \pi_1(SO(2n)/U(n)) \to 0.$$

As $\ker(\mathbb{Z} \to \mathbb{Z}_2) \cong \mathbb{Z}$, regardless of the map, we see that $\pi_2(SO(2n)/U(n)) \cong \mathbb{Z}$. So either $\mathbb{Z} \to \mathbb{Z}_2$ is given by $1 \mapsto 1$, in which case $\pi_1(SO(2n)/U(n)) = 0$, or $1 \mapsto 0$, in which case $\pi_1(SO(2n)/U(n)) = 0$. Using the five lemma, we can show the following.

Lemma. *For $n > 1$, $\pi_1(SO(2n)/U(n)) = 0$ and $\pi_2(SO(2n)/U(n)) \cong \mathbb{Z}$.*

In fact, we see that $\pi_1(SO(2n)/U(n)) \cong \pi_1(SO(4)/U(2))$ and $\pi_2(SO(2n)/U(n)) \cong \pi_2(SO(4)/U(2))$ for all $n > 1$ (then use the fact that $SO(4)/U(2) = S^2$). More generally, $\pi_i(SO(2n+2)/U(n+1)) \cong \pi_i(SO(2n)/U(n))$ for $i \leq 2n-2$. This is called the stable range (pass to the direct limit SO/U which is $(\Omega O)_0$ by Bott periodicity).

Therefore, the first obstruction to a lift g lies in $H^3(B; \mathbb{Z})$. What is it? This is the hardest part of obstruction theory, actually *identifying* the obstructions. The following result gets us started, see Theorem 5.7 of [3].

Theorem. *The first non-trivial obstruction is natural.*

This means that the first obstruction to lifting $f : B \to BSO(2n)$ to $BU(n)$ is the pullback by f of the first obstruction to lifting $\mathrm{id} : BSO(2n) \to BSO(2n)$ to $BU(n)$, i.e. the obstruction to finding a section of $BU(n) \to BSO(2n)$. This obstruction lies in $H^3(BSO(2n); \mathbb{Z})$.

By the Universal Coefficient Theorem,

$$H^3(BSO(2n); \mathbb{Z}) \cong \mathrm{Hom}(H_3(BSO(2n); \mathbb{Z}), \mathbb{Z}) \oplus \mathrm{Ext}(H_2(BSO(2n); \mathbb{Z}), \mathbb{Z}).$$

As $H_3(BSO(2n); \mathbb{Q}) \cong H^3(BSO(2n); \mathbb{Q}) = 0$, we see that $H_3(BSO(2n); \mathbb{Z})$ is torsion, so the first summand is zero. On the other hand, $\pi_1(BSO(2n)) = \pi_0(SO(2n)) = 0$, and $\pi_2(BSO(2n)) = \pi_1(SO(2n)) = \mathbb{Z}_2$ as $n > 1$, so by Hurewicz, $H_2(BSO(2n); \mathbb{Z}) \cong \mathbb{Z}_2$. So $H^3(BSO(2n); \mathbb{Z}) \cong \mathbb{Z}_2$. What is the non-zero element?

Consider the short exact sequence of abelian groups $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$. This induces a long exact sequence in cohomology

$$\dots \to H^2(BSO(2n); \mathbb{Z}) \xrightarrow{\times 2} H^2(BSO(2n); \mathbb{Z}) \xrightarrow{\rho} H^2(BSO(2n); \mathbb{Z}_2) \xrightarrow{\beta} H^3(BSO(2n); \mathbb{Z}) \to \dots$$

where ρ is reduction modulo 2, and β is the coboundary map which is called the Bockstein associated to the coefficient sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$. By exactness, $x \in H^2(BSO(2n); \mathbb{Z})$ satisfies $\beta(x) = 0$ if and only if there is $y \in H^2(BSO(2n); \mathbb{Z})$ such that $\rho(y) = x$; we usually write $y \equiv x \bmod 2$

[1]Note, if G is a topological group, $\pi_2(G)$ is not necessarily zero. For example, ΩX has the homotopy type of a topological group for any space X and $\pi_2(\Omega X) = \pi_3(X)$ which can be arbitrary.

and say y an *integral lift* for x. Recall, $w_2 \in H^2(BSO(2n);\mathbb{Z})$ is non-zero and $H^2(BSO(2n);\mathbb{Z}) \cong \mathrm{Hom}(H_2(BSO(2n);\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}(H_1(BSO(2n);\mathbb{Z}),\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}_2,\mathbb{Z}) \oplus \mathrm{Ext}(0,\mathbb{Z}) = 0$ so w_2 has no integral lift, and therefore $W_3 := \beta(w_2) \neq 0$ and hence must be the non-zero element of $H^3(BSO(2n);\mathbb{Z})$.

It turns out that the first obstruction to the existence of a section of $BU(n) \to BSO(2n)$ is W_3, the argument will be given later (see the section on the six-dimensional case). Therefore, the first obstruction to the existence of an almost complex structure on an orientable real rank $2n$ vector bundle E is f^*W_3 where $f : B \to BSO(2n)$ is any classifying map. As the Bockstein is natural, $f^*W_3 = f^*\beta(w_2) = \beta(f^*w_2) = \beta(w_2(E)) =: W_3(E)$. Note that $W_3(E) = 0$ if and only if $w_2(E)$ has an integral lift. Note, this shouldn't be completely surprising as $c_1(E) \equiv w_2(E) \bmod 2$ (so $W_3(E) = 0$ is clearly a necessary condition). What wasn't clear from the beginning is that this is all that's required to lift $B^{(3)} \to BSO(2n)$ to $B^{(3)} \to BU(n)$, there could have been other conditions.

Theorem. *Let M^{2n} be an orientable smooth manifold with $n > 1$. The first obstruction to M admitting an almost complex structure is $W_3(M)$.*

Note, if $g : B^{(3)} \to BU(n)$ is defined, then $c := g^*c_1 \in H^2(B^{(3)};\mathbb{Z}) \cong H^2(B;\mathbb{Z})$. This is important as further obstructions will be phrased in terms of c. In particular, if $g : B \to BU(n)$ can be defined, then c will be the first Chern class of the corresponding complex vector bundle.

One might predict that the other obstructions will just be the necessary conditions $w_{2i+1}(E) = 0$ and $W_{2i+1}(E) = 0$ (i.e. $w_{2i}(E)$ has an integral lift). However, these are not sufficient. For example, they are satisfied by $E = TS^{2n}$ for every n, but the only spheres which admit almost complex structures are S^2 and S^6.

Now let's stick to a smooth manifold M and let f classify its tangent bundle.

Four-dimensional case

In this case, $SO(4)/U(2) = S^2$. So there is one more potential obstruction in $H^4(M;\pi_3(S^2)) = H^4(M;\mathbb{Z})$. As M is assumed to be oriented, this group is zero if M is not closed, otherwise it is $\mathbb{Z}$ if it is closed. So, if M is a non-compact, orientable four-manifold, it admits an almost complex structure if and only $W_3(M) = 0$.

If M is closed, then there is a genuine second obstruction. It is $c_1^2 - (2e(M) + p_1(M))$. Said another way, c must satisfy $\int_M c^2 = 2\chi(M) + 3\tau(M)$. Again, it is not hard to see that this condition is necessary using the Hirzebruch signature theorem.

Note, in the closed case, the first obstruction always vanishes (M is spin^c), so you can always find c with $c \equiv w_2(M) \bmod 2$, however, it may not be possible to choose one such that the second obstruction vanishes. This is the case for $M = S^4$ for example: c must be 0, so $\int_M c^2 = 0$ while $2\chi(S^4) + 3\sigma(S^4) = 4$.

Theorem. **(Wu)** *Let M be a closed oriented smooth four-manifold. Then M admits an almost complex structure with $c_1(M) = c$ if and only if*

- $c \equiv w_2(M) \bmod 2$
- $\int_M c^2 = 2\chi(M) + 3\tau(M)$.

Six-dimensional case

In this case $SO(6)/U(3) = \mathbb{CP}^3$. From the fibration $S^1 \to S^7 \to \mathbb{CP}^3$, we see that $\pi_i(\mathbb{CP}^3) = \pi_i(S^7) = 0$ for $i = 3, 4, 5, 6$. So there are no further obstructions.

Theorem. *Let M be an orientable six-manifold. Then M admits an almost complex structure if and only if $W_3(M) = 0$.*

Unlike in the four-dimensional case, the vanishing of W_3 is not automatic in six-dimensions. One example is $S^1 \times (SU(3)/SO(3))$; the manifold $SU(3)/SO(3)$ is known as the Wu manifold.

Now we can finally justify why the first obstruction to the existence of a section of $BU(n) \to BSO(2n)$ is W_3. If it weren't, the obstruction would vanish and hence every orientable six-manifold would admit an almost complex structure, including $S^1 \times (SU(3)/SO(3))$. But then $w_2(S^1 \times (SU(3)/SO(3)))$ would have an integral lift (given by the first Chern class), but this is impossible.

One example where the obstruction vanishes is S^6. This is one explanation for the existence of an almost complex structure on S^6.

The primary obstruction always vanishes for spheres (i.e. S^{2n} is spin^c), but only S^2 and S^6 admit almost complex structures, so we see that in dimensions other than 2 and 6, there are always additional obstructions.

References

[1] Andrade, R., *Example of fiber bundle that is not a fibration.* [online] MathOverflow. Available at: `https://mathoverflow.net/q/119115/21564` [accessed 20 Sept. 2018].
[2] Davis, J.F. and Kirk, P., 2001. *Lecture notes in algebraic topology* (Vol. 35). American Mathematical Soc.
[3] Griffiths, P., and Morgan, J.W., 1981. *Rational homotopy theory and differential forms* (Vol. 16). Boston: Birkhuser.
[4] Hatcher, A., 2002. *Algebraic topology.* 2002. Cambridge UP, Cambridge, 606(9).
[5] Milnor, J., 1956. *Construction of universal bundles, II.* Annals of Mathematics, pp.430-436.