

**Derived Equivalences of Irregular Varieties and
Constraints on Hodge Numbers**

BY

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THESIS

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LIST OF ABBREVIATIONS

\mathcal{O}_X	Structure sheaf
ω_X	Canonical bundle
Ω_X^p	Bundle of holomorphic p -forms
L^{-1}	Dual line bundle
L^m	m -th tensor power of a line bundle
$H^i(X, \mathcal{F})$	i -th sheaf cohomology group
$h^i(X, \mathcal{F})$	Dimension of the i -th sheaf cohomology group
$h^{p,q}(X)$	Hodge numbers
1_X	Identity map
Δ_X	Diagonal embedding
$R(X)$	Canonical ring
$\kappa(X)$	Kodaira dimension
$q(X)$	Irregularity
$\text{Pic}^0(X)$	Picard variety
$\text{Aut}^0(X)$	Neutral component of the automorphism group
$\text{Alb}(X)$	Albanese variety
alb_X	Albanese map

LIST OF ABBREVIATIONS (Continued)

$\dim \text{alb}_X(X)$	Albanese dimension
$HH^*(X)$	Hochschild cohomology
$HH_*(X)$	Hochschild homology
$V_r^i(\mathcal{F})$	Cohomological support loci associated to \mathcal{F}
$V_r^i(\mathcal{F})_0$	Cohomological support loci associated to \mathcal{F} around the origin
$\mathbf{D}(X)$	Bounded derived category of coherent sheaves on X
$\Phi_{\mathcal{E}}, \Psi_{\mathcal{E}}$	Fourier-Mukai functors with kernel \mathcal{E}
$\Psi_{\mathcal{E}_R}, \Psi_{\mathcal{E}_L}$	Right and left adjoints to $\Phi_{\mathcal{E}}$
$F_{\mathcal{E}}$ (or F)	Rouquier isomorphism

SUMMARY

We study derived equivalences of smooth projective irregular varieties. More specifically, as suggested by a conjecture of Popa, we investigate the behavior of cohomological support loci associated to the canonical bundle (around the origin) under derived equivalence. We approach this problem in two ways. In the first approach we establish and apply the derived invariance of a “twisted” version of Hochschild homology taking into account an isomorphism due to Rouquier and related to autoequivalences of derived categories. In the second approach we relate the derived invariance of cohomological support loci to the derived invariance of Hodge numbers. As a result, we obtain the derived invariance of the first two and the last two cohomological support loci, leading to interesting geometric applications. For instance, we deduce the derived invariance of a few numerical quantities attached to irregular varieties, and furthermore we describe the geometry of Fourier-Mukai partners of Fano fibrations, and hence of Mori fiber spaces, fibered over curves of genus at least two.

Finally, we also study constraints on Hodge numbers of special classes of irregular compact Kähler manifolds. More specifically, we write down inequalities for all the Hodge numbers by studying the exactness of BGG complexes associated to bundles of holomorphic p -forms and by using classical results in the theory of vector bundles on projective spaces. As an application of our techniques, we bound the regularity of cohomology modules in terms of the defect of semismallness of the Albanese map.

CHAPTER 1

INTRODUCTION

The present dissertation is divided into two parts. The first part is devoted to the study of equivalences of bounded derived categories of coherent sheaves on complex smooth irregular projective varieties, while the second part focuses on constraints on Hodge numbers of irregular compact Kähler manifolds.

We present now the first part of this thesis consisting of §§3, 4, 5 and 6, mainly discussing results in (1) and (2).

The bounded derived category $\mathbf{D}(X) := \mathcal{D}^b(\mathit{Coh}(X))$ of coherent sheaves on a smooth projective variety X is a categorical invariant carrying several information about the variety itself. For instance, the dimension, the canonical ring and the Hochschild structure are objects that are invariant under an equivalence of derived categories. One of the goals of this work is the study of *derived equivalences* $\mathbf{D}(X) \simeq \mathbf{D}(Y)$ when X and Y are *irregular* varieties, i.e. having non-vanishing first Betti number. In particular we are interested in:

- understanding the behavior of numerical quantities under derived equivalence, such as the Hodge numbers, the Albanese dimension and the holomorphic Euler characteristic;
- understanding the geometry of Fourier-Mukai partners, for instance by looking at the behavior of their fibrations.

In the study of irregular varieties, the *cohomological support loci associated to the canonical bundle* $V^i(\omega_X) := \{\alpha \in \text{Pic}^0(X) \mid H^i(X, \omega_X \otimes \alpha) \neq 0\}$ play a fundamental role. For instance, they are the main characters in the proof of Ueno’s conjecture K (cf. (3)), and moreover they control several aspects of the birational geometry of an irregular variety, such as the Albanese map, fibrations and pluricanonical maps. Going back to our purposes, it is then important to analyze the behavior of these loci under derived equivalence. This behavior has been conjectured by Popa in (4):

Conjecture 1.0.1. *Let X and Y be complex smooth projective varieties such that $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. Then for all $i \geq 0$ there exist isomorphisms $V^i(\omega_X)_0 \simeq V^i(\omega_Y)_0$ (here the subscript 0 denotes the union of all irreducible components passing through the origin).*

We approach this conjecture in two ways:

- by studying a “twisted” version of Hochschild homology taking into account an isomorphism due to Rouquier and related to autoequivalences of derived categories (cf. §§3, 4 and 5);
- by establishing a relationship between the derived invariance of Hodge numbers and the derived invariance of cohomological support loci (cf. §6).

To begin with we introduce the above-mentioned twisted Hochschild homology (cf. §3). For any pair $(\varphi, \alpha) \in \text{Aut}^0(X) \times \text{Pic}^0(X)$ we define the spaces $HH_*(X, \varphi, \alpha) := \bigoplus_i HH_i(X, \varphi, \alpha)$ where

$$HH_i(X, \varphi, \alpha) := \text{Ext}_{X \times X}^i(\Delta_{X*} \mathcal{O}_X, (1_X, \varphi)_*(\omega_X \otimes \alpha)),$$

Δ_X is the diagonal embedding, and $(1_X, \varphi)$ is the embedding $x \mapsto (x, \varphi(x))$. It is easy to see that, via composition of morphisms, $HH_*(X, \varphi, \alpha)$ has a natural structure of graded module over the Hochschild cohomology $HH^*(X) := \bigoplus_i \text{Ext}_{X \times X}^i(\Delta_{X*} \mathcal{O}_X, \Delta_{X*} \mathcal{O}_X)$. We think of these spaces as “twisted” versions of Hochschild homology $HH_*(X) := \bigoplus_i \text{Ext}_{X \times X}^i(\Delta_{X*} \mathcal{O}_X, \Delta_{X*} \omega_X)$. Moreover, we recall that an equivalence $\Phi : \mathbf{D}(X) \simeq \mathbf{D}(Y)$ between the derived categories of two smooth projective varieties X and Y induces an isomorphism of algebraic groups

$$F : \text{Aut}^0(X) \times \text{Pic}^0(X) \longrightarrow \text{Aut}^0(Y) \times \text{Pic}^0(Y)$$

called *Rouquier isomorphism* (an explicit description of F is in Theorem 2.1.8). One of our main results is the verification of the compatibility of twisted Hochschild homology with the Rouquier isomorphism (cf. Theorem 3.0.6).

Theorem 1.0.2. *Let X and Y be smooth projective varieties defined over an algebraically closed field and let $\Phi : \mathbf{D}(X) \simeq \mathbf{D}(Y)$ be an equivalence. If $F(\varphi, \alpha) = (\psi, \beta)$ (where F is the induced Rouquier isomorphism), then there is an isomorphism of graded modules $HH_*(X, \varphi, \alpha) \simeq HH_*(Y, \psi, \beta)$.*

In this way, by specializing to the case $(\varphi, \alpha) = (1_X, \mathcal{O}_X)$, we recover previous results of Căldăraru and Orlov concerning the derived invariance of Hochschild homology (*cf.* (5) and (6)).

In §4 we analyze the impact of the derived invariance of twisted Hochschild homology towards the study of the behavior of cohomological support loci under derived equivalence. We will see that this invariance, in combination with Brion's structural results on actions of non-affine groups on smooth projective varieties (*cf.* (7)), and a version of the Hochschild-Kostant-Rosenberg isomorphism due to Yekutieli (*cf.* (8)), leads to the following isomorphisms (*cf.* Propositions 4.2.1 and 4.4.1 and Corollary 4.4.3):

Theorem 1.0.3. *Let X and Y be complex smooth projective varieties. If $\mathbf{D}(X) \simeq \mathbf{D}(Y)$, then the Rouquier isomorphism induces isomorphisms of algebraic sets: i) $V^0(\omega_X) \simeq V^0(\omega_Y)$; ii) $V^1(\omega_X)_0 \simeq V^1(\omega_Y)_0$; and iii) $V^0(\omega_X) \cap V^1(\omega_X) \simeq V^0(\omega_Y) \cap V^1(\omega_Y)$.*

Next we study Conjecture 1.0.1 in dimension three (the case of curves is trivial while in dimension two Popa himself proves and extends Conjecture 1.0.1; *cf.* (4)). The additional ingredient which allows us to make further progress in dimension three is the derived invariance of all the Hodge numbers. In this way we verify Conjecture 1.0.1 and furthermore we establish, in most cases, full isomorphisms $V^i(\omega_X) \simeq V^i(\omega_Y)$ for all $i \geq 0$. Moreover, we also obtain other results which we summarize in the following theorem (*cf.* §5).

Theorem 1.0.4. *Let X and Y be complex smooth projective threefolds such that $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. Then the Rouquier isomorphism F induces isomorphisms of algebraic sets $V^i(\omega_X)_0 \simeq V^i(\omega_Y)_0$ for all $i \geq 0$. Moreover:*

(i). If either X is of maximal Albanese dimension, or $\mathrm{Aut}^0(X)$ is affine, or $\chi(\omega_X) \neq 0$, then F induces isomorphisms $V^i(\omega_X) \simeq V^i(\omega_Y)$ for all $i \geq 0$.

(ii). If $q(X) \geq 2$, then we have $\dim V^i(\omega_X) = \dim V^i(\omega_Y)$ for all $i \geq 0$. Moreover, if $q(X) = 2$, then F induces isomorphisms $V^i(\omega_X) \simeq V^i(\omega_Y)$ for all $i \geq 0$.

Another important problem that we investigate in this work is the behavior of the Albanese dimension under derived equivalence. For varieties of Kodaira dimension zero, this invariance easily follows from a classical result of Kawamata on the surjectivity of the Albanese map for this class of varieties together with a result of Popa and Schnell establishing the derived invariance of the irregularity $q(X) := \dim H^0(X, \Omega_X^1)$ (cf. (9)). Our contribution towards this problem is the derived invariance of the Albanese dimension for smooth projective varieties having *positive* Kodaira dimension $\kappa(X)$ (cf. Theorem 4.5.2; see also Remark 4.5.3).

Theorem 1.0.5. *Let X and Y be complex smooth projective varieties such that $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. If either $\dim X \leq 3$, or $\dim X > 3$ and $\kappa(X) \geq 0$, then $\dim \mathrm{alb}_X(X) = \dim \mathrm{alb}_Y(Y)$.*

To prove this theorem we use the result of Popa and Schnell mentioned above together with techniques coming from birational geometry. In particular we use a result of Chen, Hacon and Pardini studying the Albanese map of an irregular variety via its Iitaka fibration.

Turning to applications, the results described above can be used in combination with generic vanishing theory developed in (10) and (11) to yield the derived invariance of further numerical quantities of irregular varieties. More specifically, we show the derived invariance of the holomorphic Euler characteristic for varieties having large Albanese dimension, and hence the

derived invariance of the Hodge numbers $h^{0,2}(X)$ and $h^{1,3}(X)$ for fourfolds having large Albanese dimension (cf. §4.6).

Corollary 1.0.6. *Let X and Y be complex smooth projective varieties such that $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. Suppose that either $\dim \operatorname{alb}_X(X) = \dim X$, or $\dim \operatorname{alb}_X(X) = \dim X - 1$ and $\kappa(X) \geq 0$. Then we have $\chi(\omega_X) = \chi(\omega_Y)$. Moreover, if in addition $\dim X = 4$, then we have $h^{0,2}(X) = h^{0,2}(Y)$ and $h^{1,3}(X) = h^{1,3}(Y)$.*

Now we present the main results of §6. In this chapter we relate Conjecture 1.0.1 to the conjectured derived invariance of Hodge numbers (cf. Problem 2.1.6). As a byproduct, we obtain the derived invariance of a further cohomological support locus, namely $V^{\dim X - 1}(\omega_X)_0$, making possible the study of fibrations onto smooth curves under derived equivalence.

Turning to details, we prove that whenever a Hodge number of type $h^i(X, \omega_X)$ is a derived invariant, then the corresponding locus $V^i(\omega_X)_0$ is a derived invariant as well. More precisely we have the following (cf. Theorem 6.2.1 for the most general statement):

Theorem 1.0.7. *Let X and Y be complex smooth projective varieties of dimension n such that $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. Assume that for some $i \geq 0$ the Hodge number $h^i(X, \omega_X)$ is invariant under an arbitrary derived equivalence of complex smooth projective varieties of dimension n . Then the Rouquier isomorphism induces an isomorphism of algebraic sets $V^i(\omega_X)_0 \simeq V^i(\omega_Y)_0$. In particular, the locus $V^{n-1}(\omega_X)_0$ is a derived invariant (as the irregularity $q(X) = h^{n-1}(X, \omega_X)$ is a derived invariant).*

To prove this theorem we study derived equivalences of étale cyclic covers associated to topologically trivial torsion line bundles. This is inspired by a paper of Bridgeland and Maciocia (*cf.* (12)) where the authors consider equivalences between derived categories of canonical covers of varieties having torsion canonical bundles. Another key ingredient in the proof of Theorem 1.0.7 is a theorem of Green, Lazarsfeld and Simpson describing the cohomological support loci as unions of *torsion* translates of abelian subvarieties (*cf.* Theorem 2.2.2 (i)).

In the second part of §6 we present geometric consequences of Theorem 1.0.7. More precisely, we describe the behavior of special types of fibrations under derived equivalence, providing a further tool towards the understanding of the birational geometry of Fourier-Mukai partners (*cf.* (4), (2), (13)). In particular we show that if X admits a fibration onto a smooth curve of genus $\geq g \geq 2$, then any Fourier-Mukai partner Y of X admits a fibration onto a smooth curve of genus $\geq g$ (*cf.* Theorem 6.3.1). Furthermore, we study Fourier-Mukai partners of Fano fibrations and Mori fiber spaces over curves of genus ≥ 2 (*cf.* Theorem 6.3.4).

Theorem 1.0.8. *Let X and Y be complex smooth projective varieties of dimension n such that $\mathbf{D}(X) \simeq \mathbf{D}(Y)$ and let $f : X \rightarrow C$ be a fibration onto a smooth curve of genus $g(C) \geq 2$. Then:*

- (i). *Y admits a fibration onto a curve of genus $\geq g(C)$.*
- (ii). *If f is a Fano fibration (i.e. the general fiber of f is Fano), then Y admits a Fano fibration onto the same curve C . Moreover, X and Y are birational (in fact K -equivalent).*
- (iii). *If ω_X^{-1} is f -ample (e.g. f is a Mori fiber space), then $X \simeq Y$.*

We now present the second part of this thesis. This part consists of §7 and discusses the results in (14) concerning inequalities for the Hodge numbers of irregular compact Kähler manifolds.

Finding relations among the Hodge numbers is a problem that attracted the attention of several mathematicians. The first non trivial relation of this sort was found by Castelnuovo and De Franchis at the beginning of the twentieth century: if a surface X admits no fibrations onto smooth projective curves of genus at least two, then $h^{0,2}(X) \geq 2h^{0,1}(X) - 3$ (*cf.* (15)). Subsequently, this inequality has been generalized to higher dimensional varieties, for instance by Catanese by means of sophisticated arguments involving the exterior algebra of holomorphic forms, and by Pareschi and Popa by means of generic vanishing theory and the Evans-Griffith syzygy theorem.

A further approach to this problem is the work of Lazarsfeld and Popa (*cf.* (16)). Their approach relies on the study of a global version of the *derivative complex* associated to the structure sheaf, and by means of the theory of vector bundles on projective spaces. In this way they turn a purely geometric problem into a more manageable problem of homological algebra (a similar technique also appears in (17)). It is important to note that the inequalities of Catanese, Pareschi-Popa and Lazarsfeld-Popa mainly involve Hodge numbers of type $h^{p,0}(X)$, representing the dimensions of spaces of holomorphic p -forms. By extending the techniques of (16) to *all* bundles of holomorphic p -forms Ω_X^p , we write down inequalities for all the Hodge numbers $h^{p,q}(X)$ for special classes of irregular compact Kähler manifolds.

We turn now to a more detailed presentation of our results. We denote by X an irregular compact Kähler manifold of dimension n and irregularity q , and by $\mathbf{P} := \mathbf{P}(V)$ the projective space over $V := H^1(X, \mathcal{O}_X)$. Any element $0 \neq v \in V$ determines a complex

$$0 \longrightarrow H^0(X, \Omega_X^p) \xrightarrow{\cup v} H^1(X, \Omega_X^p) \xrightarrow{\cup v} \dots \xrightarrow{\cup v} H^n(X, \Omega_X^p) \longrightarrow 0$$

called derivative complex associated to Ω_X^p . As v varies in V , we can arrange all these complexes in a complex of locally free sheaves on \mathbf{P} :

$$\begin{aligned} \underline{\mathbf{L}}_X^p : \quad 0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-n) \otimes H^0(X, \Omega_X^p) \longrightarrow \mathcal{O}_{\mathbf{P}}(-n+1) \otimes H^1(X, \Omega_X^p) \longrightarrow \dots \\ \dots \longrightarrow \mathcal{O}_{\mathbf{P}}(-1) \otimes H^{n-1}(X, \Omega_X^p) \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^n(X, \Omega_X^p) \longrightarrow 0. \end{aligned} \quad (1.1)$$

The exactness of $\underline{\mathbf{L}}_X^p$ has been studied in (16) in the case $p = 0$. If k denotes the dimension of a general fiber of the Albanese map $a : X \rightarrow \text{Alb}(X)$, then $\underline{\mathbf{L}}_X^0$ is exact at the first $n - k$ steps from the left. In §7.1 we consider the remaining cases $p > 0$ by showing that the exactness of $\underline{\mathbf{L}}_X^p$ depends on the non-negative integer $m(X) := \min\{\text{codim } Z(\omega) \mid 0 \neq \omega \in H^0(X, \Omega_X^1)\}$, representing the least codimension of the zero locus of a non-zero holomorphic one-form. More precisely, we prove that if $m(X) > p$, then $\underline{\mathbf{L}}_X^p$ is exact at the first $(m(X) - p)$ -terms from the left and that the first $m(X) - p$ maps are of constant rank. This is enough to ensure that the cokernel of the map

$$\mathcal{O}_{\mathbf{P}}(m(X) - n - p - 1) \otimes H^{m(X)-p-1}(X, \Omega_X^p) \rightarrow \mathcal{O}_{\mathbf{P}}(m(X) - n - p) \otimes H^{m(X)-p}(X, \Omega_X^p)$$

is a locally free sheaf. This in turn leads to new inequalities for the Hodge numbers thanks to the Evans-Griffith theorem and to the fact that the Chern classes of a globally generated locally free sheaf are non-negative. To give a flavor of our inequalities, we present the case $m(X) = \dim X$. We refer to Theorems 7.2.1 and 7.2.2 for the most general statements.

Theorem 1.0.9. *Let X be a compact Kähler manifold of dimension n and irregularity $q \geq n$. Suppose that $m(X) = n$ and let $0 \leq p < n$ be an integer. Then $\sum_{j=0}^{n-p} (-1)^{n-p+j} h^{p,j}(X) \geq q - n + p$. Moreover, if $n = 3$, then $h^{1,1}(X) \geq 2q$ and $h^{1,2}(X) \geq h^{1,1}(X) - 2$.*

Another problem studied in §7 concerns the *regularity of cohomology modules* $P_X^p := \bigoplus_i H^i(X, \Omega_X^p)$ over the exterior algebra $E = \bigwedge^* H^1(X, \mathcal{O}_X)$ of a complex smooth projective variety X of dimension n . The case $p = n$ has been studied in (16). There the authors show that the regularity of P_X^n is determined by the dimension of the generic fiber of the Albanese map $a : X \rightarrow \text{Alb}(X)$. Using the Bernstein-Gel'fand-Gel'fand (BGG) correspondence (cf. §7.4.1), and generic vanishing theorems for bundles of holomorphic p -forms (cf. Theorem 2.2.3), we analyze the remaining cases $p < n$ by bounding the regularity of P_X^p in terms of the defect of semismallness of the Albanese map $\delta(a) := \max_{l \in \mathbb{N}} \{2l - \dim X + \dim \text{Alb}(X)_l\}$ where $\text{Alb}(X)_l := \{y \in \text{Alb}(X) \mid \dim a^{-1}(y) \geq l\}$.

Theorem 1.0.10. *Let X be a complex smooth projective variety of dimension n and let $a : X \rightarrow \text{Alb}(X)$ be the Albanese map of X . If $p \geq \delta(a)$, then P_X^p is $(n - p + \delta(a))$ -regular.*

CHAPTER 2

DERIVED INVARIANTS AND GENERIC VANISHING THEORY

In this chapter we recall background material about equivalences of derived categories of coherent sheaves and generic vanishing theory. Work of Bondal, Căldăraru, Kawamata, Orlov, Popa, Schnell et al. shows that the bounded derived category of coherent sheaves attached to a smooth projective variety carries several information about the variety itself. For instance, the dimension, Kodaira dimension, and Hochschild (co)homology are all examples of objects that are invariant under an equivalence of derived categories. Here we briefly summarize those derived invariants that are needed in the sequel in order to investigate derived equivalences of *irregular* varieties, namely the ones having non-vanishing first Betti number.

Furthermore, we recall fundamental results concerning the structure and geometry of the main characters of our investigation, namely the cohomological support loci associated to the canonical bundle. The study of these loci is one of the main topics in Generic Vanishing Theory as introduced by Green and Lazarsfeld and further developed by Arapura, Hacon, Pareschi, Popa, Schnell et al. We refer to (6), (18) and (19) for the theory of derived categories of coherent sheaves, and to the papers (10), (20), (21), (22), (11) and (23) for the theory of generic vanishing.

In §2.1 we work over an algebraically closed field K unless otherwise specified, whereas in §2.2 we work over the field of complex numbers.

2.1 Derived invariants

Let X be a smooth projective variety defined over an algebraically closed field K . We denote by $\mathbf{D}(X) := \mathcal{D}^b(\mathcal{C}oh(X))$ the *derived category of X* , namely the bounded derived category of the abelian category $\mathcal{C}oh(X)$ of coherent sheaves on X (cf. for instance (6) and (19)). We recall that $\mathbf{D}(X)$ is a K -linear triangulated category, so that $\mathbf{D}(X)$ is endowed with a shift functor $[-] : \mathbf{D}(X) \rightarrow \mathbf{D}(X)$ and the spaces of morphisms $\mathrm{Hom}_{\mathbf{D}(X)}(\mathcal{F}, \mathcal{G})$ are finite dimensional K -vector spaces for all objects \mathcal{F} and \mathcal{G} in $\mathbf{D}(X)$. Finally, when there is no possibility of ambiguity, we denote functors and derived functors with the same symbols.

Let Y be another smooth projective variety defined over K and $p : X \times Y \rightarrow X$ and $q : X \times Y \rightarrow Y$ be the projections from $X \times Y$ onto the first and second factor respectively. An object \mathcal{E} in $\mathbf{D}(X \times Y)$ defines functors

$$\Phi_{\mathcal{E}} : \mathbf{D}(X) \longrightarrow \mathbf{D}(Y), \quad \mathcal{F} \mapsto q_*(p^*\mathcal{F} \otimes \mathcal{E})$$

$$\Psi_{\mathcal{E}} : \mathbf{D}(Y) \longrightarrow \mathbf{D}(X), \quad \mathcal{G} \mapsto p_*(q^*\mathcal{G} \otimes \mathcal{E})$$

called *Fourier-Mukai functors*.

We will be interested in equivalences of derived categories; to keep notation as simple as possible we introduce the notion of D -equivalence.

Definition 2.1.1. *Two smooth projective varieties X and Y are D -equivalent if there exists a K -linear exact equivalence $\Phi : \mathbf{D}(X) \simeq \mathbf{D}(Y)$. In this case we say that X and Y are Fourier-Mukai partners and that Φ is a derived equivalence or simply an equivalence.*

Next we present a fundamental result of Orlov concerning equivalences of derived categories which will be tacitly used throughout this work (cf. (24), (6) and (19)).

Theorem 2.1.2. *Let X and Y be smooth projective varieties defined over an algebraically closed field K . If $\Phi : \mathbf{D}(X) \simeq \mathbf{D}(Y)$ is a derived equivalence, then $\Phi \simeq \Phi_{\mathcal{E}}$ for some object \mathcal{E} in $\mathbf{D}(X \times Y)$. Moreover, \mathcal{E} is unique up to isomorphism and is called kernel.*

We now briefly recall a few quantities, objects and properties that are preserved under derived equivalence.

Dimension. The first basic invariant under an equivalence of derived categories is the dimension of a variety (cf. (6) and (25)). This follows, for instance, by the fact that the left and right adjoints to a Fourier-Mukai functor are again of Fourier-Mukai type. To see this we define the objects

$$\mathcal{E}_L := \mathcal{E}^{\vee} \otimes q^* \omega_Y[\dim Y] \quad \text{and} \quad \mathcal{E}_R := \mathcal{E}^{\vee} \otimes p^* \omega_X[\dim X]$$

where \mathcal{E} is an object in $\mathbf{D}(X \times Y)$ and $\mathcal{E}^{\vee} := \mathcal{H}om_{\mathbf{D}(X \times Y)}(\mathcal{E}, \mathcal{O}_{X \times Y})$ is the derived dual of \mathcal{E} . Then, by using the Grothendieck-Verdier duality and the projection formula, we have the following (cf. (6) Lemma 2.1.1, and (19) Proposition 5.9 and Corollary 5.21):

Theorem 2.1.3. *Let X and Y be smooth projective varieties and let $\Phi_{\mathcal{E}} : \mathbf{D}(X) \rightarrow \mathbf{D}(Y)$ be a Fourier-Mukai functor. Then the left and right adjoints to $\Phi_{\mathcal{E}}$ are respectively the Fourier-*

Mukai functors $\Psi_{\mathcal{E}_L}$ and $\Psi_{\mathcal{E}_R}$. Moreover, if $\Phi_{\mathcal{E}}$ is an equivalence, then \mathcal{E}_L and \mathcal{E}_R are isomorphic objects. Consequently

$$\dim X = \dim Y$$

and

$$p^*\omega_X \otimes \mathcal{E} \simeq q^*\omega_Y \otimes \mathcal{E}. \quad (2.1)$$

It is also important to mention that whenever $\Phi_{\mathcal{E}}$ is an equivalence, then $\Psi_{\mathcal{E}_L}$ and $\Psi_{\mathcal{E}_R}$ are isomorphic equivalences such that $\Psi_{\mathcal{E}_L} \circ \Phi_{\mathcal{E}} \simeq \Psi_{\mathcal{E}_R} \circ \Phi_{\mathcal{E}} \simeq 1_{\mathbf{D}(X)}$ and $\Phi_{\mathcal{E}} \circ \Psi_{\mathcal{E}_L} \simeq \Phi_{\mathcal{E}} \circ \Psi_{\mathcal{E}_R} \simeq 1_{\mathbf{D}(Y)}$.

Canonical ring and Kodaira dimension. We denote by

$$R(X) = R(X, \omega_X) := \bigoplus_{m \geq 0} H^0(X, \omega_X^m) \quad \text{and} \quad R(X, \omega_X^{-1}) := \bigoplus_{m \geq 0} H^0(X, \omega_X^{-m})$$

the canonical and anti-canonical rings of a smooth projective variety X . Moreover, we denote by $\kappa(X) = \kappa(X, \omega_X)$ and $\kappa(X, \omega_X^{-1})$ the Kodaira and anti-Kodaira dimensions of X .

Theorem 2.1.4. *Let X and Y be smooth projective varieties and $\Phi_{\mathcal{E}} : \mathbf{D}(X) \simeq \mathbf{D}(Y)$ be an equivalence. Then $\Phi_{\mathcal{E}}$ induces an isomorphism of rings $R(X, \omega_X) \simeq R(Y, \omega_Y)$. Moreover, if in addition the characteristic of the ground field is zero, then $\kappa(X, \omega_X) = \kappa(Y, \omega_Y)$. (Analogous statements also hold for the anti-canonical rings and anti-Kodaira dimensions.)*

We sketch here the proof of the previous theorem since its general strategy will be used later to establish the derived invariance of a twisted version of Hochschild homology (we refer to (6) Corollary 2.1.9 for the complete proof of Theorem 2.1.4).

Let $\Delta_X : X \hookrightarrow X \times X$ and $\Delta_Y : Y \hookrightarrow Y \times Y$ be the diagonal embeddings of X and Y , and let p_i and p_{ij} be the projections from the product $X \times X \times Y \times Y$ onto the i -th and (i, j) -th factor respectively. Furthermore, we set $\mathcal{E}_R \boxtimes \mathcal{E} := p_{13}^* \mathcal{E}_R \otimes p_{24}^* \mathcal{E}$ so that if $\Phi_{\mathcal{E}} : \mathbf{D}(X) \rightarrow \mathbf{D}(Y)$ is an equivalence, then

$$\Phi_{\mathcal{E}_R \boxtimes \mathcal{E}} : \mathbf{D}(X \times X) \longrightarrow \mathbf{D}(Y \times Y) \quad (2.2)$$

is an equivalence as well satisfying $\Phi_{\mathcal{E}_R \boxtimes \mathcal{E}}(\Delta_{X*} \omega_X^m) \simeq \Delta_{Y*} \omega_Y^m$ for any $m \in \mathbf{Z}$ (*cf.* for instance (6) Proposition 2.1.7 or the following Lemma 3.0.5). Hence we obtain a series of isomorphisms

$$\begin{aligned} H^0(X, \omega_X^m) = \mathrm{Hom}_{X \times X}(\mathcal{O}_X, \omega_X^m) &\simeq \mathrm{Hom}_{X \times X}(\Delta_{X*} \mathcal{O}_X, \Delta_{X*} \omega_X^m) \\ &\simeq \mathrm{Hom}_{Y \times Y}(\Phi_{\mathcal{E}_R \boxtimes \mathcal{E}}(\Delta_{X*} \mathcal{O}_X), \Phi_{\mathcal{E}_R \boxtimes \mathcal{E}}(\Delta_{X*} \omega_X^m)) \\ &\simeq \mathrm{Hom}_{Y \times Y}(\Delta_{Y*} \mathcal{O}_Y, \Delta_{Y*} \omega_Y^m) = H^0(Y, \omega_Y^m). \end{aligned}$$

At this point, since $\Phi_{\mathcal{E}_R \boxtimes \mathcal{E}}$ is a functor, it follows in particular that $\Phi_{\mathcal{E}_R \boxtimes \mathcal{E}}$ induces an isomorphism of rings. Moreover, the equality $\kappa(X) = \kappa(Y)$ easily follows as one can read off the Kodaira dimension from the canonical ring (*cf.* (26) (1.3)).

Hochschild (co)homology. Let X be a smooth projective variety of dimension n and $\Delta_X : X \hookrightarrow X \times X$ be the diagonal embedding. The *Hochschild cohomology and homology of X* are defined as $HH^*(X) := \bigoplus_i HH^i(X)$ and $HH_*(X) := \bigoplus_i HH_i(X)$ where

$$HH^i(X) := \text{Ext}_{X \times X}^i(\Delta_{X*}\mathcal{O}_X, \Delta_{X*}\mathcal{O}_X) \quad \text{and} \quad HH_i(X) := \text{Ext}_{X \times X}^i(\Delta_{X*}\mathcal{O}_X, \Delta_{X*}\omega_X)$$

(cf. (6) p. 535, (8), (5), and (27)). Via composition of morphisms, Hochschild cohomology inherits a graded ring structure, while Hochschild homology inherits a graded $HH^*(X)$ -module structure. Important results of Căldăraru and Orlov show that Hochschild (co)homology is invariant under derived equivalence (cf. (5) Theorem 8.1 and (6) Theorem 2.1.8).

Theorem 2.1.5. *Let X and Y be smooth projective varieties and $\Phi_{\mathcal{E}} : \mathbf{D}(X) \rightarrow \mathbf{D}(Y)$ be an equivalence. Then $\Phi_{\mathcal{E}}$ induces an isomorphism of graded rings $HH^*(X) \simeq HH^*(Y)$, and an isomorphism of graded modules $HH_*(X) \simeq HH_*(Y)$ compatible with the isomorphism $HH^*(X) \simeq HH^*(Y)$.*

The proof of the previous theorem again uses the equivalence $\Phi_{\mathcal{E}_R \boxtimes \mathcal{E}}$ defined in (2.2). In fact, since $\Phi_{\mathcal{E}_R \boxtimes \mathcal{E}}(\Delta_{X*}\mathcal{O}_X) \simeq \Delta_{Y*}\mathcal{O}_Y$ and $\Phi_{\mathcal{E}_R \boxtimes \mathcal{E}}(\Delta_{X*}\omega_X) \simeq \Delta_{Y*}\omega_Y$, in particular it follows that

$$\begin{aligned} HH^i(X) &= \text{Ext}_{X \times X}^i(\Delta_{X*}\mathcal{O}_X, \Delta_{X*}\omega_X) \simeq \text{Ext}_{Y \times Y}^i(\Phi_{\mathcal{E}_R \boxtimes \mathcal{E}}(\Delta_{X*}\mathcal{O}_X), \Phi_{\mathcal{E}_R \boxtimes \mathcal{E}}(\Delta_{X*}\omega_X)) \\ &\simeq \text{Ext}_{Y \times Y}^i(\Delta_{Y*}\mathcal{O}_Y, \Delta_{Y*}\omega_Y) = HH^i(Y) \end{aligned}$$

and similarly for $HH_i(X) \simeq HH_i(Y)$.

Hodge numbers. We denote the Hodge numbers of a complex smooth projective variety X (or more in general of a compact Kähler manifold) by $h^{p,q}(X) := \dim_{\mathbf{C}} H^q(X, \Omega_X^p)$ where $\Omega_X^p = \bigwedge^p \Omega_X^1$ is the bundle of holomorphic p -forms. One of the main motivation of this work is the study of the following problem concerning the derived invariance of Hodge numbers. (This problem was first attributed to Kontsevich.)

Problem 2.1.6. *Is it true that $h^{p,q}(X) = h^{p,q}(Y)$ for all $p, q \geq 0$ whenever X and Y are complex smooth projective D -equivalent varieties?*

Now we present an application of Theorem 2.1.5 in view of the previous problem. In case the ground field K has characteristic zero, Hochschild homology admits a geometric interpretation in terms of sheaf cohomology via the *Hochschild-Kostant-Rosenberg isomorphism* (cf. (8) Corollary 4.7 and (27) Corollary 2.6):

$$HH_i(X) \simeq \bigoplus_{q=0}^i H^{i-q}(X, \Omega_X^{n-q}) \quad \text{for any } i \geq 0. \quad (2.3)$$

Therefore, Theorem 2.1.5 immediately leads to the following

Corollary 2.1.7. *If X and Y are complex smooth projective D -equivalent varieties, then for any $i \geq 0$ we have*

$$\sum_{q=0}^i h^{n-q, i-q}(X) = \sum_{q=0}^i h^{n-q, i-q}(Y). \quad (2.4)$$

In particular, D -equivalent surfaces have the same Hodge numbers.

We point out that Problem 2.1.6 is trivially true in the case of curves as D -equivalent curves are isomorphic (*cf.* (19) Corollary 4.13 and p. 134). Moreover, as we will see in a moment, the derived invariance of Hodge numbers also holds for varieties of dimension three and for varieties of general type (*cf.* Theorem 2.1.10 and the comment soon after Theorem 2.1.12).

The Rouquier isomorphism. We denote by $\text{Pic}(X)$ the Picard group of a smooth projective variety X and by $\text{Aut}(X)$ its group of automorphisms. Moreover, we denote by $\text{Pic}^0(X)$ and $\text{Aut}^0(X)$ their corresponding neutral components.

An important result of Rouquier establishes the invariance of the product $\text{Aut}^0(-) \times \text{Pic}^0(-)$ under derived equivalence.

Theorem 2.1.8. *Let X and Y be smooth projective varieties and $\Phi_{\mathcal{E}} : \mathbf{D}(X) \simeq \mathbf{D}(Y)$ be an equivalence. Then $\Phi_{\mathcal{E}}$ induces an isomorphism of algebraic groups called Rouquier isomorphism*

$$F_{\mathcal{E}} : \text{Aut}^0(X) \times \text{Pic}^0(X) \longrightarrow \text{Aut}^0(Y) \times \text{Pic}^0(Y) \quad (2.5)$$

satisfying

$$F_{\mathcal{E}}(\varphi, \alpha) = (\psi, \beta) \iff p^* \alpha \otimes (\varphi \times 1_Y)^* \mathcal{E} \simeq q^* \beta \otimes (1_X \times \psi)_* \mathcal{E}. \quad (2.6)$$

The basic idea behind the Rouquier isomorphism is the following. An equivalence $\Phi_{\mathcal{E}} : \mathbf{D}(X) \simeq \mathbf{D}(Y)$ naturally induces an isomorphism $\text{Auteq}^0(\mathbf{D}(X)) \simeq \text{Auteq}^0(\mathbf{D}(Y))$ between

the neutral components of the groups of autoequivalences of $\mathbf{D}(X)$ and $\mathbf{D}(Y)$ via the formula $\phi \mapsto \Phi_{\mathcal{E}} \circ \phi \circ \Phi_{\mathcal{E}}^{-1}$, which in turn are identified to $\mathrm{Aut}^0(X) \times \mathrm{Pic}^0(X)$ and $\mathrm{Aut}^0(Y) \times \mathrm{Pic}^0(Y)$ respectively. The proof of Theorem 2.1.8 can be found in (28) Théorème 4.18, (9) Lemma 3.1, and (9) footnote at p. 531. We also refer to (29) Theorem 3.1 and (19) Proposition 9.45 for further proofs. As a matter of notation, whenever the kernel of an equivalence $\Phi : \mathbf{D}(X) \simeq \mathbf{D}(Y)$ is not specified, we denote by F the Rouquier isomorphism induced by Φ . The Rouquier isomorphism induced by a right (or left) adjoint is computed in the following lemma.

Lemma 2.1.9. *Let X and Y be smooth projective varieties, $\Phi_{\mathcal{E}} : \mathbf{D}(X) \simeq \mathbf{D}(Y)$ be an equivalence and $F_{\mathcal{E}} : \mathrm{Aut}^0(X) \times \mathrm{Pic}^0(X) \rightarrow \mathrm{Aut}^0(Y) \times \mathrm{Pic}^0(Y)$ be the induced Rouquier isomorphism. If $\Psi_{\mathcal{F}} : \mathbf{D}(Y) \simeq \mathbf{D}(X)$ is a quasi-inverse to $\Phi_{\mathcal{E}}$ and $F_{\mathcal{F}}$ is the Rouquier isomorphism induced by $\Psi_{\mathcal{F}}$, then $F_{\mathcal{F}} = F_{\mathcal{E}}^{-1}$.*

Proof. Without loss of generality we can suppose that $\Psi_{\mathcal{F}} = \Psi_{\mathcal{E}_R}$ is the right adjoint to $\Phi_{\mathcal{E}}$ where $\mathcal{E}_R = \mathcal{E}^{\vee} \otimes p^* \omega_X[\dim X] \in \mathbf{D}(X \times Y)$. We note that if \mathcal{G} is an object in $\mathbf{D}(X \times Y)$, then $\Psi_{\mathcal{G}} \simeq \Phi_{\rho^* \mathcal{G}}$ where $\rho : Y \times X \rightarrow X \times Y$ is the inversion morphism $\rho(y, x) = (x, y)$. Therefore we denote by $F_{\rho^* \mathcal{E}_R}$ the Rouquier isomorphism induced by $\Psi_{\mathcal{E}_R} \simeq \Phi_{\rho^* \mathcal{E}_R}$. We denote by s_1 and s_2 the projections from $Y \times X$ onto the first and second factor respectively. By (2.6) the condition $F_{\rho^* \mathcal{E}_R}(\psi, \beta) = (\varphi, \alpha)$ is equivalent to an isomorphism

$$s_1^* \beta \otimes (\psi \times 1_X)^* \rho^* \mathcal{E}_R \simeq s_2^* \alpha \otimes (1_Y \times \varphi)_* \rho^* \mathcal{E}_R.$$

Since $s_1 = q \circ \rho$ and $s_2 = p \circ \rho$, we then have

$$\rho^* q^* \beta \otimes \rho^*(1_X \times \psi)^* \mathcal{E}_R \simeq \rho^* p^* \alpha \otimes \rho^*(\varphi \times 1_Y)_* \mathcal{E}_R.$$

Furthermore, since $\varphi^* \omega_X \simeq \omega_X$, we obtain $q^* \beta \otimes (1_X \times \psi)^* \mathcal{E}^\vee \simeq p^* \alpha \otimes (\varphi \times 1_Y)_* \mathcal{E}^\vee$, and by dualizing we get $p^* \alpha^{-1} \otimes (\varphi^{-1} \times 1_Y)^* \mathcal{E} \simeq q^* \beta^{-1} \otimes (1_Y \times \psi^{-1})_* \mathcal{E}$. Thus we have proved that $F_{\rho^* \mathcal{E}_R}(\psi, \beta) = (\varphi, \alpha)$ if and only if $F_{\mathcal{E}}(\varphi^{-1}, \alpha^{-1}) = (\psi^{-1}, \beta^{-1})$, i.e. $F_{\mathcal{E}}(\varphi, \alpha) = (\psi, \beta)$. \square

Picard variety. In general the Rouquier isomorphism $F_{\mathcal{E}} : \text{Aut}^0(X) \times \text{Pic}^0(X) \rightarrow \text{Aut}^0(Y) \times \text{Pic}^0(Y)$ defined in Theorem 2.1.8 mixes the two factors. For instance, this happens if X is an abelian variety and Y is its dual (*cf.* (19) Example 9.38 (v)). Despite this technical issue, Popa and Schnell describe the behavior of the Picard variety under equivalences of derived categories. This is an important result opening up the way to the study of derived equivalences of varieties having non-vanishing first Betti number $b_1(X)$, also known as *irregular* varieties. We recall that the *irregularity* of a complex smooth projective variety X is the non-negative integer:

$$q(X) := h^{1,0}(X) = \dim_{\mathbb{C}} H^0(X, \Omega_X^1) = \dim \text{Pic}^0(X) = b_1(X)/2. \quad (2.7)$$

Theorem 2.1.10. *If X and Y are complex smooth projective D -equivalent varieties, then $\text{Pic}^0(X)$ and $\text{Pic}^0(Y)$ are isogenous abelian varieties. Moreover, $\text{Pic}^0(X)$ and $\text{Pic}^0(Y)$ are isomorphic unless X and Y are étale locally trivial fibrations over isogenous positive-dimensional*

abelian varieties (hence $\chi(\omega_X) = \chi(\omega_Y) = 0$). In particular we have $q(X) = q(Y)$. Moreover, if in addition X and Y are threefolds, then $h^{p,q}(X) = h^{p,q}(Y)$ for all $p, q \geq 0$.

The proof of the previous theorem relies on the study of the Rouquier isomorphism by means of results on actions of non-affine groups of Nishi-Matsumura and Brion (cf. (7) and (30)), together with other classical results in the theory of derived categories and Mukai's description of semi-homogeneous vector bundles on abelian varieties. The complete proof of Theorem 2.1.10 can be found in (9) Theorem A and Corollary B. We only point out that, once the derived invariance of the irregularity is proved, the derived invariance of Hodge numbers in dimension three follows from Corollary 2.1.7.

Positivity of the canonical bundle. Work of Bondal, Kawamata and Orlov shows that many properties of the canonical bundle are preserved under derived equivalence. Moreover, the positivity of the canonical bundle strongly impacts the geometry of Fourier-Mukai partners leading to interesting relationships between D -equivalent and K -equivalent varieties.

Definition 2.1.11. *Two complex smooth projective varieties X and Y are K -equivalent if there exists a smooth projective variety Z and birational morphisms $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ such that $f^*\omega_X \simeq g^*\omega_Y$.*

Theorem 2.1.12. *Let X and Y be complex smooth projective D -equivalent varieties.*

- (i). *If ω_X or ω_X^{-1} is ample, then $X \simeq Y$.*
- (ii). *If $\kappa(X, \omega_X) = \dim X$ or $\kappa(X, \omega_X^{-1}) = \dim X$, then X and Y are K -equivalent.*
- (iii). *ω_X (resp. ω_X^{-1}) is nef if and only if ω_Y (resp. ω_Y^{-1}) is nef.*

The first statement of the previous theorem is the celebrated reconstruction theorem of Bondal and Orlov (31) (we note that it holds, more in general, for varieties defined over an arbitrary field). The other two statements are due to Kawamata and are deduced by studying the support of the kernel of an equivalence (*cf.* (25)). To conclude we note that, thanks to work of Kontsevich on motivic integration, K -equivalent complex smooth projective varieties have the same Hodge numbers (*cf.* (32) and (33)). Therefore, it is a consequence of the previous theorem that D -equivalent complex smooth projective varieties of general type have the same Hodge numbers.

2.2 Generic vanishing theory

Let X be a complex smooth projective variety and $q(X) := \dim_{\mathbf{C}} H^0(X, \Omega_X^1)$ be its irregularity. We say that X is *irregular* if it admits a non-constant morphism to an abelian variety, or equivalently if $q(X) > 0$. We denote by

$$\mathrm{Alb}(X) := H^0(X, \Omega_X^1)^\vee / H_1(X, \mathbf{Z})$$

the *Albanese variety* of X , and by

$$\mathrm{alb}_X : X \longrightarrow \mathrm{Alb}(X)$$

the *Albanese map* of X defined via integration of holomorphic one-forms. We note that $\mathrm{Alb}(X)$ is an abelian variety of dimension $q(X)$ and that alb_X induces an isomorphism $H^0(X, \Omega_X^1) \simeq H^0(\mathrm{Alb}(X), \Omega_{\mathrm{Alb}(X)}^1)$. Furthermore, we denote by $\dim \mathrm{alb}_X(X)$ the *Albanese dimension* of X

which is the dimension of the image of the Albanese map. Finally, we say that X is of *maximal Albanese dimension* if $\dim \text{alb}_X(X) = \dim X$, or equivalently if alb_X is generically finite onto its image.

In the study of irregular varieties we consider distinguished subsets of the Picard variety:

Definition 2.2.1. *The cohomological support loci associated to the canonical bundle ω_X are defined as*

$$V_r^i(\omega_X) := \{\alpha \in \text{Pic}^0(X) \mid h^i(X, \omega_X \otimes \alpha) \geq r\}$$

where $i \geq 0$ and $r \geq 1$ are integers. Moreover, we denote by $V_r^i(\omega_X)_0$ the union of all irreducible components of $V_r^i(\omega_X)$ passing through the origin of $\text{Pic}^0(X)$. Finally, we set $V^i(\omega_X) := V_1^i(\omega_X)$.

By semicontinuity, the loci $V_r^i(\omega_X)$ are algebraic closed subsets of $\text{Pic}^0(X)$. They have been introduced by Green and Lazarsfeld in (10) to study generic vanishing type theorems. Subsequently they have been used to study the (birational) geometry of irregular varieties, for instance to investigate problems concerning fibrations, pluricanonical maps and numerical invariants (cf. for instance (20), (21), (11) and (34)). In the following theorem we summarize fundamental results regarding cohomological support loci.

Theorem 2.2.2. *Let X be a complex smooth projective irregular variety of dimension n and let $\text{alb}_X : X \rightarrow \text{Alb}(X)$ be its Albanese map.*

(i). *If Z is an irreducible component of $V_r^i(\omega_X)$ for some $i \geq 0$ and $r \geq 1$, then Z is a torsion translate of an abelian subvariety in $\text{Pic}^0(X)$.*

(ii). If $\dim \text{alb}_X(X) = n - k$, then $\text{codim}_{\text{Pic}^0(X)} V^i(\omega_X) \geq i - k$ for all $i > 0$. In particular, if $k = 0$ then $\chi(\omega_X) = h^0(X, \omega_X \otimes \alpha) \geq 0$ for generic $\alpha \in \text{Pic}^0(X)$.

(iii). If for some integer $k \geq 0$ we have $\text{codim}_{\text{Pic}^0(X)} V^i(\omega_X)_0 \geq i - k$ for all $i > 0$, then $\dim \text{alb}_X(X) \geq n - k$. In particular, we have the following formula for the Albanese dimension

$$\dim \text{alb}_X(X) = \min_{j=0, \dots, n} \{n - j + \text{codim}_{\text{Pic}^0(X)} V^j(\omega_X)_0\}. \quad (2.8)$$

(iv). Let Z be a component of $V^i(\omega_X)$ and $\alpha \in Z$ be a smooth point of $V^i(\omega_X)$. If $v \in H^1(X, \mathcal{O}_X) \simeq T_\alpha \text{Pic}^0(X)$ is not tangent to Z , then the following complex

$$H^{i-1}(X, \omega_X \otimes \alpha) \xrightarrow{\cup v} H^i(X, \omega_X \otimes \alpha) \xrightarrow{\cup v} H^{i+1}(X, \omega_X \otimes \alpha)$$

is exact. Moreover, if $\alpha \in Z$ is a general point and v is tangent to Z , then both maps of the above complex are zero.

(v). If \mathcal{P} is a Poincaré line bundle on $X \times \text{Pic}^0(X)$ and $\pi_2 : X \times \text{Pic}^0(X) \rightarrow \text{Pic}^0(X)$ is the projection onto the second factor, then for any $\alpha \in \text{Pic}^0(X)$

$$\text{codim}_\alpha V^i(\omega_X) \geq i - k \text{ for all } i > 0 \iff (R^i \pi_{2*} \mathcal{P})_\alpha = 0 \text{ for all } i < n - k.$$

(vi). If $\dim \text{alb}_X(X) = n - k$, then

$$V^k(\omega_X) \supset V^{k+1}(\omega_X) \supset \dots \supset V^n(\omega_X) = \{\mathcal{O}_X\}. \quad (2.9)$$

Proof. For the proof we refer to literature. Point (i), (ii) and (iv) are seminal results in generic vanishing theory mostly due to Green and Lazarsfeld. For point (i) we refer to (20) Theorem 0.1 and (35). We note that Green and Lazarsfeld prove that the $V_r^i(\omega_X)$ are unions of translates of abelian subvarieties, while Simpson shows that they are torsion translates. For the proof of point (ii) we refer to (10) Theorem 1, and for (iv) we refer to (10), (21) Theorem 1.2 and (36) Theorem 2.9. Point (iii) is due to Lazarsfeld and Popa and is proved in (16) Remark 2.4. For the formula (2.8) we refer to (4) p. 7. The proof of (v) is due to Hacon in the projective case (*cf.* (22) Theorem 4.1), and to Pareschi and Popa in the Kähler case (*cf.* (34) Theorem C). Finally, point (vi) is due to Pareschi and Popa (*cf.* (11) Proposition 3.14), and to Ein, Green and Lazarsfeld for the case $k = 0$ (*cf.* (21) Lemma 1.8). We remark that most of the conclusions of this theorem are also valid in the Kähler case. \square

We remark that point (ii) of the previous theorem is a generic vanishing type theorem for the canonical bundle. In particular it says that if $\dim \text{alb}_X(X) = n - k$, then $H^i(X, \omega_X \otimes \alpha) = 0$ for generic $\alpha \in \text{Pic}^0(X)$ and all $i > k$. Besides the canonical bundle, generic vanishing type theorems have also been proved for other special classes of sheaves, such as bundles of holomorphic p -forms, higher direct images of the canonical bundle and pluricanonical bundles. Before presenting the above-mentioned results, we introduce some more notation. In general, to any coherent sheaf \mathcal{F} on X and integers $i \geq 0$ and $r \geq 1$, we associate the cohomological support loci

$$V_r^i(\mathcal{F}) := \{\alpha \in \text{Pic}^0(X) \mid h^i(X, \mathcal{F} \otimes \alpha) \geq r\}. \quad (2.10)$$

As before the loci $V_r^i(\mathcal{F})$ are algebraic closed subsets and we denote by $V_r^i(\mathcal{F})_0$ the union of all irreducible components of $V_r^i(\mathcal{F})$ passing through the origin. Moreover, we set $V^i(\mathcal{F}) := V_1^i(\mathcal{F})$. Following Pareschi and Popa (*cf.* (11)) we say that \mathcal{F} is a *GV-sheaf* if $\text{codim}_{\text{Pic}^0(X)} V^i(\mathcal{F}) \geq i$ for all $i > 0$. Finally, we define the *defect of semismallness of the Albanese map* $\text{alb}_X : X \rightarrow \text{Alb}(X)$ as:

$$\delta(\text{alb}_X) := \max_{l \in \mathbf{N}} \{2l - \dim X + \dim \text{Alb}(X)_l\}$$

where $\text{Alb}(X)_l := \{y \in \text{Alb}(X) \mid \dim \text{alb}_X^{-1}(y) \geq l\}$.

Theorem 2.2.3. *Let X be a complex smooth projective variety of dimension n and let $\text{alb}_X : X \rightarrow \text{Alb}(X)$ be its Albanese map.*

(i). *For every $i, j \in \mathbf{N}$ we have $\text{codim}_{\text{Pic}^0(X)} V^i(\Omega_X^j) \geq |i + j - n| - \delta(\text{alb}_X)$. Moreover, there exist two indexes i and j such that the above inequality is an equality.*

(ii). *If the zero locus of a non-zero holomorphic one-form ω is of codimension $\geq k$, then the complex*

$$H^j(X, \Omega_X^{i-1}) \xrightarrow{\wedge \omega} H^j(X, \Omega_X^i) \xrightarrow{\wedge \omega} H^j(X, \Omega_X^{i+1})$$

is exact whenever $i + j < k$. Moreover, if there exists a non-zero holomorphic one-form ω such that $\text{codim}_X Z(\omega) \geq k$, then $H^j(X, \Omega_X^i \otimes \alpha) = 0$ for generic $\alpha \in \text{Pic}^0(X)$ and whenever $i + j < k$.

(iii). *Let $f : X \rightarrow Y$ be a surjective morphism with Y smooth projective and of maximal Albanese dimension. Then $\text{codim}_{\text{Pic}^0(Y)} V^i(R^j f_* \omega_X) \geq i$ for all $i > 0$ and $j \geq 0$.*

(iv). If X is minimal and of maximal Albanese dimension, then $\text{codim}_{\text{Pic}^0(X)} V^i(\omega_X^m) \geq i$ for all $i > 0$ and $m \geq 2$.

(v). If \mathcal{P} is a Poincaré line bundle on $X \times \text{Pic}^0(X)$ and π_1 and π_2 are the projections from $X \times \text{Pic}^0(X)$ onto the first and second factor respectively, then

$$\text{codim}_\alpha V^i(\Omega_X^j) \geq i + j - n - \delta(\text{alb}_X) \text{ for all } i > 0 \iff \quad (2.11)$$

$$(R^i \pi_{2*}(\pi_1^* \Omega_X^{n-j} \otimes \mathcal{P}))_\alpha = 0 \text{ for all } i < j - \delta(\text{alb}_X).$$

Proof. We refer to literature. Point (i) is due to Popa and Schnell and is proved in (23) Theorem 3.2. Point (ii) is due to Green and Lazarsfeld and is proved in (10) §3. Point (iii) is due to Pareschi and Popa (*cf.* (11) Theorem 5.8), and to Hacon in the case Y is an abelian variety (*cf.* (22) Corollary 4.2). Point (iv) is due to Pareschi and Popa and is proved in (11) Corollary 5.5 (we note that the minimality condition in (iv) is necessary as shown in (11) Example 5.6). Finally, point (v) is a consequence of point (i) and (34) Theorem 2.2. We remark that most of the results of this theorem can be proved in the Kähler setting as well, for instance point (ii). □

Cohomological support loci and fibrations. We now recall a relationship between cohomological support loci associated to the canonical bundle and fibrations over varieties of maximal

Albanese dimension. First, we introduce some more notation. Let $f : X \rightarrow C$ be a morphism with connected fibers onto a smooth curve C and general fiber F . We denote by

$$\mathrm{Pic}^0(X, f) := \mathrm{Ker} (\mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}^0(F)) \quad (2.12)$$

the kernel of the pull-back of the inclusion morphism $F \hookrightarrow X$ (cf. (37) p. 4).

Theorem 2.2.4. *Let X be a compact Kähler manifold of dimension n .*

(i). *If $T \subset V^i(\omega_X)$ is a positive-dimensional irreducible component with $i > 0$, then there exists a dominant analytic map $f : X \rightarrow Z$ with connected fibers such that: a) Z is a normal analytic variety with $0 < \dim Z \leq n - i$; b) $T \subset f^*\mathrm{Pic}^0(Z) + \gamma$ for some $\gamma \in \mathrm{Pic}^0(X)$; c) any smooth model of Z is of maximal Albanese dimension.*

(ii). *Let $\{f_i : X \rightarrow C_i\}$ be the set of all fibrations of X onto smooth curves C_i of genus $g(C_i) \geq 1$ for $i = 1, \dots, k$. Then $V^{n-1}(\omega_X)$ consists of a finite set of points, and of $\mathrm{Pic}^0(X, f_i)$ if $g(C_i) \geq 2$, and of $\mathrm{Pic}^0(X, f_i) \setminus f_i^*\mathrm{Pic}^0(C_i)$ if $g(C_i) = 1$.*

Proof. Point (i) is due to Green and Lazarsfeld and is proved in (20) Theorem 0.2. Point (ii) is due to Beauville and is proved in (37) Corollaire 2.3. □

CHAPTER 3

TWISTED HOCHSCHILD HOMOLOGY

The goal of this chapter is to prove the derived invariance of a twisted version of Hochschild homology taking into account the Rouquier isomorphism. Applications of this invariance will be discussed in §§4 and 5. In the following we freely use notation from §2.

Let X be a smooth projective variety of dimension n defined over an algebraically closed field. For any triple $(\varphi, \alpha, m) \in \text{Aut}^0(X) \times \text{Pic}^0(X) \times \mathbf{Z}$ we define the spaces $HH_*(X, \varphi, \alpha, m) := \bigoplus_i HH_i(X, \varphi, \alpha, m)$ where

$$HH_i(X, \varphi, \alpha, m) := \text{Ext}_{X \times X}^i(\Delta_{X*} \mathcal{O}_X, (1_X, \varphi)_*(\omega_X^m \otimes \alpha)),$$

Δ_X is the diagonal embedding, and $(1_X, \varphi)$ is the embedding $x \mapsto (x, \varphi(x))$. The spaces $HH_i(X, \varphi, \alpha, m)$ have a natural structure of graded $HH^*(X)$ -module given by composition of morphisms. We refer to them as *twisted versions of Hochschild homology*.

Before proving the compatibility of twisted Hochschild homology with the Rouquier isomorphism, we show a lemma extending previous computations of Căldăraru and Orlov (*cf.* (5) Proposition 8.1 and (6) isomorphism (10)).

Lemma 3.0.5. *Let X and Y be smooth projective varieties of dimension n defined over an algebraically closed field and let $\Phi_{\mathcal{E}} : \mathbf{D}(X) \simeq \mathbf{D}(Y)$ be an equivalence. Denote by $F = F_{\mathcal{E}}$ the induced Rouquier isomorphism and let $m \in \mathbf{Z}$. If $F(\varphi, \alpha) = (\psi, \beta)$, then*

$$\Phi_{\mathcal{E}_R \boxtimes \mathcal{E}}((1_X, \varphi)_*(\omega_X^m \otimes \alpha)) \simeq ((1_Y, \psi)_*(\omega_Y^m \otimes \beta)).$$

Proof. We denote by p_i and p_{ij} the projections from $X \times X \times Y \times Y$ onto the i -th and (i, j) -th factor respectively. Similarly, we denote by t_i and t_{ij} the projections from $Y \times X \times Y$. Moreover, we define the morphism $\lambda : Y \times X \times Y \rightarrow X \times X \times Y \times Y$ as $\lambda(y_1, x, y_2) = (x, \varphi(x), y_1, y_2)$, and we look at the following fiber product diagram

$$\begin{array}{ccc} Y \times X \times Y & \xrightarrow{\lambda} & X \times X \times Y \times Y \\ \downarrow t_2 & & \downarrow p_{12} \\ X & \xrightarrow{(1_X, \varphi)} & X \times X \end{array}$$

so that, by base change and the projection formula, we get

$$\begin{aligned} \Phi_{\mathcal{E}_R \boxtimes \mathcal{E}}((1_X, \varphi)_*(\omega_X^m \otimes \alpha)) &= p_{34*} \left(p_{12}^* (1_X, \varphi)_*(\omega_X^m \otimes \alpha) \otimes (\mathcal{E}_R \boxtimes \mathcal{E}) \right) & (3.1) \\ &\simeq p_{34*} \left(\lambda_* t_2^*(\omega_X^m \otimes \alpha) \otimes p_{13}^* \mathcal{E}_R \otimes p_{24}^* \mathcal{E} \right) \\ &\simeq p_{34*} \lambda_* \left(t_2^*(\omega_X^m \otimes \alpha) \otimes \lambda^* p_{13}^* \mathcal{E}_R \otimes \lambda^* p_{24}^* \mathcal{E} \right) \\ &\simeq t_{13*} \left(t_2^*(\omega_X^m \otimes \alpha) \otimes t_{21}^* \mathcal{E}_R \otimes t_{23}^*(\varphi \times 1_Y)^* \mathcal{E} \right) \end{aligned}$$

(where in the last isomorphism we used the fact that $p_{24} \circ \lambda = (\varphi \times 1_Y) \circ t_{23}$).

As noted in (2.1), the equivalence $\Phi_{\mathcal{E}}$ induces an isomorphism $\mathcal{E} \otimes p^* \omega_X \simeq \mathcal{E} \otimes q^* \omega_Y$ (where p and q are the projections from $X \times Y$ onto the first and second factor respectively), and by (9) Lemma 3.1 it induces further isomorphisms

$$(\varphi \times 1_Y)^* \mathcal{E} \otimes p^* \alpha \simeq (1_X \times \psi)_* \mathcal{E} \otimes q^* \beta \quad \text{whenever} \quad F(\varphi, \alpha) = (\psi, \beta). \quad (3.2)$$

Therefore we have $p^*(\omega_X^m \otimes \alpha) \otimes (\varphi \times 1_Y)^* \mathcal{E} \simeq q^*(\omega_Y^m \otimes \beta) \otimes (1_X \times \psi)_* \mathcal{E}$, and by pulling back via $t_{23} : Y \times X \times Y \rightarrow X \times Y$ we get

$$t_2^*(\omega_X^m \otimes \alpha) \otimes t_{23}^*(\varphi \times 1_Y)^* \mathcal{E} \simeq t_3^*(\omega_Y^m \otimes \beta) \otimes t_{23}^*(1_X \times \psi)_* \mathcal{E}. \quad (3.3)$$

At this point we rewrite the morphism $t_3 : Y \times X \times Y \rightarrow Y$ as $t_3 = \sigma_2 \circ t_{13}$ where $\sigma_2 : Y \times Y \rightarrow Y$ is the projection onto the second factor. Moreover, we denote by $\rho : Y \times X \rightarrow X \times Y$ the inversion morphism $(y, x) \mapsto (x, y)$. Then by (3.1) and (3.3) we obtain

$$\begin{aligned} \Phi_{\mathcal{E}_R \boxtimes \mathcal{E}} \left((1_X, \varphi)_* (\omega_X^m \otimes \alpha) \right) &\simeq t_{13*} \left(t_2^*(\omega_X^m \otimes \alpha) \otimes t_{21}^* \mathcal{E}_R \otimes t_{23}^*(\varphi \times 1_Y)^* \mathcal{E} \right) \\ &\simeq t_{13*} \left(t_3^*(\omega_Y^m \otimes \beta) \otimes t_{21}^* \mathcal{E}_R \otimes t_{23}^*(1_X \times \psi)_* \mathcal{E} \right) \\ &\simeq t_{13*} \left(t_{13}^* \sigma_2^*(\omega_Y^m \otimes \beta) \otimes t_{21}^* \mathcal{E}_R \otimes t_{23}^*(1_X \times \psi)_* \mathcal{E} \right) \\ &\simeq \sigma_2^*(\omega_Y^m \otimes \beta) \otimes t_{13*} \left(t_{21}^* \mathcal{E}_R \otimes t_{23}^*(1_X \times \psi)_* \mathcal{E} \right) \\ &\simeq \sigma_2^*(\omega_Y^m \otimes \beta) \otimes t_{13*} \left(t_{12}^* \rho^* \mathcal{E}_R \otimes t_{23}^*(1_X \times \psi)_* \mathcal{E} \right). \end{aligned}$$

Finally, by (6) Proposition 2.1.2 or (19) Proposition 5.10, we note that the object $t_{13*}\left(t_{12}^*\rho^*\mathcal{E}_R \otimes t_{23}^*(1_X \times \psi)_*\mathcal{E}\right)$ is the kernel of the composition

$$\Phi_{(1_X \times \psi)_*\mathcal{E}} \circ \Phi_{\rho^*\mathcal{E}_R} \simeq \psi_* \circ \Phi_{\mathcal{E}} \circ \Psi_{\mathcal{E}_R} \simeq \psi_* \circ 1_{\mathbf{D}(Y)} \simeq \psi_*.$$

On the other hand, since the kernel of the derived functor $\psi_* : \mathbf{D}(Y) \rightarrow \mathbf{D}(Y)$ is the structure sheaf of the graph of ψ , i.e. $\mathcal{O}_{\Gamma_\psi} \simeq (1_Y, \psi)_*\mathcal{O}_Y$ (cf. (19) Example 5.4), we have an isomorphism $t_{13*}\left(t_{12}^*\rho^*\mathcal{E}_R \otimes t_{23}^*(1_X \times \psi)_*\mathcal{E}\right) \simeq (1_Y, \psi)_*\mathcal{O}_Y$ induced by the uniqueness of the Fourier-Mukai kernel. To recap

$$\begin{aligned} \Phi_{\mathcal{E}_R \boxtimes \mathcal{E}}\left((1_X, \varphi)_*(\omega_X^m \otimes \alpha)\right) &\simeq \sigma_2^*(\omega_Y^m \otimes \beta) \otimes t_{13*}\left(t_{12}^*\rho^*\mathcal{E}_R \otimes t_{23}^*(1_X \times \psi)_*\mathcal{E}\right) \\ &\simeq \sigma_2^*(\omega_Y^m \otimes \beta) \otimes (1_Y, \psi)_*\mathcal{O}_Y \\ &\simeq (1_Y, \psi)_*\left((1_Y, \psi)^*\sigma_2^*(\omega_Y^m \otimes \beta)\right) \\ &\simeq (1_Y, \psi)_*\left(\psi^*(\omega_Y^m \otimes \beta)\right) \\ &\simeq (1_Y, \psi)_*(\omega_Y^m \otimes \beta) \end{aligned}$$

where the last isomorphism follows as the action of $\text{Aut}^0(X)$ on $\text{Pic}^0(X)$ is trivial (cf. (9) footnote at p. 531). □

We are now ready to prove the invariance of twisted Hochschild homology under derived equivalence.

Theorem 3.0.6. *Let X and Y be smooth projective varieties defined over an algebraically closed field and let $\Phi_{\mathcal{E}} : \mathbf{D}(X) \simeq \mathbf{D}(Y)$ be an equivalence. Denote by $F_{\mathcal{E}}$ the induced Rouquier isomorphism and let $m \in \mathbf{Z}$. If $F_{\mathcal{E}}(\varphi, \alpha) = (\psi, \beta)$, then $\Phi_{\mathcal{E}_R \boxtimes \mathcal{E}}$ induces an isomorphism of graded modules $HH_*(X, \varphi, \alpha, m) \simeq HH_*(Y, \psi, \beta, m)$ compatible with the isomorphism $HH^*(X) \simeq HH^*(Y)$.*

Proof. By Lemma 3.0.5, the equivalence $\Phi_{\mathcal{E}_R \boxtimes \mathcal{E}}$ induces isomorphisms on the graded components of $HH_*(X, \varphi, \alpha, m)$ and $HH_*(Y, \psi, \beta, m)$:

$$\begin{aligned} \mathrm{Ext}_{X \times X}^i(\Delta_{X*} \mathcal{O}_X, (1_X, \varphi)_*(\omega_X^m \otimes \alpha)) &\simeq \mathrm{Ext}_{Y \times Y}^i(\Phi_{\mathcal{E}_R \boxtimes \mathcal{E}}(\Delta_{X*} \mathcal{O}_X), \Phi_{\mathcal{E}_R \boxtimes \mathcal{E}}((1_X, \varphi)_*(\omega_X^m \otimes \alpha))) \\ &\simeq \mathrm{Ext}_{Y \times Y}^i(\Delta_{Y*} \mathcal{O}_Y, (1_Y, \psi)_*(\omega_Y^m \otimes \beta)). \end{aligned}$$

Since $\Phi_{\mathcal{E}_R \boxtimes \mathcal{E}}$ is a functor, it follows in particular that $\Phi_{\mathcal{E}_R \boxtimes \mathcal{E}}$ induces an isomorphism of graded modules. □

If the characteristic of the ground field is zero, then Theorem 3.0.6 has a nice consequence when both φ and ψ are the identity automorphisms. In fact, in this case, we can rewrite the isomorphism $HH_i(X, 1_X, \alpha, m) \simeq HH_i(Y, 1_Y, \beta, m)$ in terms of sheaf cohomology thanks to a theorem of Yekutieli extending the Hochschild-Kostant-Rosenberg isomorphism (2.3). We recall here Yekutieli's result for the benefit of the reader (*cf.* (8) Corollary 4.7 or (27) Corollary 2.6 for a proof over the field of complex numbers).

Theorem 3.0.7. *Let X be a smooth projective variety of dimension n defined over an algebraically closed field of characteristic zero. If \mathcal{F} is an \mathcal{O}_X -module, then*

$$\mathrm{Ext}_{X \times X}^i(\Delta_{X*}\mathcal{O}_X, \Delta_{X*}\mathcal{F}) \simeq \bigoplus_{q=0}^i H^{i-q}(X, \Omega_X^{n-q} \otimes \omega_X^{-1} \otimes \mathcal{F}) \quad \text{for any } i \geq 0.$$

Theorem 3.0.6 will be often used in the following weaker form:

Corollary 3.0.8. *Let X and Y be smooth projective D -equivalent varieties of dimension n defined over an algebraically closed field K of characteristic zero. If F denotes the induced Rouquier isomorphism and if $F(1_X, \alpha) = (1_Y, \beta)$, then for any integers m and $i \geq 0$ there exist isomorphisms*

$$\bigoplus_{q=0}^i H^{i-q}(X, \Omega_X^{n-q} \otimes \omega_X^m \otimes \alpha) \simeq \bigoplus_{q=0}^i H^{i-q}(Y, \Omega_Y^{n-q} \otimes \omega_Y^m \otimes \beta). \quad (3.4)$$

In particular we have $H^0(X, \omega_X^m \otimes \alpha) \simeq H^0(Y, \omega_Y^m \otimes \beta)$. Moreover, if $K = \mathbf{C}$ is the field of complex numbers, then $h^1(X, \omega_X \otimes \alpha) = h^1(Y, \omega_Y \otimes \beta)$. Finally, if in addition $n = 3$, then we also get $h^2(X, \omega_X \otimes \alpha) = h^2(Y, \omega_Y \otimes \beta)$.

Proof. The first statement is a direct consequence of Theorems 3.0.6 and 3.0.7. The second statement follows from the first by setting $i = 0$. To prove the third statement we recall a special case of Serre duality $H^p(X, \Omega_X^q \otimes \gamma) \simeq H^{n-p}(X, \Omega_X^{n-q} \otimes \gamma^{-1})^\vee$, and the Hodge linear-

conjugate isomorphism $H^p(X, \Omega_X^q \otimes \gamma) \simeq \overline{H^q(X, \Omega_X^p \otimes \gamma^{-1})}$ where $\gamma \in \text{Pic}^0(X)$. Therefore we have

$$h^0(X, \Omega_X^{n-1} \otimes \alpha) = h^1(X, \omega_X \otimes \alpha) \quad \text{and} \quad h^0(Y, \Omega_Y^{n-1} \otimes \beta) = h^1(Y, \omega_Y \otimes \beta).$$

Moreover, thanks to the isomorphisms (3.4) for $i = 1$, we have

$$H^1(X, \omega_X \otimes \alpha) \oplus H^0(X, \Omega_X^{n-1} \otimes \alpha) \simeq H^1(Y, \omega_Y \otimes \beta) \oplus H^0(Y, \Omega_Y^{n-1} \otimes \beta)$$

which immediately leads to $h^1(X, \omega_X \otimes \alpha) = h^1(Y, \omega_Y \otimes \beta)$.

To prove the last statement we use the derived invariance of the holomorphic Euler characteristic in dimension three (*cf.* Theorem 2.1.10), and the fact that $\chi(\omega_X)$ is invariant under deformations, i.e. $\chi(\omega_X) = \chi(\omega_X \otimes \gamma)$ for all $\gamma \in \text{Pic}^0(X)$. Since $h^3(X, \omega_X \otimes \alpha) = h^0(X, \alpha^{-1})$ is either zero if $\alpha \neq \mathcal{O}_X$, or one if α is trivial, we have that $h^3(X, \omega_X \otimes \alpha) = h^3(Y, \omega_Y \otimes \beta)$ whenever $F(1_X, \alpha) = (1_Y, \beta)$ (this is in fact true in arbitrary dimension). At this point we conclude thanks to the equalities $\chi(\omega_X \otimes \alpha) = \chi(\omega_X) = \chi(\omega_Y) = \chi(\omega_Y \otimes \beta)$, $h^0(X, \omega_X \otimes \alpha) = h^0(Y, \omega_Y \otimes \beta)$ and $h^1(X, \omega_X \otimes \alpha) = h^1(Y, \omega_Y \otimes \beta)$. \square

Furthermore, under the assumptions of Corollary 3.0.8, $\Phi_{\mathcal{E}_R \boxtimes \mathcal{E}}$ induces isomorphisms between the twisted (anti-)canonical rings of X and Y :

$$\begin{aligned} \bigoplus_{m \geq 0} H^0(X, \omega_X^m \otimes \alpha) &\simeq \bigoplus_{m \geq 0} H^0(Y, \omega_Y^m \otimes \beta) \\ \bigoplus_{m \leq 0} H^0(X, \omega_X^m \otimes \alpha) &\simeq \bigoplus_{m \leq 0} H^0(Y, \omega_Y^m \otimes \beta). \end{aligned}$$

CHAPTER 4

DERIVED INVARIANTS OF IRREGULAR VARIETIES

In this chapter we apply the derived invariance of twisted Hochschild homology established in Theorem 3.0.6 to study a conjecture of Popa concerning the behavior of cohomological support loci $V^i(\omega_X) := \{\alpha \in \text{Pic}^0(X) \mid H^i(X, \omega_X \otimes \alpha) \neq 0\}$ under derived equivalence. In particular, we will see that the derived invariance of twisted Hochschild homology naturally leads to the derived invariance of $V^0(\omega_X)$ (*cf.* Proposition 4.2.1), and furthermore to the derived invariance of $V^1(\omega_X)_0$ thanks to Brion's results on actions of non-affine groups on smooth projective varieties (*cf.* Proposition 4.4.1).

Before turning to applications, we establish the derived invariance of the Albanese dimension for varieties having either non-negative Kodaira dimension or nef anticanonical bundle (*cf.* Theorem 4.5.2 and Remark 4.5.3). Finally, as an application of the above-mentioned results, in §4.6 we study the behavior of other numerical quantities under derived equivalence, such as the holomorphic Euler characteristic and the Hodge numbers of special classes of irregular varieties.

In this chapter we work over the field of complex numbers.

4.1 A conjecture of Popa

Let X be a complex smooth projective variety. We start by recalling the definition of cohomological support loci associated to the canonical bundle

$$V_r^i(\omega_X) := \{\alpha \in \text{Pic}^0(X) \mid h^i(X, \omega_X \otimes \alpha) \geq r\}$$

where $i \geq 0$ and $r \geq 1$ are integers. Moreover, we denote by $V_r^i(\omega_X)_0$ the union of all irreducible components of $V_r^i(\omega_X)$ passing through the origin and we set $V^i(\omega_X) := V_1^i(\omega_X)$. We also recall that one can define cohomological support loci associated to an arbitrary coherent sheaf (cf. (2.10)).

The behavior of cohomological support loci associated to the canonical bundle under derived equivalence is predicted by Popa in (4):

Conjecture 4.1.1. *Let X and Y be complex smooth projective D -equivalent varieties. Then for all $i \geq 0$ there exist isomorphisms $V^i(\omega_X)_0 \simeq V^i(\omega_Y)_0$.*

We also consider a variant of the previous conjecture:

Variante 4.1.2. *Let X and Y be complex smooth projective D -equivalent varieties. Then for all $i \geq 0$ there exist isomorphisms $V^i(\omega_X) \simeq V^i(\omega_Y)$.*

Since many properties of a variety are in fact captured by the “smaller” loci $V^i(\omega_X)_0$, for the applications we are interested in, it will be sufficient to study Conjecture 4.1.1 rather than Variante 4.1.2. We recall that Variante 4.1.2 (and therefore Conjecture 4.1.1) holds in dimension

up to two (cf. (4)) and for varieties of general type. To see this last point we note that the loci $V^i(\omega_X)$ are birationally invariant while D -equivalent varieties of general type are K -equivalent, and therefore birational (cf. Theorem 2.1.12). Moreover, the case $i = \dim X$ of Variant 4.1.2 is easy to verify as Serre duality yields $V^{\dim X}(\omega_X) = \{\mathcal{O}_X\}$ and $V^{\dim Y}(\omega_Y) = \{\mathcal{O}_Y\}$.

4.2 Derived invariance of the zero-th cohomological support locus

In this section we prove the derived invariance of loci of the form $V_r^0(\omega_X^m)$ for any integers m and $r \geq 1$.

Proposition 4.2.1. *Let X and Y be smooth projective varieties and $\Phi_{\mathcal{E}} : \mathbf{D}(X) \simeq \mathbf{D}(Y)$ be an equivalence. Let $F_{\mathcal{E}}$ be the induced Rouquier isomorphism and let $m, r \in \mathbf{Z}$ with $r \geq 1$. If $\alpha \in V_r^0(\omega_X^m)$ and $F_{\mathcal{E}}(1_X, \alpha) = (\psi, \beta)$, then we have $\psi = 1_Y$, $\beta \in V_r^0(\omega_Y^m)$ and $H^0(X, \omega_X^m \otimes \alpha) \simeq H^0(Y, \omega_Y^m \otimes \beta)$. Moreover, $F_{\mathcal{E}}$ induces an isomorphism of algebraic sets $V_r^0(\omega_X^m) \simeq V_r^0(\omega_Y^m)$.*

Proof. Let $\alpha \in V_r^0(\omega_X^m)$ be an arbitrary point. By Theorem 3.0.6 and adjunction formula we have

$$\begin{aligned} r \leq h^0(X, \omega_X^m \otimes \alpha) &= \dim \operatorname{Hom}_{X \times X}(\Delta_{X*} \mathcal{O}_X, \Delta_{X*}(\omega_X^m \otimes \alpha)) \\ &= \dim \operatorname{Hom}_{Y \times Y}(\Delta_{Y*} \mathcal{O}_Y, (1_Y, \psi)_*(\omega_Y^m \otimes \beta)) \\ &= \dim \operatorname{Hom}_Y((1_Y, \psi)^* \Delta_{Y*} \mathcal{O}_Y, \omega_Y^m \otimes \beta). \end{aligned}$$

Since $(1_Y, \psi)^* \Delta_{Y*} \mathcal{O}_Y$ is supported on the locus of fixed points of ψ (which is of codimension ≥ 1 if $\psi \neq 1_Y$), and since there are no non-zero morphisms from a torsion sheaf to a locally free

sheaf, we must have $\psi = 1_Y$ and $\beta \in V_r^0(\omega_Y^m)$. Thus we have $F_{\mathcal{E}}(1_X, V_r^0(\omega_X^m)) \subset (1_Y, V_r^0(\omega_Y^m))$ as α is arbitrary in $V_r^0(\omega_X^m)$.

In order to show the reverse inclusion, we consider the right adjoint $\Psi_{\mathcal{E}_R}$ to $\Phi_{\mathcal{E}}$ so that $\Psi_{\mathcal{E}_R} \circ \Phi_{\mathcal{E}} \simeq 1_{\mathbf{D}(X)}$ and $\Phi_{\mathcal{E}} \circ \Psi_{\mathcal{E}_R} \simeq 1_{\mathbf{D}(Y)}$. We note that, by Lemma 2.1.9, the Rouquier isomorphism induced by $\Psi_{\mathcal{E}_R}$ is $F_{\mathcal{E}}^{-1}$. Hence, by repeating the previous argument, we get the inclusion $F_{\mathcal{E}}^{-1}(1_Y, V_r^0(\omega_Y^m)) \subset (1_X, V_r^0(\omega_X^m))$ inducing the desired isomorphism. \square

4.3 On the image of the Rouquier isomorphism

In this section we provide sufficient conditions on a line bundle $\alpha \in \text{Pic}^0(X)$ for which the Rouquier isomorphism $F : \text{Aut}^0(X) \times \text{Pic}^0(X) \rightarrow \text{Aut}^0(Y) \times \text{Pic}^0(Y)$ maps the pair $(1_X, \alpha)$ to a pair $(1_Y, \beta)$ for some $\beta \in \text{Pic}^0(Y)$. For instance, we have seen in Proposition 4.2.1 that this is the case as soon as $\alpha \in V^0(\omega_X^m)$. This problem turns out to be strictly related to the geometry of the algebraic group $\text{Aut}^0(X)$.

Lemma 4.3.1. *Let X and Y be smooth projective D -equivalent varieties and let F be the induced Rouquier isomorphism. If $\text{Aut}^0(X)$ is affine, then $F(1_X, \text{Pic}^0(X)) = (1_Y, \text{Pic}^0(Y))$.*

Proof. We define the morphism $\pi_A : \text{Pic}^0(Y) \rightarrow \text{Aut}^0(X)$ as $\pi_A(\beta) = \text{pr}_1(F^{-1}(1, \beta))$ (here pr_1 denotes the projection onto the first factor from $\text{Aut}^0(X) \times \text{Pic}^0(X)$). Moreover, we set $A := \text{Im } \pi_A$ so that, in order to show the lemma, it would be enough to prove that A is trivial. But this is the case as A is an abelian variety in $\text{Aut}^0(X)$ which is an affine group by hypotheses.

\square

In case $\text{Aut}^0(X)$ is not affine, then the abelian variety $A = \text{Im } \pi_A$ defined in the proof of Lemma 4.3.1 might be of positive dimension. If this is the case, then X admits a very special structure thanks to a theorem of Brion concerning actions of non-affine groups on smooth projective varieties (*cf.* (7)).

Lemma 4.3.2. *Let X be a smooth projective variety and A be a positive-dimensional abelian subvariety of $\text{Aut}^0(X)$. Then there exists a finite subgroup $H \subset A$ and an A -equivariant morphism $\xi : X \rightarrow A/H$ inducing the following fiber-product diagram*

$$\begin{array}{ccc} A \times Z & \xrightarrow{\gamma} & X \\ \downarrow & & \downarrow \xi \\ A & \xrightarrow{\bar{\gamma}} & A/H, \end{array} \quad (4.1)$$

where Z is the smooth and connected fiber of ξ over the origin of A/H , γ is the restriction of the action of A on X to Z , and $\bar{\gamma}$ is the quotient map. In particular $\chi(\mathcal{O}_X) = 0$.

Proof. There are different proofs of this fact. It was first proved by Brion in (7) §3. Other proofs appear in (9) Lemmas 2.4, 2.5, and in (38) Proposition 34. Finally, the last statement “ $\chi(\mathcal{O}_X) = 0$ ” is proved in (9) Corollary 2.6. \square

Lemma 4.3.3. *Let X and Y be smooth projective D -equivalent varieties and F be the induced Rouquier isomorphism. If $\alpha \in V^i(\Omega_X^p \otimes \omega_X^m)_0$ for some integers i, m, p with $i, p \geq 0$ and $F(1_X, \alpha) = (\psi, \beta)$, then $\psi = 1_Y$.*

Proof. Let $A := \text{Im } \pi_A$ be as in the proof of Lemma 4.3.1. If A is trivial then the lemma follows from Lemma 4.3.1. We assume then $\dim A > 0$ so that, by Lemma 4.3.2, we have a commutative diagram as in (4.1). Let $z_0 \in Z$ be an arbitrary point and let $f_1 : A \rightarrow X$ be the orbit map $\varphi \mapsto \varphi(z_0)$. In (9) p. 533 it is shown that

$$\alpha \in (\text{Ker } f_1^*)_0 \implies F(1_X, \alpha) = (1_Y, \beta) \quad \text{for some } \beta \in \text{Pic}^0(Y)$$

(here $(\text{Ker } f_1^*)_0$ denotes the neutral component of $\text{Ker } f_1^*$). So it is enough to show that $\alpha \in (\text{Ker } f_1^*)_0$. This is achieved by computing cohomology spaces on $A \times Z$ via the étale morphism γ and by using the fact that these computations are straightforward on $A \times Z$ thanks to Künneth's formula.

Let p_1 and p_2 be the projections from $A \times Z$ onto the first and second factor respectively. By denoting by $\nu : A \times \{z_0\} \hookrightarrow A \times Z$ the inclusion morphism, we have $\gamma \circ \nu = f_1$. Moreover, via the isomorphism $\text{Pic}^0(A \times Z) \simeq \text{Pic}^0(A) \times \text{Pic}^0(Z)$ we obtain $\gamma^* \alpha \simeq p_1^* \alpha_1 \otimes p_2^* \alpha_2$ where $\alpha_1 \in \text{Pic}^0(A)$ and $\alpha_2 \in \text{Pic}^0(Z)$. Note also that $f_1^* \alpha \simeq \nu^* \gamma^* \alpha \simeq \alpha_1$. At this point, by (39) Lemma 4.1.14, we have

$$H^i(A \times Z, \omega_{A \times Z}^m \otimes \Omega_{A \times Z}^p \otimes \gamma^* \alpha) \supset H^i(X, \omega_X^m \otimes \Omega_X^p \otimes \alpha) \neq 0, \quad (4.2)$$

and if l denotes the dimension of A , then there is a decomposition

$$\Omega_{A \times Z}^p \simeq \bigoplus_{e_1 + e_2 = p} (p_1^* \Omega_A^{e_1} \otimes p_2^* \Omega_Z^{e_2}) \simeq \bigoplus_{e_1 + e_2 = p} (p_2^* \Omega_Z^{e_2})^{\oplus \binom{l}{e_1}}.$$

Therefore by Künneth's formula we obtain

$$\begin{aligned}
0 &\neq h^i(A \times Z, \Omega_{A \times Z}^p \otimes \omega_{A \times Z}^m \otimes p_1^* f_1^* \alpha \otimes p_2^* \alpha_2) = \\
h^i(A \times Z, (\bigoplus_{e_1+e_2=p} (p_2^* \Omega_Z^{e_2})^{\oplus \binom{l}{e_1}}) \otimes \omega_{A \times Z}^m \otimes p_1^* f_1^* \alpha \otimes p_2^* \alpha_2) &= \\
h^i(A \times Z, \bigoplus_{e_1+e_2=p} (p_1^* f_1^* \alpha \otimes p_2^* (\Omega_Z^{e_2} \otimes \omega_Z^m \otimes \alpha_2))^{\oplus \binom{l}{e_1}}) &= \\
\sum_{e_1+e_2=p} \binom{l}{e_1} h^i(A \times Z, p_1^* f_1^* \alpha \otimes p_2^* (\Omega_Z^{e_2} \otimes \omega_Z^m \otimes \alpha_2)) &= \\
\sum_{e_1+e_2=p} \sum_{e_3+e_4=i} \binom{l}{e_1} h^{e_3}(A, f_1^* \alpha) \cdot h^{e_4}(Z, \Omega_Z^{e_2} \otimes \omega_Z^m \otimes \alpha_2). &
\end{aligned}$$

The last sum is non-zero only if $f_1^* \alpha \simeq \mathcal{O}_A$, i.e. $\alpha \in \text{Ker } f_1^*$. This shows that $V^i(\Omega_X^p \otimes \omega_X^m) \subset \text{Ker } f_1^*$ and hence that $V^i(\Omega_X^p \otimes \omega_X^m)_0 \subset (\text{Ker } f_1^*)_0$.

□

Corollary 4.3.4. *Under the assumptions of Lemma 4.3.2 we also have $\chi(\Omega_X^p \otimes \omega_X^m) = 0$ for all integers m and p with $p \geq 0$.*

Proof. Fix m and p . While proving Lemma 4.3.3 we noted the inclusions $V^i(\Omega_X^p \otimes \omega_X^m) \subset \text{Ker } f_1^*$ for all $i \geq 0$. Moreover, $\text{Ker } f_1^* \subsetneq \text{Pic}^0(X)$ since $\dim A > 0$ and the pull-back $f_1^* : \text{Pic}^0(X) \rightarrow \text{Pic}^0(A)$ introduced in the proof of Lemma 4.3.3 is surjective by a theorem of Nishi-Matsumura (cf. (30) or (9) Theorem 2.3). Therefore we have $V^i(\Omega_X^p \otimes \omega_X^m) \subsetneq \text{Pic}^0(X)$ for all $i \geq 0$.

Finally, since the holomorphic Euler characteristic is invariant under deformations, we have $\chi(\Omega_X^p \otimes \omega_X^m) = \chi(\Omega_X^p \otimes \omega_X^m \otimes \alpha)$ for any $\alpha \in \text{Pic}^0(X)$. Therefore the corollary follows as soon as we pick $\alpha \notin \bigcup_{i \geq 0} V^i(\Omega_X^p \otimes \omega_X^m)$. \square

We remark that a further case where the Rouquier isomorphism maps a pair $(1_X, \alpha)$ with $\alpha \in V^1(\omega_X)$ to a pair $(1_Y, \beta)$ with $\beta \in \text{Pic}^0(Y)$ occurs when X is of maximal Albanese dimension (*cf.* Corollary 4.5.4).

4.4 Derived invariance of higher cohomological support loci

The invariance of twisted Hochschild homology, in combination with the results of §4.3 about the image of the Rouquier isomorphism, yields the derived invariance of $V^1(\omega_X)_0$.

Proposition 4.4.1. *Let X and Y be smooth projective varieties and $\Phi_{\mathcal{E}} : \mathbf{D}(X) \simeq \mathbf{D}(Y)$ be an equivalence. Let $F_{\mathcal{E}}$ be the induced Rouquier isomorphism and $r \geq 1$ be an integer. If $\alpha \in V_r^1(\omega_X)_0$ and $F_{\mathcal{E}}(1_X, \alpha) = (\psi, \beta)$, then we have $\psi = 1_Y$, $\beta \in V_r^1(\omega_Y)_0$ and $h^1(X, \omega_X \otimes \alpha) = h^1(Y, \omega_Y \otimes \beta)$. Moreover, $F_{\mathcal{E}}$ induces isomorphisms of algebraic sets $V_r^1(\omega_X)_0 \simeq V_r^1(\omega_Y)_0$ for any $r \geq 1$.*

Proof. Let α be a point in $V_r^1(\omega_X)_0$ so that, by Lemma 4.3.3, we have $F_{\mathcal{E}}(1_X, \alpha) = (1_Y, \beta)$ for some $\beta \in \text{Pic}^0(Y)$. Then, by Corollary 3.0.8, we have $r \leq h^1(X, \omega_X \otimes \alpha) = h^1(Y, \omega_Y \otimes \beta)$ and therefore $\beta \in V_r^1(\omega_Y)$. Moreover, since β is the image of an element in $V_r^1(\omega_X)_0$, in particular we have $\beta \in V_r^1(\omega_Y)_0$. Finally, since the previous argument can be repeated for an arbitrary element $\alpha \in V_r^1(\omega_X)_0$, we have the inclusions $F_{\mathcal{E}}(1_X, V_r^1(\omega_X)_0) \subset (1_Y, V_r^1(\omega_Y)_0)$ for all $r \geq 1$.

In order to show the reverse inclusions, we consider the right adjoint $\Psi_{\mathcal{E}_R}$ to $\Phi_{\mathcal{E}}$ so that $\Psi_{\mathcal{E}_R} \circ \Phi_{\mathcal{E}} \simeq 1_{\mathbf{D}(X)}$ and $\Phi_{\mathcal{E}} \circ \Psi_{\mathcal{E}_R} \simeq 1_{\mathbf{D}(Y)}$. At this point we note that, by Lemma 2.1.9, the Rouquier isomorphism induced by $\Psi_{\mathcal{E}_R}$ is $F_{\mathcal{E}}^{-1}$, and therefore that it is possible to repeat the previous argument to get the desired inclusions $F_{\mathcal{E}}^{-1}(1_Y, V_r^1(\omega_Y)_0) \subset (1_X, V_r^1(\omega_X)_0)$. \square

The proof of the previous proposition naturally extends to cohomological support loci associated to bundles of holomorphic p -forms Ω_X^p .

Proposition 4.4.2. *Let X and Y be smooth projective varieties of dimension n , $\Phi_{\mathcal{E}} : \mathbf{D}(X) \simeq \mathbf{D}(Y)$ be an equivalence, and $F_{\mathcal{E}}$ be the induced Rouquier isomorphism. Then, for any integers m and $i \geq 0$, $F_{\mathcal{E}}$ induces isomorphisms of algebraic sets*

$$\bigcup_{q=0}^i V^{i-q}(\Omega_X^{n-q} \otimes \omega_X^m)_0 \simeq \bigcup_{q=0}^i V^{i-q}(\Omega_Y^{n-q} \otimes \omega_Y^m)_0.$$

Proof. To begin with we recall some notation and results from (9). Let $\pi_A : \text{Pic}^0(Y) \rightarrow \text{Aut}^0(X)$ and $\pi_B : \text{Pic}^0(X) \rightarrow \text{Aut}^0(Y)$ be morphisms defined as

$$\pi_A(\beta) = \text{pr}_1(F_{\mathcal{E}}^{-1}(1_Y, \beta)), \quad \pi_B(\alpha) = \text{pr}_1(F_{\mathcal{E}}(1_X, \alpha))$$

(here pr_1 denotes the projection onto the first factor from $\text{Aut}^0(-) \times \text{Pic}^0(-)$). Moreover we set $A := \text{Im } \pi_A$ and $B := \text{Im } \pi_B$. We recall that A and B are isogenous abelian varieties (cf. (9) p. 533).

We consider first the case when A is trivial. Then, by Lemma 4.3.1, we have $F_{\mathcal{E}}(1_X, \text{Pic}^0(X)) = (1_Y, \text{Pic}^0(Y))$, and hence by Corollary 3.0.8 we get inclusions

$$F_{\mathcal{E}}(1_X, \bigcup_q V^{i-q}(\Omega_X^{n-q} \otimes \omega_X^m)) \subset (1_Y, \bigcup_q V^{i-q}(\Omega_Y^{n-q} \otimes \omega_Y^m)) \quad \text{for any } i \geq 0.$$

In order to prove the reverse inclusions, we note that B is trivial as well and that the Rouquier isomorphism induced by the right adjoint $\Psi_{\mathcal{E}_R}$ to $\Phi_{\mathcal{E}}$ is $F_{\mathcal{E}}^{-1}$ (cf. Lemma 2.1.9). Therefore a second application of Corollary 3.0.8 yields

$$F_{\mathcal{E}}^{-1}(1_Y, \bigcup_q V^{i-q}(\Omega_Y^{n-q} \otimes \omega_Y^m)) \subset (1_X, \bigcup_q V^{i-q}(\Omega_X^{n-q} \otimes \omega_X^m)) \quad \text{for any } i \geq 0.$$

We suppose now that both A and B are non trivial. Then by Lemma 4.3.3

$$F_{\mathcal{E}}(1_X, \bigcup_q V^{i-q}(\Omega_X^{n-q} \otimes \omega_X^m)_0) \subset (1_Y, \text{Pic}^0(Y)) \quad \text{for any } i \geq 0$$

and therefore, by Corollary 3.0.8, the Rouquier isomorphism maps

$$1_X \times \bigcup_q V^{i-q}(\Omega_X^{n-q} \otimes \omega_X^m)_0 \mapsto 1_Y \times \bigcup_q V^{i-q}(\Omega_Y^{n-q} \otimes \omega_Y^m)_0.$$

Similarly, since B is not trivial, Lemma 4.3.3 also yields

$$\beta \in \bigcup_q V^{i-q}(\Omega_Y^{n-q} \otimes \omega_Y^m)_0 \implies F_{\mathcal{E}}^{-1}(1_Y, \beta) = (1_X, \alpha) \text{ for some } \alpha \in \text{Pic}^0(X).$$

This concludes the proof as, by Corollary 3.0.8, for any $i \geq 0$, $F_{\mathcal{E}}^{-1}$ maps

$$1_Y \times \bigcup_q V^{i-q}(\Omega_Y^{n-q} \otimes \omega_Y^m)_0 \mapsto 1_X \times \bigcup_q V^{i-q}(\Omega_X^{n-q} \otimes \omega_X^m)_0.$$

□

Corollary 4.4.3. *Under the assumptions of Theorem 4.4.2, the Rouquier isomorphism $F_{\mathcal{E}}$ induces isomorphisms of algebraic sets for any integers l, m, r, s with $r, s \geq 1$*

$$\begin{aligned} V_r^0(\omega_X^m) \cap \left(\bigcup_q V^{i-q}(\Omega_X^{n-q} \otimes \omega_X^l) \right) &\simeq V_r^0(\omega_Y^m) \cap \left(\bigcup_q V^{i-q}(\Omega_Y^{n-q} \otimes \omega_Y^l) \right) \\ V_r^0(\omega_X^m) \cap V_s^1(\omega_X) &\simeq V_r^0(\omega_Y^m) \cap V_s^1(\omega_Y). \end{aligned}$$

Proof. In Proposition 4.2.1 we have seen that if $\alpha \in V_r^0(\omega_X^m)$, then $F_{\mathcal{E}}(1_X, \alpha) = (1_Y, \beta)$ for some $\beta \in V_r^0(\omega_Y^m)$. We argue then as in the proofs of Propositions 4.4.1 and 4.4.2. □

Remark 4.4.4. *We note that the proof of Proposition 4.4.2 can be adapted to yield the derived invariance of more refined cohomological support loci*

$$\bigcup_{r_0 + \dots + r_i = r} \bigcup_{q=0}^i V_{r_q}^{i-q}(\Omega_X^{n-q} \otimes \omega_X^m)_0 \simeq \bigcup_{r_0 + \dots + r_i = r} \bigcup_{q=0}^i V_{r_q}^{i-q}(\Omega_Y^{n-q} \otimes \omega_Y^m)_0$$

for integers $m, r \geq 1$ and $i \geq 0$.

Remark 4.4.5. *It is important to note that, whenever $F_{\mathcal{E}}(1_X, \text{Pic}^0(X)) = (1_Y, \text{Pic}^0(Y))$, the proofs of Propositions 4.4.1 and 4.4.2 yield full isomorphisms*

$$\bigcup_q V^{i-q}(\Omega_X^{n-q} \otimes \omega_X^m) \simeq \bigcup_q V^{i-q}(\Omega_Y^{n-q} \otimes \omega_Y^m) \quad \text{for any } i \geq 0$$

$$V_r^1(\omega_X) \simeq V_r^1(\omega_Y).$$

By Lemmas 4.3.1 and 4.3.3, this occurs either if $\text{Aut}^0(X)$ is affine, or when $V^p(\Omega_X^q \otimes \omega_X^m) = \text{Pic}^0(X)$ for some integers p, q, m with $p, q \geq 0$.

Remark 4.4.6. *We point out that a stronger version of Proposition 4.4.2 holds. If $Z \subset \text{Pic}^0(X)$ is an algebraic closed subset, we denote by Z^0 the connected component of Z passing through the origin. If in the proof of Proposition 4.4.2 we replace*

$$\bigcup_q V^{i-q}(\Omega_X^{n-q} \otimes \omega_X^m)_0 \quad \text{with} \quad \left(\bigcup_q V^{i-q}(\Omega_X^{n-q} \otimes \omega_X^m) \right)^0,$$

and

$$\bigcup_q V^{i-q}(\Omega_Y^{n-q} \otimes \omega_Y^m)_0 \quad \text{with} \quad \left(\bigcup_q V^{i-q}(\Omega_Y^{n-q} \otimes \omega_Y^m) \right)^0,$$

then we see that $F_{\mathcal{E}}$ induces further isomorphisms

$$\left(\bigcup_q V^{i-q}(\Omega_X^{n-q} \otimes \omega_X^m) \right)^0 \simeq \left(\bigcup_q V^{i-q}(\Omega_Y^{n-q} \otimes \omega_Y^m) \right)^0 \quad \text{for all } i \geq 0.$$

Moreover, as in the proof of Proposition 4.4.1, we also get

$$V_r^1(\omega_X)^0 \simeq V_r^1(\omega_Y)^0 \quad \text{for any } r \geq 1.$$

4.5 Behavior of the Albanese dimension under derived equivalence

In this section we establish the derived invariance of the Albanese dimension for varieties having non-negative Kodaira dimension. Our main tool is a generalization of a result due to Chen-Hacon-Pardini saying that if $f : X \rightarrow Z$ is a non-singular representative of the Iitaka fibration of a smooth projective variety X of maximal Albanese dimension, then $q(X) - q(Z) = \dim X - \dim Z$ (cf. (36) Proposition 2.1 and (40) Corollary 3.6). We generalize this fact in two ways: we consider all possible values of the Albanese dimension of X , and we replace the Iitaka fibration of X with a more general class of morphisms.

Lemma 4.5.1. *Let X and Z be smooth projective varieties and $f : X \rightarrow Z$ be a surjective morphism with connected fibers. If the general fiber of f is a smooth variety with surjective Albanese map, then $q(X) - q(Z) = \dim \text{alb}_X(X) - \dim \text{alb}_Z(Z)$.*

Proof. We follow (36) Proposition 2.1 and (40) Corollary 3.6. Due to the functoriality of the Albanese map, there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\text{alb}_X} & \text{Alb}(X) \\ \downarrow f & & \downarrow f_* \\ Z & \xrightarrow{\text{alb}_Z} & \text{Alb}(Z). \end{array}$$

where f_* is surjective since f is so (cf. (41) Remark V.14). Furthermore, f_* has connected fibers. To see this we denote by K the connected component of $\text{Ker } f_*$ through the origin and set $A := \text{Alb}(X)/K$. Then the natural map $\nu : A \rightarrow \text{Alb}(Y)$ is étale and f factors through the induced map $Y \times_{\text{Alb}(Y)} A \rightarrow Y$, which is étale of the same degree as ν . Since f has connected fibers, we see that ν is an isomorphism and $K = \text{Ker } f_*$.

We now show that the image of a general fiber P of f via alb_X is a translate of $\text{Ker } f_*$. Since alb_P is surjective, the image of P via alb_X is a translate of a sub-torus of $\text{Ker } f_*$. Furthermore, since P moves in a continuous family, such images are all translates of a fixed sub-torus $T \subset \text{Ker } f_*$. Our next step is to show $T = \text{Ker } f_*$. By setting $B := \text{Alb}(X)/T$, we see that the induced morphism $X \rightarrow B$ maps a general fiber of f to a point. Therefore it induces a rational map $h : Z \dashrightarrow B$ which is a morphism since B is an abelian variety. Furthermore, $h(Z)$ generates the abelian variety B since the image of the Albanese map generates the Albanese variety. This leads to the inequality

$$\dim B \leq q(Z) = q(X) - \dim \text{Ker } f_*,$$

which in turn yields $\dim T \geq \dim \text{Ker } f_*$ as $\dim B = q(X) - \dim T$. For dimension reasons we get then $T = \text{Ker } f_*$. In particular, this says that $\text{alb}_X(X)$ is fibered in tori of dimension $q(X) - q(Z)$ over $\text{alb}_Z(Z)$, and by the theorem on the dimension of the fibers of a morphism we get the stated equality. \square

Now we prove the derived invariance of the Albanese dimension.

Theorem 4.5.2. *Let X and Y be smooth projective D -equivalent varieties with $\kappa(X) \geq 0$. Then $\dim \operatorname{alb}_X(X) = \dim \operatorname{alb}_Y(Y)$.*

Proof. We note that $\kappa(X) = \kappa(Y)$ since the Kodaira dimension is invariant under derived equivalence (cf. Theorem 2.1.4). If $\kappa(X) = \kappa(Y) = 0$, then the Albanese maps of X and Y are surjective by a theorem of Kawamata (42) Theorem 1. Thus the Albanese dimensions of X and Y are respectively $q(X)$ and $q(Y)$, which are equal by Theorem 2.1.10.

We now suppose $\kappa(X) = \kappa(Y) > 0$. Since the problem is invariant under birational modification, with a little abuse of notation, we consider non-singular representatives $f : X \rightarrow Z$ and $g : Y \rightarrow W$ of the Iitaka fibrations of X and Y respectively (cf. (26) (1.10)). As the canonical rings of X and Y are isomorphic (cf. Theorem 2.1.4), it turns out that Z and W are birational varieties (cf. (26) Proposition 1.4 or (13) p. 13). By (42) Theorem 1, the morphisms f and g satisfy the hypotheses of Lemma 4.5.1 which yields

$$q(X) - \dim \operatorname{alb}_X(X) = q(Z) - \dim \operatorname{alb}_Z(Z) = q(W) - \dim \operatorname{alb}_W(W) = q(Y) - \dim \operatorname{alb}_Y(Y).$$

We conclude as $q(X) = q(Y)$. □

Remark 4.5.3. *The previous theorem also holds for D -equivalent varieties X and Y such that ω_X^{-1} is nef. In fact, in this case ω_Y^{-1} is nef as well (cf. Theorem 2.1.12), and moreover the main result of Zhang in (43) (part of a conjecture of Demailly-Peternell-Schneider) ensures that the Albanese map of a variety with nef anticanonical bundle is surjective. Hence $\dim \operatorname{alb}_X(X) = q(X) = q(Y) = \dim \operatorname{alb}_Y(Y)$.*

Moreover, in Corollaries 5.1.3 and 5.2.2 we will establish the derived invariance of the Albanese dimension also for varieties of dimension up to three with no restrictions on the Kodaira dimension. This is achieved by means of different techniques, mainly relying on the derived invariance of cohomological support loci around the origin in dimension up to three.

The following corollary is an application of Theorem 4.5.2.

Corollary 4.5.4. *Let X and Y be smooth projective D -equivalent varieties with X of maximal Albanese dimension. If F denotes the induced Rouquier isomorphism and $F(1_X, \alpha) = (\psi, \beta)$ with $\alpha \in V_r^1(\omega_X)$, then we have $\psi = 1_Y$, $\beta \in V_r^1(\omega_Y)$, and $h^1(X, \omega_X \otimes \alpha) = h^1(Y, \omega_Y \otimes \beta)$. Moreover, F induces isomorphisms of algebraic sets $V_r^1(\omega_X) \simeq V_r^1(\omega_Y)$ for all $r \geq 1$.*

Proof. By Theorem 4.5.2 Y is of maximal Albanese dimension, and by (2.9) there are inclusions $V_r^1(\omega_X) \subset V^0(\omega_X)$ and $V_r^1(\omega_Y) \subset V^0(\omega_Y)$. Hence, the first assertion follows from Proposition 4.2.1 while the second by Proposition 4.4.1. Finally, the last assertion is a consequence of Corollary 4.4.3. \square

4.6 Applications

In this section we apply our results concerning the derived invariance of cohomological support loci and of the Albanese dimension to deduce the derived invariance of particular numerical invariants, such as the holomorphic Euler characteristic and certain Hodge numbers for varieties having large Albanese dimension.

4.6.1 Holomorphic Euler characteristic

The holomorphic Euler characteristic $\chi(\omega_X) = \sum_{i=0}^{\dim X} (-1)^i h^{\dim X, i}(X)$ is known to be a derived invariant for varieties of dimension up to three and for varieties of general type (cf. §2.1). Moreover, since Hodge numbers are expected to be preserved under derived equivalence (cf. Problem 2.1.6), the same is expected for $\chi(\omega_X)$. Here we provide further classes of varieties for which this is true.

Corollary 4.6.1. *Let X and Y be smooth projective D -equivalent varieties of dimension n . Suppose that either (i) X of maximal Albanese dimension; or (ii) $\dim \text{alb}_X(X) = n - 1$ and $\kappa(X) \geq 0$; or (iii) $\dim \text{alb}_X(X) = n - 1$ and ω_X^{-1} is nef. Then we have $\chi(\omega_X) = \chi(\omega_Y)$.*

Proof. Let F be the induced Rouquier isomorphism. We begin with the case $\dim \text{alb}_X(X) = n$. By Theorem 4.5.2 Y is of maximal Albanese dimension as well, and by (2.8) we get inequalities

$$\text{codim } V^1(\omega_X) \geq 1 \quad \text{and} \quad \text{codim } V^1(\omega_Y) \geq 1.$$

We distinguish two cases: $V^0(\omega_X) \subsetneq \text{Pic}^0(X)$ and $V^0(\omega_X) = \text{Pic}^0(X)$. If $V^0(\omega_X) \subsetneq \text{Pic}^0(X)$, then there are inclusions $\text{Pic}^0(X) \supsetneq V^0(\omega_X) \supset V^1(\omega_X) \supset \cdots \supset V^n(\omega_X) = \{\mathcal{O}_X\}$, and similarly for the loci $V^i(\omega_Y)$ (cf. (2.9)). Therefore we get $h^i(X, \omega_X \otimes \alpha) = h^i(Y, \omega_Y \otimes \beta) = 0$ for all $i \geq 0$ and for any $\alpha \notin V^0(\omega_X)$ and $\beta \notin V^0(\omega_Y)$. Finally, by using the fact that the holomorphic Euler characteristic is invariant under deformations, we have that

$$\chi(\omega_X) = \chi(\omega_X \otimes \alpha) = 0 = \chi(\omega_Y \otimes \beta) = \chi(\omega_Y).$$

On the other hand, if $V^0(\omega_X) = \text{Pic}^0(X)$, then by Proposition 4.2.1 we have that the Rouquier isomorphism F maps $F(1_X, \text{Pic}^0(X)) = (1_Y, \text{Pic}^0(Y))$, and thus that

$$\exists \alpha_0 \in V^0(\omega_X) \setminus (\cup_{i=1}^n V^i(\omega_X)) \text{ such that } F(1_X, \alpha_0) = (1_Y, \beta_0) \text{ with } \beta_0 \in V^0(\omega_Y) \setminus (\cup_{i=1}^n V^i(\omega_Y)).$$

Hence, by using Corollary 3.0.8 with $m = i = 0$, we have

$$\chi(\omega_X) = \chi(\omega_X \otimes \alpha_0) = h^0(X, \omega_X \otimes \alpha_0) = h^0(Y, \omega_Y \otimes \beta_0) = \chi(\omega_Y \otimes \beta_0) = \chi(\omega_Y).$$

We suppose now $\dim \text{alb}_X(X) = n - 1$ and either $\kappa(X) \geq 0$ or ω_X^{-1} nef. By Theorem 4.5.2 and Remark 4.5.3 we have $\dim \text{alb}_Y(Y) = n - 1$, and therefore there are inclusions $V^1(\omega_X) \supset V^2(\omega_X) \supset \dots \supset V^n(\omega_X)$ and $V^1(\omega_Y) \supset V^2(\omega_Y) \supset \dots \supset V^n(\omega_Y)$. We distinguish four cases.

The first case is when $V^0(\omega_X) = V^1(\omega_X) = \text{Pic}^0(X)$. By Propositions 4.2.1 and 4.4.1 we have that $V^0(\omega_Y) = V^1(\omega_Y) = \text{Pic}^0(Y)$ as well. We claim that

$$\exists \mathcal{O}_X \neq L_1 \in V^0(\omega_X) \setminus V^2(\omega_X) \text{ such that } F(1_X, L_1) = (1_Y, M_1) \text{ with } \mathcal{O}_Y \neq M_1 \in V^0(\omega_Y) \setminus V^2(\omega_Y).$$

In fact, by Remark 4.4.5, we have $F(1_X, \text{Pic}^0(X)) = (1_Y, \text{Pic}^0(Y))$ and then it is enough to choose the image, under F^{-1} , of a generic element $(1_Y, \beta)$ with $\beta \notin V^2(\omega_Y)$. By using Corollary 3.0.8 we obtain

$$\begin{aligned} \chi(\omega_X) &= \chi(\omega_X \otimes \alpha_1) = h^0(X, \omega_X \otimes \alpha_1) - h^1(X, \omega_X \otimes \alpha_1) = \\ &h^0(Y, \omega_Y \otimes \beta_1) - h^1(Y, \omega_Y \otimes \beta_1) = \chi(\omega_Y \otimes \beta_1) = \chi(\omega_Y). \end{aligned}$$

The second case is when $V^0(\omega_X) = \text{Pic}^0(X)$ and $V^1(\omega_X) \subsetneq \text{Pic}^0(X)$. By Propositions 4.2.1 and 4.4.1 we have $V^0(\omega_Y) = \text{Pic}^0(Y)$ and $V^1(\omega_Y) \subsetneq \text{Pic}^0(Y)$. As before $F(1_X, \text{Pic}^0(X)) = (1_Y, \text{Pic}^0(Y))$ and hence we can pick an element

$$\mathcal{O}_X \neq \alpha_2 \in V^0(\omega_X) \setminus V^1(\omega_X) \text{ such that } F(1_X, \alpha_2) = (1_Y, \beta_2) \text{ with } \mathcal{O}_Y \neq \beta_2 \in V^0(\omega_Y) \setminus V^1(\omega_Y).$$

Therefore we obtain equalities $\chi(\omega_X) = \chi(\omega_X \otimes \alpha_2) = h^0(X, \omega_X \otimes \alpha_2) = h^0(Y, \omega_Y \otimes \beta_2) = \chi(\omega_Y \otimes \beta_2) = \chi(\omega_Y)$.

The third case is when $V^0(\omega_X) \subsetneq \text{Pic}^0(X)$ and $V^1(\omega_X) = \text{Pic}^0(X)$. It is easy to see that $V^0(\omega_Y) \subsetneq \text{Pic}^0(Y)$ and $V^1(\omega_Y) = \text{Pic}^0(Y)$ by using Propositions 4.2.1 and 4.4.1. Moreover, we note that $F(1_X, \text{Pic}^0(X)) = (1_Y, \text{Pic}^0(Y))$ by Remark 4.4.5. Similarly to the previous cases, there exists a pair $(\alpha_3, \beta_3) \neq (\mathcal{O}_X, \mathcal{O}_Y)$ such that

$$F(1_X, \alpha_3) = (1_Y, \beta_3) \text{ with } \alpha_3 \notin V^0(\omega_X) \cup V^2(\omega_X) \text{ and } \beta_3 \notin V^0(\omega_Y) \cup V^2(\omega_Y).$$

Therefore, by Corollary 3.0.8, we have $\chi(\omega_X) = \chi(\omega_X \otimes \alpha_3) = -h^1(X, \omega_X \otimes \alpha_3) = -h^1(Y, \omega_Y \otimes \beta_3) = \chi(\omega_Y \otimes \beta_3) = \chi(\omega_Y)$.

The last case is when both $V^0(\omega_X)$ and $V^1(\omega_X)$ are proper subvarieties of $\text{Pic}^0(X)$. Then $V^0(\omega_Y)$ and $V^1(\omega_Y)$ are proper subvarieties as well and hence $\chi(\omega_X) = \chi(\omega_Y) = 0$. \square

We also study the behavior of Euler characteristics of pluricanonical bundles.

Corollary 4.6.2. *Let X and Y be smooth projective D -equivalent varieties. Suppose that X is minimal and of maximal Albanese dimension. Then we have $\chi(\omega_X^m) = \chi(\omega_Y^m)$ for all $m \geq 2$.*

Proof. By Theorem 2.2.3 (iv), we have $\text{codim}_{\text{Pic}^0(X)} V^i(\omega_X^m) \geq i$ and $\text{codim}_{\text{Pic}^0(Y)} V^i(\omega_Y^m) \geq i$ for all $i \geq 1$. Since $V^0(\omega_X^m) \simeq V^0(\omega_Y^m)$ by Proposition 4.2.1, we argue then as in the proof of Corollary 4.6.1 after having noted that $V^0(\omega_X^m) \supset V^1(\omega_X^m)$ and $V^0(\omega_Y^m) \supset V^1(\omega_Y^m)$ (cf. (11) Proposition 3.14). \square

4.6.2 Hodge numbers

As mentioned in §2.1, Hodge numbers are known to be invariant under derived equivalence in dimension up to three and for varieties of general type. In dimension four there are partial results provided in (9) where the authors show the derived invariance of $h^{1,2}(X)$. Furthermore, they show the derived invariance of $h^{0,2}(X)$ and $h^{1,3}(X)$ if in addition one assumes that $\text{Aut}^0(X)$ is not affine. Here we prove the derived invariance of $h^{0,2}(X)$ and $h^{1,3}(X)$ for fourfolds satisfying one of the hypotheses of Corollary 4.6.1. Finally, we recall that the Hodge numbers of type $h^{0,i}(X)$ with $i = 0, 1, \dim X - 1, \dim X$ are known to be invariant under an arbitrary derived equivalence thanks to the derived invariance of Hochschild homology and Theorem 2.1.10.

Corollary 4.6.3. *Let X and Y be smooth projective D -equivalent fourfolds. Suppose that either (i) X of maximal Albanese dimension; or (ii) $\dim \operatorname{alb}_X(X) = 3$ and $\kappa(X) \geq 0$; or (iii) $\dim \operatorname{alb}_X(X) = 3$ and ω_X^{-1} is nef. Then we have $h^{0,2}(X) = h^{0,2}(Y)$ and $h^{1,3}(X) = h^{1,3}(Y)$.*

Proof. By the invariance of Hochschild homology $HH_i(X) \simeq HH_i(Y)$ for $i = 0, 1$ we have $h^0(X, \omega_X) = h^0(Y, \omega_Y)$ and $h^1(X, \omega_X) = h^1(Y, \omega_Y)$ (cf. also Corollary 3.0.8 with α and β trivial). Then Corollary 4.6.1 implies $h^{0,2}(X) = h^{0,2}(Y)$ since $h^3(X, \omega_X) = q(X) = q(Y) = h^3(Y, \omega_Y)$ and $h^4(X, \omega_X) = 1 = h^4(Y, \omega_Y)$. The second equality of the corollary follows at once from the isomorphism $HH_2(X) \simeq HH_2(Y)$ and the Hochschild-Kostant-Rosenberg isomorphism (2.3). (When X is of maximal Albanese dimension, the derived invariance of $h^{0,2}(X)$ and of $h^{1,3}(X)$ is independent by the invariance of cohomological support loci. In fact, if X is of general type, then we have already noticed that X and Y have the same Hodge numbers thanks to work of Kontsevich on motivic integration. On the other hand, if X is not of general type, then $\chi(\omega_X) = \chi(\omega_Y) = 0$ (cf. (44) Theorem 1) and $h^{0,2}(-)$ can be computed in terms of the other Hodge numbers which are known to be derived invariant. We thank Professor Mihnea Popa for this observation.) □

CHAPTER 5

POPA'S CONJECTURES IN DIMENSION TWO AND THREE

In this chapter we give a closer look to Conjecture 4.1.1 and Variant 4.1.2 in dimension two and three. We will see that our techniques, together with the derived invariance of Hodge numbers in lower dimension, lead to more precise statements about the derived invariance of cohomological support loci.

More specifically, in dimension two we present a second proof of Variant 4.1.2 (the first proof is due to Popa in (4)) making explicit the isomorphisms between cohomological support loci (*cf.* Theorem 5.1.2). In dimension three we verify the derived invariance of loci $V^p(\Omega_X^q)_0$ for $p, q \geq 0$, which in particular yields Conjecture 4.1.1 (*cf.* Theorem 5.2.1). Moreover, we establish Variant 4.1.2 in most cases, namely for varieties of maximal Albanese dimension varieties, or having affine automorphism group, or having non vanishing holomorphic Euler characteristic (*cf.* Propositions 5.2.3 and 5.2.5). Furthermore, we verify Variant 4.1.2 in its full generality in the special case when the irregularity is equal to two (*cf.* Proposition 5.2.15). Finally, for the cases we were not able to prove Variant 4.1.2, we provide a strong evidence towards it by proving the derived invariance of the dimensions of cohomological support loci associated to the canonical bundle. We point out that the main obstruction to prove Variant 4.1.2 in its full generality is the possible presence of non-trivial automorphisms.

In this chapter we work over the field of complex numbers.

5.1 The case of surfaces

We start by proving a general fact about varieties having negative Kodaira dimension.

Lemma 5.1.1. *Let X be a smooth projective variety with $\kappa(X) = -\infty$. Then $V^0(\omega_X^m) = \emptyset$ for all $m \geq 1$.*

Proof. If, by contradiction, $V^0(\omega_X^m) \neq \emptyset$ for some $m \geq 1$, then by Theorem 2.2.4 there would exist an element $\alpha \in V^0(\omega_X^m)$ having finite order $e := \text{ord } \alpha$. But this yields a contradiction since a non-zero section $\mathcal{O}_X \rightarrow \omega_X^m \otimes \alpha$ induces a non-zero section $\mathcal{O}_X \rightarrow \omega_X^{me}$. \square

Theorem 5.1.2. *Let X and Y be smooth projective D -equivalent surfaces and let F be the induced Rouquier isomorphism. Then F induces isomorphisms of algebraic sets $V_r^p(\Omega_X^q) \simeq V_r^p(\Omega_Y^q)$ for all $p, q \geq 0$ and $r \geq 1$.*

Proof. Since Hodge numbers are derived invariants in dimension two, we can assume $q(X) > 0$ (cf. Corollary 2.1.7). The isomorphisms $V_r^0(\omega_X) \simeq V_r^0(\omega_Y)$ are proved in Proposition 4.2.1, while the isomorphisms $V_r^2(\omega_X) \simeq V_r^2(\omega_Y)$ are trivial. To prove the isomorphisms $V_r^1(\omega_X) \simeq V_r^1(\omega_Y)$ we distinguish three cases.

Case I: X is not ruled and $q(X) \geq 2$. By (41) Theorem X.4 we have $\chi(\omega_X) \geq 0$, and consequently $V^0(\omega_X) \supset V^1(\omega_X)$. Hence the desired isomorphisms $V_r^1(\omega_X)_0 \simeq V_r^1(\omega_Y)_0$ follow from Proposition 4.2.1 and Corollary 4.4.3.

Case II: X is not ruled and $q(X) = 1$. As before we have $\chi(\omega_X) \geq 0$ and hence the inclusion $V^0(\omega_X) \setminus \{\mathcal{O}_X\} \supset V^1(\omega_X) \setminus \{\mathcal{O}_X\}$. Therefore Proposition 4.2.1 and Corollary 4.4.3 yield isomor-

phisms $V_r^i(\omega_X) \setminus \{\mathcal{O}_X\} \simeq V_r^i(\omega_Y) \setminus \{\mathcal{O}_Y\}$ for $i = 0, 1$ and we conclude then by using the derived invariance of Hodge numbers.

Case III: X is ruled. We have $\kappa(X) = -\infty$, and hence $V^0(\omega_X) = \emptyset$ by Lemma 5.1.1. If $q(X) = 1$, then X is birational to $E \times \mathbf{P}^1$ where E is a curve of genus 1. Therefore we have $\chi(\omega_X) = 0$, $V^1(\omega_X) = \{\mathcal{O}_X\}$ and $V_r^1(\omega_X) = \emptyset$ for $r \geq 2$. Since the Kodaira dimension and the irregularity are derived invariants, the same analysis holds for Y .

Suppose now $q(X) \geq 2$. Then X is birational to $C \times \mathbf{P}^1$ where C is a curve of genus ≥ 2 . Hence we have $\chi(\omega_X) < 0$ and furthermore $V^1(\omega_X) = \text{Pic}^0(X)$ since $V^0(\omega_X) = \emptyset$. Thus, by Proposition 4.4.1, the Rouquier isomorphism induces isomorphisms $\text{Pic}^0(X) = V^1(\omega_X) = V^1(\omega_X)_0 \simeq V^1(\omega_Y)_0 = V^1(\omega_Y) = \text{Pic}^0(Y)$. At this point the isomorphisms $V_r^1(\omega_X) \simeq V_r^1(\omega_Y)$ for $r > 1$ easily follow from Corollary 3.0.8.

We now show the isomorphisms $V_r^1(\Omega_X^1) \simeq V_r^1(\Omega_Y^1)$ for all $r \geq 1$. We distinguish two cases: $\chi(\Omega_X^1) \geq 0$ and $\chi(\Omega_X^1) < 0$. Let us start with $\chi(\Omega_X^1) \geq 0$. By Serre duality and the Hodge linear-conjugate isomorphism we have

$$0 \leq \chi(\Omega_X^1 \otimes \alpha) = h^1(X, \omega_X \otimes \alpha) - h^1(X, \Omega_X^1 \otimes \alpha) + h^1(X, \omega_X \otimes \alpha^{-1})$$

for any $\alpha \in \text{Pic}^0(X)$. Thus, if $\alpha \in V_r^1(\Omega_X^1)$, then either α or α^{-1} belongs to $V^1(\omega_X)$. By noting that $V^1(\omega_X) \setminus \{\mathcal{O}_X\} = V^0(\omega_X) \setminus \{\mathcal{O}_X\}$, we have $F(1_X, V_r^1(\Omega_X^1) \setminus \{\mathcal{O}_X\}) \subset (1_Y, \text{Pic}^0(Y))$. Hence,

by Corollary 3.0.8 (with $m = 0$ and $i = 2$), we get $F(1_X, V_r^1(\Omega_X^1) \setminus \{\mathcal{O}_X\}) \subset (1_Y, V_r^1(\Omega_Y^1) \setminus \{\mathcal{O}_Y\})$, and since $h^1(X, \Omega_X^1) = h^1(Y, \Omega_Y^1)$ we have

$$F(1_X, V_r^1(\Omega_X^1)) \subset (1_Y, V_r^1(\Omega_Y^1)).$$

At this point, it is enough to show that F^{-1} maps $V_r^1(\Omega_Y^1)$ to $V_r^1(\Omega_X^1)$. But this follows as $\chi(\Omega_Y^1) = \chi(\Omega_X^1) \geq 0$, and hence by Lemma 2.1.9 we can repeat the previous argument for F^{-1} .

We now study the other case: $\chi(\Omega_X^1) < 0$. We distinguish two subcases: $V^1(\omega_X) = \text{Pic}^0(X)$ and $V^1(\omega_X) \subsetneq \text{Pic}^0(X)$. If $V^1(\omega_X) = \text{Pic}^0(X)$, then it is sufficient to apply Lemma 4.3.3 and Corollary 3.0.8 to get the desired isomorphisms $V_r^1(\Omega_X^1) \simeq V_r^1(\Omega_Y^1)$. We then suppose the second subcase $V^1(\omega_X) \subsetneq \text{Pic}^0(X)$. Since for any $\alpha \in \text{Pic}^0(X)$

$$\chi(\Omega_X^1 \otimes \alpha) = h^1(X, \omega_X \otimes \alpha) - h^1(X, \Omega_X^1 \otimes \alpha) + h^1(X, \omega_X \otimes \alpha^{-1}) < 0,$$

we have $V^1(\omega_X)^c \cap (-V^1(\omega_X))^c \subset V^1(\Omega_X^1)$ (the superscript c denotes the complement set). Hence $V^1(\Omega_X^1)$ contains an open dense set as $V^1(\omega_X)$ is of codimension at least one. Therefore we get $V^1(\Omega_X^1) = \text{Pic}^0(X)$. Moreover, by Lemma 4.3.3, we have $F(1_X, \text{Pic}^0(X)) = (1_Y, \text{Pic}^0(Y))$ and by Corollary 3.0.8 (with $m = 0$ and $i = 2$) we obtain inclusions $F(1_X, V_r^1(\Omega_X^1)) \subset (1_Y, V_r^1(\Omega_Y^1))$. Finally, as we did before, we get the reverse inclusions by repeating the same argument for F^{-1} thanks to Lemma (2.1.9).

□

Corollary 5.1.3. *Let X and Y be smooth projective D -equivalent surfaces. Then $\dim \operatorname{alb}_X(X) = \dim \operatorname{alb}_Y(Y)$.*

Proof. The corollary is a consequence of the previous theorem and of the fact that the Albanese dimension is determined by the dimensions of cohomological support loci around the origin (cf. (2.8)). \square

We now specialize to the case of surfaces of maximal Albanese dimension and of Kodaira dimension one. Following (37) we recall how to attach an invariant to this class of surfaces. Moreover, we will prove its invariance under derived equivalence.

Let X be a surface of maximal Albanese dimension and with $\kappa(X) = 1$ (i.e. an isotrivial elliptic surface fibered onto a smooth curve of genus ≥ 2). Since $\chi(\omega_X) = 0$ (cf. (45)), we have $V^0(\omega_X) \subsetneq \operatorname{Pic}^0(X)$ by (2.9). Moreover, X admits a unique fibration $f : X \rightarrow C$ onto a curve of genus $g(C) = q(X) - 1 \geq 2$ (cf. for instance (4)). Beauville in (37) p. 4 shows that there exists a group $\Gamma^0(f)$ such that $\operatorname{Pic}^0(X, f) \simeq \Gamma^0(f) \times f^*\operatorname{Pic}^0(C)$ (see (2.12) for the definition of $\operatorname{Pic}^0(X, f)$). The group $\Gamma^0(f)$ is the invariant mentioned above; it is identified to the group of connected components of $\operatorname{Pic}^0(X, f)$, and hence to the group of connected components of the kernel of the canonical homomorphism $\operatorname{Alb}(F) \rightarrow \operatorname{Alb}(X)$ where F is a general fiber of f .

If Y is a Fourier-Mukai partner of X , then Y is an elliptic surface of maximal Albanese dimension and of Kodaira dimension one admitting a unique fibration $g : Y \rightarrow C$ (cf. (46) Proposition 4.4). In his dissertation (47) Theorem 5.2.7, Pham proves the invariance of $\Gamma^0(f)$ under derived equivalence, i.e. the existence of an isomorphism $\Gamma^0(f) \simeq \Gamma^0(g)$. Here we show that this isomorphism is also a consequence of the derived invariance of the zeroth cohomological

support locus. To see this, first of all we note that $V^0(\omega_X) = V^1(\omega_X)$ since $\chi(\omega_X) = 0$, and secondly that, by Theorem 2.2.4 (ii), $V^1(\omega_X) = \text{Pic}^0(X, f) \cup \{\text{finite number of points}\}$. Moreover, if by contradiction there would exist an isolated point in $V^1(\omega_X)$, then $\chi(\omega_X) \geq q(X) - 2 + 1 \geq 2$ by (34) Remark 4.3. This yields a contradiction and therefore $V^0(\omega_X) \simeq V^1(\omega_X) = \text{Pic}^0(X, f)$. In the same way we have $V^0(\omega_Y) = V^1(\omega_Y) \simeq \text{Pic}^0(Y, g)$. Hence, by Proposition 4.2.1, we obtain $\Gamma^0(f) \times f^*\text{Pic}^0(C) \simeq \Gamma^0(g) \times g^*\text{Pic}^0(C)$ leading to $\Gamma^0(f) \simeq \Gamma^0(g)$.

In alternative to Beauville's theorem, one can show that $V^1(\omega_X)$ is of pure codimension one by adapting an argument of (21) involving derivative complexes. To prove this we argue by contradiction and we suppose that $Z \subset V^1(\omega_X) = V^0(\omega_X)$ is a positive-dimensional irreducible component of codimension ≥ 2 . Let $\mathcal{O}_X \neq \alpha^{-1} \in Z$ be a general point and consider the following complex on $\mathbf{P} := \mathbf{P}(H^1(X, \mathcal{O}_X))$:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-1) \otimes H^1(X, \alpha) \xrightarrow{\cup v} \mathcal{O}_{\mathbf{P}} \otimes H^2(X, \alpha) \longrightarrow \mathcal{F}_{\alpha} \longrightarrow 0$$

which is induced by cupping pointwise with elements $v \in H^1(X, \mathcal{O}_X)$ and where \mathcal{F}_{α} is the cokernel of $\cup v$. As in (16) p. 618, we note that this complex is everywhere exact. Moreover, by choosing a subspace $W \subset T_{\alpha^{-1}}\text{Pic}^0(X) = H^1(X, \mathcal{O}_X)$ transverse to the tangent space of Z at α^{-1} , and by restricting the previous complex to $\mathbf{P}(W)$, we get a new complex

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}(W)}(-1) \otimes H^1(X, \alpha) \xrightarrow{\cup v} \mathcal{O}_{\mathbf{P}(W)} \otimes H^2(X, \alpha) \longrightarrow G \longrightarrow 0$$

where G is a locally free sheaf of rank $\chi(\omega_X)$ (here we use (21) Theorem 1.2 and we also refer to (16) Variant 4.12 for a similar argument). Finally, since $\chi(\omega_X) = 0$, we get $G = 0$. Therefore $h^1(X, \omega_X \otimes \alpha^{-1}) = h^1(X, \alpha) = 0$ which is impossible since $\alpha^{-1} \in Z$.

5.2 The case of threefolds

The following theorem in particular proves Conjecture 4.1.1 in dimension three.

5.2.1 Conjecture 4.1.1 in dimension three

Theorem 5.2.1. *Let X and Y be smooth projective threefolds, $\Phi : \mathbf{D}(X) \simeq \mathbf{D}(Y)$ be an equivalence, and $F_{\mathcal{E}}$ be the induced Rouquier isomorphism. Then $F_{\mathcal{E}}$ induces isomorphisms of algebraic sets $V_r^p(\Omega_X^q)_0 \simeq V_r^p(\Omega_Y^q)_0$ for any $p, q \geq 0$ and $r \geq 1$.*

Proof. The isomorphisms $V_r^0(\omega_X) \simeq V_r^0(\omega_Y)$ and $V_r^1(\omega_X)_0 \simeq V_r^1(\omega_Y)_0$ have been proved in Propositions 4.2.1 and 4.4.1 respectively. We now prove the isomorphisms $V_r^2(\omega_X)_0 \simeq V_r^2(\omega_Y)_0$. Let $\alpha \in V_r^2(\omega_X)$ so that, by Lemma 4.3.3, $F_{\mathcal{E}}(1_X, \alpha) = (1_Y, \beta)$ for some $\beta \in \text{Pic}^0(Y)$. Then, by Corollary 3.0.8, we have $h^2(X, \omega_X \otimes \alpha) = h^2(Y, \omega_Y \otimes \beta)$ and consequently $\beta \in V_r^2(\omega_Y)_0$. This in turns yields inclusions $F_{\mathcal{E}}(1_X, V_r^2(\omega_X)_0) \subset (1_Y, V_r^2(\omega_Y)_0)$.

In order to show the reverse inclusions, we note that $F_{\mathcal{E}}^{-1}$ is the Rouquier isomorphism induced by the right adjoint to $\Phi_{\mathcal{E}}$ (cf. Lemma 2.1.9). Therefore we can repeat the above argument to get inclusions $F_{\mathcal{E}}^{-1}(1_Y, V_r^2(\omega_Y)_0) \subset (1_X, V_r^2(\omega_X)_0)$, and therefore the desired isomorphisms. This in turn yields isomorphisms $V_r^0(\Omega_X^1)_0 \simeq V_r^0(\Omega_Y^1)_0$ thanks to Serre duality and the Hodge linear-conjugate isomorphism.

We now establish the isomorphisms $V_r^1(\Omega_X^q)_0 \simeq V_r^1(\Omega_Y^q)_0$ for $q = 1, 2$. We start with the case $q = 2$. Let $\alpha \in V_r^2(\Omega_X)_0$ so that, by Lemma 4.3.3, $F_{\mathcal{E}}(1_X, \alpha) = (1_Y, \beta)$. By Corollary 3.0.8 (with $m = 0$ and $i = 2$) there is an isomorphism

$$\begin{aligned} H^2(X, \omega_X \otimes \alpha) \oplus H^1(X, \Omega_X^2 \otimes \alpha) \oplus H^0(X, \Omega_X^1 \otimes \alpha) \simeq \\ H^2(Y, \omega_Y \otimes \beta) \oplus H^1(Y, \Omega_Y^2 \otimes \beta) \oplus H^0(Y, \Omega_Y^1 \otimes \beta). \end{aligned}$$

Moreover, by Serre duality and the Hodge linear-conjugate isomorphism, we get equalities $h^0(X, \Omega_X^1 \otimes \alpha) = h^2(X, \omega_X \otimes \alpha)$ and $h^0(Y, \Omega_Y^1 \otimes \beta) = h^2(Y, \omega_Y \otimes \beta)$. At this point, thanks to the derived invariance of the holomorphic Euler characteristic in dimension three, we also have $h^2(X, \omega_X \otimes \alpha) = h^2(Y, \omega_Y \otimes \beta)$ (cf. Corollary 3.0.8). Therefore we have $h^1(X, \Omega_X^2 \otimes \alpha) = h^1(Y, \Omega_Y^2 \otimes \beta)$ and consequently inclusions $F_{\mathcal{E}}(1_X, V_r^1(\Omega_X^2)_0) \subset (1_Y, V_r^1(\Omega_Y^2)_0)$. Finally, as we have seen before, we can repeat the previous argument for $F_{\mathcal{E}}^{-1}$ in order to get the reverse inclusions $F_{\mathcal{E}}^{-1}(1_Y, V_r^1(\Omega_Y^2)_0) \subset (1_X, V_r^1(\Omega_X^2)_0)$.

The isomorphisms $V_r^1(\Omega_X^1)_0 \simeq V_r^1(\Omega_Y^1)_0$ follow in the same way by using Corollary 3.0.8 with $m = 0$ and $i = 3$. □

Corollary 5.2.2. *Let X and Y be smooth projective D -equivalent threefolds, then $\dim \text{alb}_X(X) = \dim \text{alb}_Y(Y)$.*

Proof. It is enough to use Theorem 5.2.1 and (2.8). □

5.2.2 Variant 4.1.2 in dimension three

In this section we study Variant 4.1.2 in dimension three. In particular we prove Variant 4.1.2 in some particular cases. We present our results in two steps.

Proposition 5.2.3. *Let X and Y be smooth projective D -equivalent threefolds and F be the induced Rouquier isomorphism. Assume that either $\text{Aut}^0(X)$ is affine, or that $V^p(\Omega_X^q \otimes \omega_X^m) = \text{Pic}^0(X)$ for some $p, q \geq 0$ and $m \in \mathbf{Z}$ (for instance this happens if $\chi(\Omega_X^q \otimes \omega_X^m) \neq 0$). Then F induces isomorphisms of algebraic sets $V_r^p(\Omega_X^q) \simeq V_r^p(\Omega_Y^q)$ for all $p, q \geq 0$ and $r \geq 1$.*

Proof. By Remark 4.4.5 we have $F(1_X, \text{Pic}^0(X)) = (1_Y, \text{Pic}^0(Y))$. The isomorphisms $V_r^0(\omega_X) \simeq V_r^0(\omega_Y)$ and $V_r^1(\omega_X) \simeq V_r^1(\omega_Y)$ hold by Proposition 4.2.1 and Remark 4.4.5 respectively. The isomorphisms $V_r^2(\omega_X) \simeq V_r^2(\omega_Y)$ follow by Corollary 3.0.8 as in Proposition 5.2.1.

We now establish the isomorphisms $V_r^1(\Omega_X^2) \simeq V_r^1(\Omega_Y^2)$. Let $\alpha \in V_r^1(\Omega_X^2)$ so that, by Lemma 4.3.3, $F(1_X, \alpha) = (1_Y, \beta)$. By Corollary 3.0.8 (with $m = 0$ and $i = 2$), Serre duality, and the Hodge linear-conjugate isomorphism, we get $h^1(X, \Omega_X^2 \otimes \alpha) = h^1(Y, \Omega_Y^2 \otimes \beta)$. This shows that F maps $1_X \times V_r^1(\Omega_X^2) \mapsto 1_Y \times V_r^1(\Omega_Y^2)$, inducing the desired isomorphisms as in Proposition 4.2.1. Finally, the isomorphisms $V_r^1(\Omega_X^1) \simeq V_r^1(\Omega_Y^1)$ follow in the same way by using Corollary 3.0.8 with $m = 0$ and $i = 3$. □

Remark 5.2.4. *The previous proposition also holds if the image of the Albanese map is not fibered in tori, as in this case $\text{Aut}^0(X)$ is affine by results of Nishi (cf. (30) p. 154).*

Proposition 5.2.5. *Let X and Y be smooth projective D -equivalent threefolds and F be the induced Rouquier isomorphism. If X is of maximal Albanese dimension, then F induces isomorphisms of algebraic sets $V_r^i(\omega_X) \simeq V_r^i(\omega_Y)$ for all $i \geq 0$ and $r \geq 1$.*

Proof. Recall that by (2.9) there are inclusions $V^0(\omega_X) \supset V^1(\omega_X) \supset V^2(\omega_X) \supset V^3(\omega_X) = \{\mathcal{O}_X\}$. Hence Proposition 4.2.1 and Corollary 4.5.4 yield isomorphisms $V_r^i(\omega_X) \simeq V_r^i(\omega_Y)$ for any $i \neq 2$. We now focus on the case $i = 2$. Since $V_r^2(\omega_X) \subset V^2(\omega_X) \subset V^0(\omega_X)$, by Proposition 4.2.1 we have that $F(1_X, V_r^2(\omega_X)) \subset (1_Y, \text{Pic}^0(Y))$. Thus, by Corollary 3.0.8, we get $h^2(X, \omega_X \otimes \alpha) = h^2(Y, \omega_Y \otimes \beta)$ for any $\alpha \in V^2(\omega_X)$ and therefore F and F^{-1} induce the desired isomorphisms. \square

5.2.3 Derived invariance of the dimensions of cohomological support loci

In this section we provide further evidence to Variant 4.1.2. We aim to prove the following

Theorem 5.2.6. *Let X and Y be smooth projective D -equivalent threefolds with $q(X) \geq 2$. Then $\dim V^i(\omega_X) = \dim V^i(\omega_Y)$ for all $i \geq 0$.*

The proof of this theorem is rather long and technical. Before jumping into technicalities, we first present the plan of its proof.

Thanks to Propositions 5.2.3 and 5.2.5 we can assume that X is a threefold with $\dim \text{alb}_X(X) \leq 2$, $V^0(\omega_X) \subsetneq \text{Pic}^0(X)$ and with non-affine $\text{Aut}^0(X)$. In particular, we can assume that X is not of general type and that $\chi(\omega_X) = 0$ by Lemma 4.3.2. Thanks to Proposition 4.2.1, Theorem 4.5.2, and (9) Theorem A (1) the Fourier-Mukai partner Y satisfies the same hypotheses as X . Hence Theorem 5.2.6 follows as soon as we classify $\dim V^i(\omega_X)$ and $\dim V^i(\omega_Y)$ in terms of de-

rived invariants. This classification is carried out in Propositions 5.2.9-5.2.13 where $\dim V^1(\omega_X)$ and $\dim V^2(\omega_X)$ are computed in terms of $\kappa(X)$, $q(X)$, $\dim \text{alb}_X(X)$ and $\dim V^0(\omega_X)$.

The main tools we use in the proofs of Propositions 5.2.9-5.2.13 are generic vanishing theory (cf. Theorems 2.2.2 and 2.2.3), Kollár's result on higher direct images of the canonical bundle (cf. Theorem 5.2.7), and the classification of smooth projective surfaces (cf. for instance (41)).

Theorem 5.2.7. *Let X and Y be complex smooth projective varieties and let $f : X \rightarrow Y$ be a surjective morphism. Then:*

(i). *The higher direct images $R^i f_* \omega_X$ are torsion-free for $i \geq 0$. Moreover, $R^i f_* \omega_X = 0$ for $i > \dim X - \dim Y$.*

(ii). *In $\mathbf{D}(Y)$ there is a decomposition $\mathbf{R}f_* \omega_X \simeq \bigoplus_i R^i f_* \omega_X[-i]$. In particular, for any $i \geq 0$, we have*

$$h^i(X, \omega_X \otimes f^* \alpha) = \sum_{p+q=i} h^p(Y, R^q f_* \omega_X \otimes \alpha).$$

(iii). *If f has connected fibers, then there exists an isomorphism $R^{\dim X - \dim Y} f_* \omega_X \simeq \omega_Y$.*

The previous theorem is due to Kollár and is proved in (48) Theorem 2.1, (49) Theorem 3.1 and Corollary 3.2. Moreover, it holds in much generality (e.g. in (i) and (ii) Y can be assumed reduced and projective). Before starting the computation of the dimensions of cohomological support loci, we show a simple useful lemma.

Lemma 5.2.8. *Let X and Y be smooth projective varieties and let $f : X \rightarrow Y$ be a surjective morphism with connected fibers. If h denotes the dimension of the general fiber of f , then $f^* V^i(\omega_Y) \subset V^{i+h}(\omega_X)$ for any $i = 0, \dots, \dim Y$.*

Proof. By Theorem 5.2.7 we have: $R^h f_* \omega_X \simeq \omega_Y$, $R^i f_* \omega_X = 0$ for $i > h$, and moreover

$$h^{i+h}(X, \omega_X \otimes f^* \alpha) = h^i(Y, \omega_Y \otimes \alpha) + \sum_{l \neq i} h^l(Y, R^{h+i-l} f_* \omega_X \otimes \alpha) \neq 0$$

for any $\alpha \in \text{Pic}^0(Y)$. At this point we get the thesis as the pull-back homomorphism $f^* : \text{Pic}^0(X) \rightarrow \text{Pic}^0(Y)$ is injective since the fibers of f are connected. \square

Proposition 5.2.9. *Let X be a smooth projective threefold such that $\kappa(X) = 2$, $\dim \text{alb}_X(X) = 2$, $\chi(\omega_X) = 0$ and $V^0(\omega_X) \subsetneq \text{Pic}^0(X)$. If $q(X) = 2$, then we have: i) $\dim V^2(\omega_X) = 0$; ii) $\dim V^1(\omega_X) = 1$ if and only if $\dim V^0(\omega_X) = 1$; and iii) $\dim V^1(\omega_X) = 0$ if and only if $\dim V^0(\omega_X) \leq 0$. If $q(X) > 2$, then we have $\dim V^1(\omega_X) = \dim V^2(\omega_X) = q(X) - 1$.*

Proof. Since the problem is invariant under birational modification, with a little abuse of notation, we consider a non-singular representative $f : X \rightarrow S$ of the Iitaka fibration of X (cf. (26) (1.10)), so that X and S are smooth varieties and f is an algebraic fiber space. We divide the proof into three cases according to the values of the Albanese dimension of S .

Case I: $\dim \text{alb}_S(S) = 2$. By the classification theory of smooth projective surfaces, S is either a surface of general type, or birational to an abelian surface, or birational to an elliptic surface fibered onto a curve of genus ≥ 2 . Moreover, by Lemma 4.5.1, we have $q(X) = q(S)$.

If S is of general type, then by Castelnuovo's Theorem (cf. (41) Theorem X.4) we have $\chi(\omega_S) > 0$ and hence $V^0(\omega_S) = \text{Pic}^0(S)$. Therefore, by Lemma 5.2.8, we get $V^1(\omega_X) = \text{Pic}^0(X)$, and consequently $V^0(\omega_X) = \text{Pic}^0(X)$ since $\chi(\omega_X) = 0$ and $V^2(\omega_X) \subsetneq \text{Pic}^0(X)$ (cf. (2.8)). This contradicts our hypotheses and hence this case does not occur.

If S is birational to an abelian surface, then we have $q(X) = q(S) = 2$ and $f^*\text{Pic}^0(S) = \text{Pic}^0(X)$. By using Theorem 5.2.7 (ii), we obtain equalities

$$h^2(X, \omega_X \otimes f^*\alpha) = h^2(S, f_*\omega_X \otimes \alpha) + h^1(S, R^1f_*\omega_X \otimes \alpha)$$

for any $\alpha \in \text{Pic}^0(S)$. Moreover, we note that $R^1f_*\omega_X \simeq \omega_S$ and $R^2f_*\omega_X = 0$ (cf. Theorem 5.2.7 (i) and (iii)). Therefore, since by Theorem 2.2.3 (ii) $f_*\omega_X$ is a *GV*-sheaf on S (i.e. $\text{codim}_{\text{Pic}^0(S)} V^i(f_*\omega_X) \geq i$ for $i > 0$), we get $\dim V^2(\omega_X) = 0$. At this point the statements ii) and iii) of the proposition follow as $\chi(\omega_X) = 0$ and $\dim V^1(\omega_X) \geq 0$ (note that $\mathcal{O}_X \in V^1(\omega_X)$ since $q(X) = 2$).

If S is birational to an elliptic surface $h : S \rightarrow C$ fibered onto a curve C of genus $g(C) = q(S) - 1 = q(X) - 1 \geq 2$, then X is fibered onto C as well. Therefore we have $V^0(\omega_C) = \text{Pic}^0(C)$, and consequently $V^2(\omega_X)$ is of codimension one in $\text{Pic}^0 X$ by Lemma 5.2.8 and (2.8). Since $\chi(\omega_X) = 0$, $V^1(\omega_X)$ is of codimension one as well.

Case II: $\dim \text{alb}_S(S) = 1$. We have $q(X) = q(S) + 1$ by Lemma 4.5.1. Moreover, alb_S has connected fibers, and by (41) Proposition V.15 $\text{alb}_S(S)$ is a smooth curve of genus $q(S)$. We distinguish two subcases: $q(S) = 1$ and $q(S) \geq 2$.

If $q(S) = 1$, then $q(X) = 2$ and alb_X is surjective. Let $X \xrightarrow{b} Z \rightarrow \text{Alb}(X)$ be the Stein factorization of alb_X , and let $b' : X' \rightarrow Z'$ be a non-singular representative of b . We note that Z' is a smooth surface with $q(Z') = 2$ and of maximal Albanese dimension. Therefore either Z' is of general type, or it is birational to an abelian surface. However, we have just seen that

Z' cannot possibly be of general type, therefore Z' is birational to an abelian surface and the same calculations of the previous case apply.

If $q(S) \geq 2$, then the Albanese map of S induces a fibration of S onto a smooth curve C of genus $g(C) = q(S)$. Therefore X is fibered onto C as well, and we conclude as in the previous case.

Case III: $\dim \text{alb}_S(S) = 0$. As we have seen in the proof of Lemma 4.5.1, the image of a general fiber of f is mapped via alb_X onto a fiber of the induced morphism $f_* : \text{Alb}(X) \rightarrow \text{Alb}(S)$. On the other hand, if $\dim \text{alb}_S(S) = 0$, then $\text{Alb}(S)$ is trivial. This yields a contradiction and therefore this case does not occur. \square

Proposition 5.2.10. *Let X be a smooth projective threefold such that $\kappa(X) = 2$, $\dim \text{alb}_X(X) = 1$, $\chi(\omega_X) = 0$ and $V^0(\omega_X) \subsetneq \text{Pic}^0(X)$. If $q(X) = 1$, then we have $\dim V^1(\omega_X) \leq 0$ and $\dim V^2(\omega_X) = 0$. On the other hand, if $q(X) > 1$ then we have $V^1(\omega_X) = V^2(\omega_X) = \text{Pic}^0(X)$.*

Proof. As in the previous proof we denote by $f : X \rightarrow S$ a non-singular representative of the Iitaka fibration of X . By Lemma 4.5.1, we have $q(X) - q(S) = 1 - \dim \text{alb}_S(S)$. We distinguish two cases: $\dim \text{alb}_S(S) = 0$ and $\dim \text{alb}_S(S) = 1$.

Case I: $\dim \text{alb}_S(S) = 0$. In this case we have $q(S) = 0$, $q(X) = 1$ and also that alb_X is surjective since $\dim \text{alb}_X(X) = 1$. Moreover, alb_X has connected fibers by (50) Lemma 2.11. We set $E := \text{Alb}(X)$ and $a := \text{alb}_X$. By Theorem 5.2.7 we have $R^2 a_* \omega_X \simeq \mathcal{O}_E$, $R^i a_* \omega_X = 0$ for $i > 2$, and equalities

$$h^2(X, \omega_X \otimes a^* \alpha) = h^1(E, R^1 a_* \omega_X \otimes \alpha) + h^0(E, \alpha)$$

for any $\alpha \in \text{Pic}^0(E) \simeq \text{Pic}^0(X)$. Finally, by Theorem 2.2.3 (ii), $R^1 a_* \omega_X$ is a *GV*-sheaf on E (i.e. $\text{codim } V^1(R^1 a_* \omega_X) \geq 1$). Hence we have $\dim V^2(\omega_X) = 0$, and therefore $V^1(\omega_X)$ is either empty or zero-dimensional as $V^0(\omega_X) \subsetneq \text{Pic}^0(X)$ and $\chi(\omega_X) = 0$.

Case II: $\dim \text{alb}_S(S) = 1$. By Lemma 4.5.1 we have $q(X) = q(S)$. We distinguish two subcases: $q(S) = 1$ and $q(S) > 1$. If $q(S) = q(X) = 1$, then the image of alb_X is an elliptic curve and the same argument of the previous case applies. Suppose now $q(S) = q(X) > 1$, so that alb_S has connected fibers and its image is a smooth curve B of genus $g(B) = q(S) > 1$. By noting that $\text{Pic}^0(X) \simeq \text{Pic}^0(S) \simeq \text{Pic}^0(B)$ and $V^0(\omega_B) = \text{Pic}^0(B)$, we remark that, by Lemma 5.2.8, there are inclusions

$$\text{alb}_S^* \text{Pic}^0(B) = \text{alb}_S^* V^0(\omega_B) \subset V^1(\omega_S) \subset \text{Pic}^0(S),$$

which force $V^1(\omega_S) = \text{Pic}^0(S)$. Finally, another application of Lemma 5.2.8 gives $f^* V^1(\omega_S) \subset V^2(\omega_X) \subset \text{Pic}^0(X)$. Therefore we have $V^2(\omega_X) = \text{Pic}^0(X)$ and consequently that $V^1(\omega_X) = \text{Pic}^0(X)$. \square

Proposition 5.2.11. *Let X be a smooth projective threefold such that $\kappa(X) = 1$, $\chi(\omega_X) = 0$ and $V^0(\omega_X) \subsetneq \text{Pic}^0(X)$.*

- (i). *Assume $\dim \text{alb}_X(X) = 2$. If $q(X) = 2$, then we have: i) $\dim V^2(\omega_X) = 0$; ii) $\dim V^1(\omega_X) = 1$ if and only if $\dim V^0(\omega_X) = 1$; and iii) $\dim V^1(\omega_X) = 0$ if and only if $\dim V^0(\omega_X) \leq 0$.*
- If $q(X) \geq 3$, then we have $\dim V^1(\omega_X) = \dim V^2(\omega_X) = q(X) - 1$.*

(ii). Assume $\dim \text{alb}_X(X) = 1$. If $q(X) = 1$, then we have $\dim V^1(\omega_X) \leq 0$ and $\dim V^2(\omega_X) = 0$. If $q(X) \geq 2$, then we obtain $V^1(\omega_X) = V^2(\omega_X) = \text{Pic}^0(X)$.

Proof. We start with the case $\dim \text{alb}_X(X) = 2$. Let $f : X \rightarrow C$ be a non-singular representative of the Iitaka fibration of X where C is a smooth curve.

If $g(C) \geq 2$, then by Lemma 4.5.1 we have $q(X) = g(C) + 1 \geq 3$, and by Lemma 5.2.8 we obtain a series of inclusions $f^*\text{Pic}^0(C) = f^*V^0(\omega_C) \subset V^2(\omega_X) \subset \text{Pic}^0(X)$. We conclude that

$$\dim V^2(\omega_X) = q(X) - 1$$

since $V^2(\omega_X) \subsetneq \text{Pic}^0(X)$ by (2.8). Therefore we see that $V^1(\omega_X) \subsetneq \text{Pic}^0(X)$ as $\chi(\omega_X) = 0$ and $V^0(\omega_X) \subsetneq \text{Pic}^0(X)$. Finally, thanks to the inclusion $V^1(\omega_X) \supset V^2(\omega_X)$ of (2.9) we obtain $\dim V^1(\omega_X) = q(X) - 1$.

If $g(C) \leq 1$, then $q(X) = 2$ and $a := \text{alb}_X$ is surjective. Let $b : X' \rightarrow Z'$ be a non-singular representative of the Stein factorization of a . Then, as we have seen in the proof Proposition 5.2.9, Z' is birational to an abelian surface, and therefore $\dim V^2(\omega_X) = 0$. Since $\mathcal{O}_X \in V^1(\omega_X)$, we obtain the statements ii) and iii) of the proposition.

We now study the case $\dim \text{alb}_X(X) = 1$. If $g(C) \geq 2$, then it turns out that $q(X) = g(C)$ and $f^*\text{Pic}^0(C) = \text{Pic}^0(X)$. Therefore, by Lemma 5.2.8, we get $V^2(\omega_X) = \text{Pic}^0(X)$, and hence we have $V^1(\omega_X) = \text{Pic}^0(X)$. On the other hand, if $g(C) \leq 1$, then $q(X) = 1$ and $\text{alb}_X : X \rightarrow \text{Alb}(X)$ is an algebraic fiber space onto an elliptic curve. We conclude then as in the proof of Proposition 5.2.10. \square

Proposition 5.2.12. *Let X be a smooth projective threefold such that $\kappa(X) = 0$ and $\chi(\omega_X) = 0$. If $\dim \text{alb}_X(X) = 2$, then we have $\dim V^1(\omega_X) = \dim V^2(\omega_X) = 0$. On the other hand, if $\dim \text{alb}_X(X) = 1$ then $\dim V^1(\omega_X) \leq 0$ and $\dim V^2(\omega_X) = 0$.*

Proof. We recall that, by (51) Lemma 3.1, $V^0(\omega_X)$ consists of at most one point. We start with the case $\dim \text{alb}_X(X) = 2$. By (42) Theorem 1, alb_X is surjective and has connected fibers. Therefore we have $q(X) = h^2(X, \omega_X) = 2$ and hence $\mathcal{O}_X \in V^1(\omega_X)$ since $\chi(\omega_X) = 0$. We set $a := \text{alb}_X$ and we note that, by (22) Corollary 4.2, $a_*\omega_X$ is a GV -sheaf, i.e.

$$\text{codim}V^1(a_*\omega_X) \geq 1 \quad \text{and} \quad \text{codim}V^2(a_*\omega_X) \geq 2.$$

By using that $R^1a_*\omega_X \simeq \mathcal{O}_{\text{Alb}(X)}$ and $R^2a_*\omega_X = 0$ (cf. Theorem 5.2.7 (i) and (iii)), and by using Theorem 5.2.7 (ii), we get isomorphisms $H^1(X, \omega_X \otimes a^*\alpha) \simeq H^1(\text{Alb}(X), a_*\omega_X \otimes \alpha) \oplus H^0(\text{Alb}(X), \alpha)$ for any $\alpha \in \text{Pic}^0(\text{Alb}(X)) \simeq \text{Pic}^0(X)$. Therefore we have

$$\text{codim}V^1(\omega_X) \geq 1 \quad \text{and} \quad \text{codim}V^2(\omega_X) \geq 2,$$

and consequently the hypothesis $\chi(\omega_X) = 0$ implies $\dim V^1(\omega_X) = 0$.

If $\dim \text{alb}_X(X) = 1$, then as in the previous case we have $\dim V^2(\omega_X) = 0$. Therefore $V^1(\omega_X)$ is either empty or zero-dimensional since $\chi(\omega_X) = 0$. \square

Proposition 5.2.13. *Let X be a smooth projective threefold with $\kappa(X) = -\infty$ and $\chi(\omega_X) = 0$.*

(i). Suppose $\dim \text{alb}_X(X) = 2$. If $q(X) = 2$, then we have $V^1(\omega_X) = V^2(\omega_X) = \{\mathcal{O}_X\}$. If $q(X) > 2$, then we obtain $\dim V^1(\omega_X) = \dim V^2(\omega_X) = q(X) - 1$.

(ii). Suppose $\dim \text{alb}_X(X) = 1$. If $q(X) = 1$, then we have $\dim V^1(\omega_X) \leq 0$ and $\dim V^2(\omega_X) = 0$. If $q(X) > 1$, then we obtain $V^1(\omega_X) = V^2(\omega_X) = \text{Pic}^0(X)$.

Proof. We start with the case $\dim \text{alb}_X(X) = 2$. Let $a : X \rightarrow S \subset \text{Alb}(X)$ be the Albanese map of X , $b : X \rightarrow S'$ be the Stein factorization of a , and let $c : X' \rightarrow S''$ be a non-singular representative of b . We can easily check that $q(X') = q(S'')$, $\dim \text{alb}_S(S) = 2$, and hence that $\kappa(S'') \geq 0$. Furthermore, we have $c_*\omega_{X'} = 0$. To see this, we point out that by Theorem 2.2.3 (ii) $c_*\omega_{X'}$ is a GV -sheaf on S'' , and moreover that by Lemma 5.1.1 $V^0(c_*\omega_{X'}) = V^0(\omega_{X'}) = V^0(\omega_X) = \emptyset$. This immediately implies $c_*\omega_{X'} = 0$ as a GV -sheaf \mathcal{F} is non-zero if and only if $V^0(\mathcal{F}) \neq \emptyset$. We distinguish now three cases according to the values of $\kappa(S'')$.

If $\kappa(S'') = 0$, then S'' is birational to an abelian surface. This forces $q(X) = q(X') = q(S'') = 2$ and $c^*\text{Pic}^0(S'') = \text{Pic}^0(X')$. Moreover, by Theorem 5.2.7, for any $\alpha \in \text{Pic}^0(S'')$ we obtain equalities

$$h^2(X', \omega_{X'} \otimes c^*\alpha) = h^1(S'', \omega_{S''} \otimes \alpha), \quad h^1(X', \omega_{X'} \otimes c^*\alpha) = h^0(S'', \omega_{S''} \otimes \alpha)$$

leading to isomorphisms $V^2(\omega_X) \simeq V^2(\omega_{X'}) = c^*V^1(\omega_{S''}) = \{\mathcal{O}_{X'}\}$ and $V^1(\omega_X) \simeq V^1(\omega_{X'}) = c^*V^0(\omega_{S''}) = \{\mathcal{O}_{X'}\}$.

If $\kappa(S'') = 1$, then S'' is birational to an elliptic surface of maximal Albanese dimension fibered onto a curve of genus $g(C) \geq 2$. Thus X is fibered onto C as well and $q(X') = q(S'') = g(C) + 1$. By Lemma 5.2.8 and (2.8), we deduce $\dim V^2(\omega_{X'}) = g(C) = q(X') - 1$, and therefore we get $\dim V^1(\omega_{X'}) = q(X') - 1$ as $\chi(\omega_{X'}) = 0$ and $V^0(\omega_{X'}) = \emptyset$.

If $\kappa(S'') = 2$, then by Castelnuovo's Theorem we have $\chi(\omega_{S''}) > 0$, which immediately yields $V^0(\omega_{S''}) = \text{Pic}^0(S'')$. By using Lemma 5.2.8, we see that $\dim V^0(\omega_{X'}) > 0$. This contradicts Lemma 5.1.1 and hence this case does not occur.

We now suppose $\dim \text{alb}_X(X) = 1$. Let $a : X \rightarrow C \subset \text{Alb}(X)$ be the Albanese map of X where $C := \text{Im } a$. Then a has connected fibers and $q(X) = g(C)$ by (50) Lemma 2.11. As in the previous case, we note that $a_*\omega_X = 0$. Moreover, by Theorem 5.2.7, we obtain equalities

$$\begin{aligned} h^1(X, \omega_X \otimes a^*\alpha) &= h^0(C, R^1a_*\omega_X \otimes \alpha) \\ h^2(X, \omega_X \otimes a^*\alpha) &= h^1(C, R^1a_*\omega_X \otimes \alpha) + h^0(C, \omega_C \otimes \alpha) \end{aligned}$$

for any $\alpha \in \text{Pic}^0(C)$. At this point we distinguish two cases: $g(C) = 1$ and $g(C) > 1$. If $g(C) = q(X) > 1$, then we have $V^0(\omega_C) = \text{Pic}^0(C)$, and by Lemma 5.2.8 we get $V^2(\omega_X) = V^1(\omega_X) = \text{Pic}^0(X)$. On the other hand, if $g(C) = q(X) = 1$, then by (22) Corollary 4.2 $R^1a_*\omega_X$ is a GV -sheaf on $C = \text{Alb}(X)$. Hence we obtain $\dim V^2(\omega_X) = 0$, and consequently we see that $\dim V^1(\omega_X) \leq 0$ since $\chi(\omega_X) = 0$ and $V^0(\omega_X) = \emptyset$. \square

Remark 5.2.14. *In the case $q(X) = 1$, the previous propositions yield the following statement: for each i , $\dim V^i(\omega_X) = 1$ if and only if $\dim V^i(\omega_Y) = 1$. In general, we have not been able*

to show that if a locus $V^i(\omega_X)$ is empty (resp. of dimension zero) then the corresponding locus $V^i(\omega_Y)$ is empty (resp. of dimension zero). This ambiguity is mainly caused by the possible presence of non-trivial automorphisms.

In the next proposition we provide a further class of threefolds for which Variant 4.1.2 holds.

Proposition 5.2.15. *If X and Y are smooth projective D -equivalent threefolds with $q(X) = 2$, then $V^i(\omega_X) \simeq V^i(\omega_Y)$ for all $i \geq 0$*

Proof. We can assume that both X and Y are not of general type, with $V^0(\omega_X) \subsetneq \text{Pic}^0(X)$, and $V^0(\omega_Y) \subsetneq \text{Pic}^0(Y)$ (cf. Proposition 5.2.3). Moreover, we can suppose that $\text{Aut}^0(X)$ is not affine, and hence that $\text{Aut}^0(Y)$ is not affine as well (cf. (9) Theorem A (1)). In particular, by (9) Corollary 2.6, we have $\chi(\omega_X) = \chi(\omega_Y) = 0$.

We proceed now with a case by case analysis according to the values of $\kappa(X)$ and $\dim \text{alb}_X(X)$. Let us start with the case $\kappa(X) = 2$. If $\dim \text{alb}_X(X) = 1$, then $\text{alb}_X(X)$ is a smooth curve of genus ≥ 2 and we conclude then by using Remark 5.2.4.

Suppose now $\dim \text{alb}_X(X) = 2$. Then, by Proposition 5.2.9, we get $\dim V^1(\omega_X) \leq 1$ and $\dim V^2(\omega_X) = 0$. Moreover, it turns out that

Claim 5.2.16. $V^0(\omega_X) \setminus \{\mathcal{O}_X\} \supset V^1(\omega_X) \setminus \{\mathcal{O}_X\} \supset V^2(\omega_X) \setminus \{\mathcal{O}_X\}$.

Proof. The inclusion $V^1(\omega_X) \setminus \{\mathcal{O}_X\} \supset V^2(\omega_X) \setminus \{\mathcal{O}_X\}$ follows from (2.9) as $\dim \text{alb}_X(X) = 2$, so we only focus on the first inclusion. We proceed by contradiction. We suppose then that there exists $\mathcal{O}_X \neq \alpha \in V^1(\omega_X) \setminus V^0(\omega_X)$ so that, since $\chi(\omega_X) = 0$,

$$\alpha \in V^2(\omega_X) \quad \text{and} \quad h^1(X, \omega_X \otimes \alpha) = h^2(X, \omega_X \otimes \alpha) > 0. \quad (5.1)$$

We now show that we reach to a contradiction by studying derivative complexes associated to $\omega_X \otimes \alpha$. To this end, for any $v \in H^1(X, \mathcal{O}_X)$, we consider the following complexes of vector spaces

$$0 \longrightarrow H^1(X, \omega_X \otimes \alpha) \xrightarrow{\cup v} H^2(X, \omega_X \otimes \alpha) \longrightarrow 0 \quad (5.2)$$

where the morphisms are given by cupping with v . Generic vanishing theory (*cf.* for instance (21) Theorem 1.2) tells us that $\cup v$ is surjective for any $0 \neq v \in H^1(X, \mathcal{O}_X)$ since α is isolated in $V^2(\omega_X)$ (recall that $\dim V^2(\omega_X) = 0$). Moreover, by (5.1), this implies that (5.2) is exact for any $v \neq 0$. We now globalize the previous complexes into a complex of locally free sheaves on $\mathbf{P} \stackrel{\text{def}}{=} \mathbf{P}(H^1(X, \mathcal{O}_X))$:

$$0 \longrightarrow H^1(X, \omega_X \otimes \alpha) \otimes \mathcal{O}_{\mathbf{P}}(-2) \xrightarrow{\cup v} H^2(X, \omega_X \otimes \alpha) \otimes \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow 0, \quad (5.3)$$

which is exact since exactness can be checked at the level of fibers. This clearly yields a contradiction. \square

The previous claim applies to Y as well, so we get $V^0(\omega_Y) \setminus \{\mathcal{O}_Y\} \supset V^1(\omega_Y) \setminus \{\mathcal{O}_Y\} \supset V^2(\omega_Y) \setminus \{\mathcal{O}_Y\}$. By Corollary 4.4.3, and by the invariance of Hodge numbers in dimension three, the Rouquier isomorphism F induces an isomorphism

$$V^1(\omega_X) = V^0(\omega_X) \cap V^1(\omega_X) \simeq V^0(\omega_Y) \cap V^1(\omega_Y) = V^1(\omega_Y).$$

Moreover, since by Proposition 4.2.1 $F(1_X, V^2(\omega_X)) \subset (1_Y, \text{Pic}^0(Y))$, the Rouquier isomorphism induces a further isomorphism $V^2(\omega_X) \simeq V^2(\omega_Y)$. This concludes the case $\kappa(X) = 2$ and $\dim \text{alb}_X(X) = 2$.

We now suppose $\kappa(X) = 1$. If $\dim \text{alb}_X(X) = 2$, then it is enough to apply the previous argument. On the other hand, if $\dim \text{alb}_X(X) = 1$, then $V^1(\omega_X) = \text{Pic}^0(X)$ and we conclude then thanks to Proposition 5.2.3.

We suppose now $\kappa(X) = 0$. By (42) Theorem 1, alb_X is surjective with connected fibers, and by Proposition 5.2.12 we have $\dim V^1(\omega_X) = \dim V^2(\omega_X) = 0$. Moreover, by arguing as in the proof of Claim 5.2.16, we can show that $V^0(\omega_X) \supset V^1(\omega_X) \setminus \{\mathcal{O}_X\}$. This allow us to conclude as in the case $\kappa(X) = 2$.

Finally, if $\kappa(X) = -\infty$, then by Proposition 5.2.13 either $V^1(\omega_X) = V^2(\omega_X) = \{\mathcal{O}_X\}$ or $V^1(\omega_X) = V^2(\omega_X) = \text{Pic}^0(X)$. Hence the Rouquier isomorphism induces the desired isomorphisms. □

CHAPTER 6

COMPARISON OF COHOMOLOGICAL SUPPORT LOCI AND FIBRATIONS

In this chapter we present a relationship between the derived invariance of Hodge numbers of type $h^{0,i}(X)$ and the derived invariance of cohomological support loci of type $V^{\dim X-i}(\omega_X)_0$. More precisely, we show that if for some i the Hodge number $h^{0,i}(X)$ is invariant under derived equivalence, then the corresponding locus $V^{\dim X-i}(\omega_X)_0$ is a derived invariant as well (*cf.* Theorem 6.2.1). The proof of this result relies on the use of liftings of derived equivalences of smooth projective varieties to derived equivalences of special types of étale cyclic covers associated to topologically trivial torsion line bundles. This approach is inspired by a theorem of Bridgeland and Maciocia studying derived liftings of varieties having torsion canonical bundles. (*cf.* Theorem 6.1.6).

As an application, we prove the derived invariance of $V^{\dim X-1}(\omega_X)_0$, the most geometric locus among the other loci as it controls fibrations onto smooth curves. In this way we prove that if X and Y are Fourier-Mukai partners and $X \rightarrow C$ is a fibration onto a smooth curve of genus $g \geq 2$, then Y admits a fibration onto a smooth curve of genus $\geq g$. In the same spirit, we study Fourier-Mukai partners of Fano fibrations and Mori fiber spaces over curves of genus at least two.

In this chapter we work over the field of complex numbers.

6.1 Derived equivalences of étale cyclic covers

We start by reviewing some properties of derived categories of étale cyclic covers associated to topologically trivial torsion line bundles.

6.1.1 Étale cyclic covers

Let X be a complex smooth projective variety and let α be a d -torsion element of $\text{Pic}^0(X)$.

We denote by

$$\pi_\alpha : X_\alpha \longrightarrow X$$

the étale cyclic cover associated to α (*cf.* e.g. (19) §7.3). Then

$$\pi_{\alpha*} \mathcal{O}_{X_\alpha} \simeq \bigoplus_{i=0}^{d-1} \alpha^{-i} \tag{6.1}$$

and there is a free action of the group $G := \mathbf{Z}/d\mathbf{Z}$ on X_α such that $X_\alpha/G \simeq X$. The following lemma is analogous to (12) Proposition 2.5. We include a proof for completeness.

Lemma 6.1.1. (i). *Let E be an object in $\mathbf{D}(X)$. Then there is an object E_α in $\mathbf{D}(X_\alpha)$ such that $\pi_{\alpha*} E_\alpha \simeq E$ if and only if $E \otimes \alpha \simeq E$.*

(ii). *Let E_α be an object in $\mathbf{D}(X_\alpha)$. Then there is an object E in $\mathbf{D}(X)$ such that $\pi_\alpha^* E \simeq E_\alpha$ if and only if $g^* E_\alpha \simeq E_\alpha$ for some generator g of G .*

Proof. (i). If $\pi_{\alpha*} E_\alpha \simeq E$, then by projection formula we have $E \otimes \alpha \simeq \pi_{\alpha*} E_\alpha \otimes \alpha \simeq \pi_{\alpha*} (E_\alpha \otimes \pi_\alpha^* \alpha) \simeq \pi_{\alpha*} E_\alpha \simeq E$.

For the other implication, let $s : E \xrightarrow{\cong} E \otimes \alpha$ be an isomorphism. We proceed by induction on the number r of non-zero cohomology sheaves of E . If E is a sheaf concentrated in degree zero, then the lemma is a standard fact. Indeed, it is well known that $\pi_{\alpha*} : \text{Coh}(X_\alpha) \rightarrow \text{Coh}(\mathcal{A})$ is an equivalence between the category of coherent \mathcal{O}_{X_α} -modules and the category of coherent $\mathcal{A} := (\bigoplus_{i=0}^{d-1} \alpha^i)$ -algebras, while a coherent sheaf E on X belongs to $\text{Coh}(\mathcal{A})$ if and only if $E \otimes \alpha \simeq E$.

Suppose now that the lemma is true for all objects having at most $r-1$ non-zero cohomology sheaves, and consider an object E with r non-zero cohomology sheaves. By shifting E , we can assume that $\mathcal{H}^i(E) = 0$ for $i \notin [-(r-1), 0]$. Since $E \otimes \alpha \simeq E$, we also have $\mathcal{H}^0(E) \otimes \alpha \simeq \mathcal{H}^0(E)$. Therefore, by the above, there exists a coherent sheaf M_α on X_α such that $\pi_{\alpha*} M_\alpha \simeq \mathcal{H}^0(E)$. Now the natural morphism $E \xrightarrow{j} \mathcal{H}^0(E)$ induces a distinguished triangle

$$E \xrightarrow{j} \mathcal{H}^0(E) \xrightarrow{f} F \longrightarrow E[1]$$

such that the object F has $r-1$ non-zero cohomology sheaves. By the commutativity of the following diagram

$$\begin{array}{ccccccc} E & \xrightarrow{j} & \mathcal{H}^0(E) & \xrightarrow{f} & F & \longrightarrow & E[1] \\ \downarrow & & \downarrow & & & & \downarrow \\ E \otimes \alpha & \xrightarrow{j \otimes \alpha} & \mathcal{H}^0(E) \otimes \alpha & \xrightarrow{f \otimes \alpha} & F \otimes \alpha & \longrightarrow & (E \otimes \alpha)[1], \end{array}$$

we obtain an isomorphism $F \simeq F \otimes \alpha$, and therefore by induction an object F_α in $\mathbf{D}(X_\alpha)$ such that $\pi_{\alpha*}F_\alpha \simeq F$.

To show the existence of an object E_α in $\mathbf{D}(X_\alpha)$ such that $\pi_{\alpha*}E_\alpha \simeq E$, we assume for a moment that there exists a morphism $f_\alpha : M_\alpha \rightarrow F_\alpha$ such that $\pi_{\alpha*}f_\alpha = f$. This is enough to conclude, since by completing f_α to a distinguished triangle

$$M_\alpha \xrightarrow{f_\alpha} F_\alpha \longrightarrow E_\alpha[1] \longrightarrow M_\alpha[1],$$

and applying $\pi_{\alpha*}$, we obtain $\pi_{\alpha*}E_\alpha \simeq E$.

We are left with showing the existence of f_α . Let $\lambda_\alpha : \pi_{\alpha*}M_\alpha \rightarrow \pi_{\alpha*}M_\alpha \otimes \alpha$ and $\mu_\alpha : \pi_{\alpha*}F_\alpha \rightarrow \pi_{\alpha*}F_\alpha \otimes \alpha$ be the isomorphisms determined by the diagram above. Note that

$$\mu_\alpha \circ f = (f \otimes \alpha) \circ \lambda_\alpha \quad \text{in} \quad \mathbf{D}(X). \quad (6.2)$$

We can replace F_α by an injective resolution

$$\cdots \longrightarrow \mathcal{I}_\alpha^{-1} \xrightarrow{d^{-1}} \mathcal{I}_\alpha^0 \xrightarrow{d^0} \mathcal{I}_\alpha^1 \xrightarrow{d^1} \cdots$$

so that f is represented (up to homotopy) by a morphism of \mathcal{O}_X -modules

$$u : \pi_{\alpha*}M_\alpha \longrightarrow \pi_{\alpha*}\mathcal{I}_\alpha^0.$$

Let V be the image of the map

$$\mathrm{Hom}(\pi_{\alpha*}M_{\alpha}, -) : \mathrm{Hom}(\pi_{\alpha*}M_{\alpha}, \pi_{\alpha*}\mathcal{I}_{\alpha}^{-1}) \longrightarrow \mathrm{Hom}(\pi_{\alpha*}M_{\alpha}, \pi_{\alpha*}\mathcal{I}_{\alpha}^0).$$

By (6.2), we have isomorphisms of \mathcal{O}_X -modules $a_1 : \pi_{\alpha*}M_{\alpha} \rightarrow \pi_{\alpha*}M_{\alpha} \otimes \alpha$ and $b_1 : \pi_{\alpha*}\mathcal{I}_{\alpha}^0 \rightarrow \pi_{\alpha*}\mathcal{I}_{\alpha}^0 \otimes \alpha$ such that

$$b_1 \circ u = (u \otimes \alpha) \circ a_1 \quad \text{up to homotopy.} \quad (6.3)$$

By setting $a_i := (a_1 \otimes \alpha^{i-1}) \circ \cdots \circ (a_1 \otimes \alpha) \circ a_1$ (for $i \geq 2$) and similarly for b_i , we define an action of $G := \mathbf{Z}/d\mathbf{Z}$ on V as

$$g^i \cdot (-) := b_i^{-1} \circ (- \otimes \alpha^i) \circ a_i,$$

where g is a generator of G . Moreover, we define operators A and B on V as

$$A := \sum_{i=0}^{d-1} g^i \cdot (-), \quad B := 1 - g \cdot (-).$$

Since $AB = 0$, we note that $\mathrm{Ker} A = \mathrm{Im} B$.

By (6.3), we have that $B(u) = u - b_1^{-1} \circ (u \otimes \alpha) \circ a_1$ is null-homotopic, and therefore $B(u) \in V$. Since $\mathrm{Ker} A = \mathrm{Im} B$, there exists a morphism $\eta \in V$ such that $B(\eta) = B(u)$. Consider the morphism $t := u - \eta \in \mathrm{Hom}(\pi_{\alpha*}M_{\alpha}, \pi_{\alpha*}\mathcal{I}_{\alpha}^0)$. It is easy to check that t is homotopic

to u and therefore it represents f as well. But now $B(t) = 0$, so $t = \pi_{\alpha^*}(v)$ for some morphism $v : M_\alpha \rightarrow \mathcal{I}_\alpha^0$, which concludes the proof.

(ii). The proof of this point is completely analogous to (12) Proposition 2.5 (a). \square

6.1.2 Equivariant lifts

We recall two definitions from (12) (*cf.* also (19) §7.3). Let \tilde{X} and \tilde{Y} be two smooth projective varieties on which the group $G := \mathbf{Z}/d\mathbf{Z}$ acts freely, and let $\pi_X : \tilde{X} \rightarrow X$ and $\pi_Y : \tilde{Y} \rightarrow Y$ be the corresponding quotient maps.

Definition 6.1.2. *A functor $\tilde{\Phi} : \mathbf{D}(\tilde{X}) \rightarrow \mathbf{D}(\tilde{Y})$ is equivariant if there exist an automorphism μ of G and isomorphisms of functors*

$$g^* \circ \tilde{\Phi} \simeq \tilde{\Phi} \circ \mu(g)^* \quad \text{for all } g \in G.$$

Definition 6.1.3. *Let $\Phi : \mathbf{D}(X) \rightarrow \mathbf{D}(Y)$ be a functor. A lift of Φ is a functor $\tilde{\Phi} : \mathbf{D}(\tilde{X}) \rightarrow \mathbf{D}(\tilde{Y})$ inducing isomorphisms*

$$\pi_{Y*} \circ \tilde{\Phi} \simeq \Phi \circ \pi_{X*} \tag{6.4}$$

$$\pi_Y^* \circ \Phi \simeq \tilde{\Phi} \circ \pi_X^*. \tag{6.5}$$

Remark 6.1.4. *If both $\Phi : \mathbf{D}(X) \rightarrow \mathbf{D}(Y)$ and $\tilde{\Phi} : \mathbf{D}(\tilde{X}) \rightarrow \mathbf{D}(\tilde{Y})$ are equivalences, then by taking the adjoints we see that (6.4) holds if and only if (6.5) holds.*

The following lemma is taken from (12) Lemma 4.3.

Lemma 6.1.5. *Let \tilde{X} be a smooth projective variety on which the group $G := \mathbf{Z}/d\mathbf{Z}$ acts freely and let $\pi : \tilde{X} \rightarrow X$ be the quotient map.*

- (i). *If $\Phi_{\tilde{\mathcal{E}}} : \mathbf{D}(\tilde{X}) \rightarrow \mathbf{D}(\tilde{X})$ is a functor such that $\pi_* \circ \Phi_{\tilde{\mathcal{E}}} \simeq \pi_*$, then $\Phi_{\tilde{\mathcal{E}}} \simeq g_*$ for some $g \in G$.*
- (ii). *If $\Phi_{\mathcal{E}} : \mathbf{D}(X) \rightarrow \mathbf{D}(X)$ is a functor such that $\Phi_{\mathcal{E}} \circ \pi_* \simeq \pi_*$, then $\Phi_{\mathcal{E}} \simeq (M \otimes -)$ for some line bundle M on X . If in addition $\pi = \pi_{\alpha}$ is the étale cyclic cover associated to a d -torsion element $\alpha \in \text{Pic}^0(X)$, then $\Phi_{\mathcal{E}} \simeq (\alpha^i \otimes -)$ for some i .*

Proof. Let $\tilde{x} \in \tilde{X}$ and $x \in X$ be points such that $\pi(\tilde{x}) = x$. Then we have $\pi_*(\Phi_{\tilde{\mathcal{E}}}(\mathcal{O}_{\tilde{x}})) \simeq \pi_*(\mathcal{O}_{\tilde{x}}) \simeq \mathcal{O}_x$ and therefore $\Phi_{\tilde{\mathcal{E}}}(\mathcal{O}_{\tilde{x}}) \simeq \mathcal{O}_{f(\tilde{x})}$ for some point $f(\tilde{x}) \in \pi^{-1}(x)$. Since this holds for an arbitrary point $\tilde{x} \in \tilde{X}$, the proof of (19) Corollary 5.23 implies that $f : \tilde{X} \rightarrow \tilde{X}$ is a morphism of varieties. Moreover, since $\pi \circ f = \pi$, we have that f is an isomorphism. We conclude then that $f = g$ for some $g \in G$. Finally, by following the same proof of *cf. loc. cit.*, we conclude that $\Phi_{\tilde{\mathcal{E}}} \simeq (L \otimes -) \circ g_*$ for some line bundle L on \tilde{X} . It remains to show that $L \simeq \mathcal{O}_{\tilde{X}}$. By taking the adjoints to $\pi_* \circ \Phi_{\tilde{\mathcal{E}}} \simeq \pi_*$, and by composing with $\Phi_{\tilde{\mathcal{E}}}$, we obtain $\pi^* \simeq \Phi_{\tilde{\mathcal{E}}} \circ \pi^*$. Therefore $(L \otimes g_* \pi^* \mathcal{O}_X) \simeq \pi^* \mathcal{O}_X \simeq \mathcal{O}_{\tilde{X}}$ and hence $L \simeq \mathcal{O}_{\tilde{X}}$.

We now prove (ii). As before, let $x \in X$ and $\tilde{x} \in \tilde{X}$ be points such that $\pi(\tilde{x}) = x$. Then we have $\Phi_{\mathcal{E}}(\mathcal{O}_x) \simeq \Phi_{\mathcal{E}}(\pi_*(\mathcal{O}_{\tilde{x}})) \simeq \pi_* \mathcal{O}_{\tilde{x}} \simeq \mathcal{O}_x$, and thus $\Phi_{\mathcal{E}} \simeq (M \otimes -)$ for some line bundle M on X . Suppose now that $\pi = \pi_{\alpha}$. By taking the adjoints to $\Phi_{\mathcal{E}} \circ \pi_{\alpha*} \simeq \pi_{\alpha*}$, and by composing with $\Phi_{\mathcal{E}}$, we get $\pi_{\alpha}^* \simeq \pi_{\alpha}^* \circ \Phi_{\mathcal{E}}$. Hence $\mathcal{O}_{\tilde{X}} \simeq \pi_{\alpha}^* M$, and by projection formula we have $\bigoplus_{i=0}^{d-1} \alpha^{-i} \simeq M \otimes \bigoplus_{i=0}^{d-1} \alpha^{-i}$. This implies $M \simeq \alpha^{-i}$ for some i . \square

Now we are ready to prove the main result of this section (*cf.* Theorem 6.1.7). It should be compared to (12) Theorem 4.5, which we here recall for the benefit of the reader.

Theorem 6.1.6. *Let X, Y be smooth projective varieties having torsion canonical bundles and let \tilde{X}, \tilde{Y} be the corresponding canonical covers. If $\Phi_{\mathcal{E}} : \mathbf{D}(X) \rightarrow \mathbf{D}(Y)$ is an equivalence, then there exists an equivariant lift $\Phi_{\tilde{\mathcal{E}}} : \mathbf{D}(\tilde{X}) \rightarrow \mathbf{D}(\tilde{Y})$. Moreover, if $\Phi_{\tilde{\mathcal{F}}} : \mathbf{D}(\tilde{X}) \rightarrow \mathbf{D}(\tilde{Y})$ is an equivariant equivalence, then $\Phi_{\tilde{\mathcal{F}}}$ is a lift of some equivalence $\Phi_{\mathcal{F}} : \mathbf{D}(X) \rightarrow \mathbf{D}(Y)$.*

Theorem 6.1.7. *Let X and Y be smooth projective varieties, and $\alpha \in \text{Pic}^0(X)$ and $\beta \in \text{Pic}^0(Y)$ be d -torsion elements. Denote by $\pi_{\alpha} : X_{\alpha} \rightarrow X$ and $\pi_{\beta} : Y_{\beta} \rightarrow Y$ the étale cyclic covers associated to α and β respectively.*

(i). *Suppose that $\Phi_{\mathcal{E}} : \mathbf{D}(X) \rightarrow \mathbf{D}(Y)$ is an equivalence and $F_{\mathcal{E}}$ is the induced Rouquier isomorphism. If $F_{\mathcal{E}}(1_X, \alpha) = (1_Y, \beta)$, then there exists an equivariant equivalence $\Phi_{\tilde{\mathcal{E}}} : \mathbf{D}(X_{\alpha}) \rightarrow \mathbf{D}(Y_{\beta})$ lifting $\Phi_{\mathcal{E}}$.*

(ii). *Suppose that $\Phi_{\tilde{\mathcal{F}}} : \mathbf{D}(X_{\alpha}) \rightarrow \mathbf{D}(Y_{\beta})$ is an equivariant equivalence. Then $\Phi_{\tilde{\mathcal{F}}}$ is the lift of some equivalence $\Phi_{\mathcal{F}} : \mathbf{D}(X) \rightarrow \mathbf{D}(Y)$.*

Proof. To see (i), we consider the following commutative diagram where $p_1, p_2, r_1, r_2, q_1, q_2$ are projection maps:

$$\begin{array}{ccccc}
 X_\alpha & \xleftarrow{q_1} & X_\alpha \times Y_\beta & \xrightarrow{q_2} & Y_\beta \\
 \parallel & & \downarrow (1_{X_\alpha} \times \pi_\beta) & & \downarrow \pi_\beta \\
 X_\alpha & \xleftarrow{r_1} & X_\alpha \times Y & \xrightarrow{r_2} & Y \\
 \downarrow \pi_\alpha & & \downarrow (\pi_\alpha \times 1_Y) & & \parallel \\
 X & \xleftarrow{p_1} & X \times Y & \xrightarrow{p_2} & Y.
 \end{array}$$

By (2.6), the condition $F_{\mathcal{E}}(1_X, \alpha) = (1_Y, \beta)$ is equivalent to an isomorphism in $\mathbf{D}(X \times Y)$:

$$p_1^* \alpha \otimes \mathcal{E} \simeq p_2^* \beta \otimes \mathcal{E}. \quad (6.6)$$

By pulling (6.6) back via the map $(\pi_\alpha \times 1_Y)$, we get an isomorphism $(\pi_\alpha \times 1_Y)^* \mathcal{E} \simeq r_2^* \beta \otimes (\pi_\alpha \times 1_Y)^* \mathcal{E}$ as $\pi_\alpha^* \alpha \simeq \mathcal{O}_{X_\alpha}$. Since the map $(1_{X_\alpha} \times \pi_\beta) : X_\alpha \times Y_\beta \rightarrow X_\alpha \times Y$ is the étale cyclic cover associated to the line bundle $r_2^* \beta$, by Lemma 6.1.1 there exists an object $\tilde{\mathcal{E}}$ such that $(1_{X_\alpha} \times \pi_\beta)_* \tilde{\mathcal{E}} \simeq (\pi_\alpha \times 1_Y)^* \mathcal{E}$. We now show the following isomorphism, which will be used in a moment

$$\pi_{\beta*} \circ \Phi_{\tilde{\mathcal{E}}} \simeq \Phi_{\mathcal{E}} \circ \pi_{\alpha*}. \quad (6.7)$$

Let E be an object in $\mathbf{D}(X_\alpha)$. Then projection formula and base change yield

$$\begin{aligned}
\pi_{\beta*}\Phi_{\tilde{\mathcal{E}}}(E) &\simeq \pi_{\beta*}q_{2*}(q_1^*E \otimes \tilde{\mathcal{E}}) \simeq r_{2*}(1_{X_\alpha} \times \pi_\beta)_*((1_{X_\alpha} \times \pi_\beta)^*r_1^*E \otimes \tilde{\mathcal{E}}) \\
&\simeq r_{2*}(r_1^*E \otimes (1_{X_\alpha} \times \pi_\beta)_*\tilde{\mathcal{E}}) \simeq r_{2*}(r_1^*E \otimes (\pi_\alpha \times 1_Y)^*\mathcal{E}) \\
&\simeq p_{2*}(\pi_\alpha \times 1_Y)_*(r_1^*E \otimes (\pi_\alpha \times 1_Y)^*\mathcal{E}) \simeq p_{2*}((\pi_\alpha \times 1_Y)_*r_1^*E \otimes \mathcal{E}) \\
&\simeq p_{2*}(p_1^*\pi_{\alpha*}E \otimes \mathcal{E}) \simeq \Phi_{\mathcal{E}}(\pi_{\alpha*}E).
\end{aligned}$$

This proves (6.7).

We now show that $\Phi_{\tilde{\mathcal{E}}}$ is an equivalence. Let $\Psi_{\mathcal{E}_R} : \mathbf{D}(Y) \rightarrow \mathbf{D}(X)$ be the right adjoint to $\Phi_{\mathcal{E}}$. From (6.6) we deduce $p_2^*\beta \otimes \mathcal{E}_R \simeq p_1^*\alpha \otimes \mathcal{E}_R$. By pulling this back via $(1_X \times \pi_\beta)$, we get $(1_X \times \pi_\beta)^*\mathcal{E}_R \simeq r_1^*\alpha \otimes (1_X \times \pi_\beta)^*\mathcal{E}_R$. Since the map $(\pi_\alpha \times 1_{Y_\beta})$ is the étale cyclic cover associated to $r_1^*\alpha$, by Lemma 6.1.1 there exists an object $\tilde{\mathcal{E}}_R$ in $\mathbf{D}(X_\alpha \times Y_\beta)$ such that $(\pi_\alpha \times 1_{Y_\beta})_*\tilde{\mathcal{E}}_R \simeq (1_X \times \pi_\beta)^*\mathcal{E}_R$. We now show that

$$\pi_{\alpha*} \circ \Psi_{\tilde{\mathcal{E}}_R} \simeq \Psi_{\mathcal{E}_R} \circ \pi_{\beta*}. \quad (6.8)$$

Consider the following diagram where the maps $p_1, p_2, s_1, s_2, t_1, t_2$ are projection maps:

$$\begin{array}{ccccc}
 X_\alpha & \xleftarrow{t_1} & X_\alpha \times Y_\beta & \xrightarrow{t_2} & Y_\beta \\
 \downarrow \pi_\alpha & & \downarrow \pi_\alpha \times 1_{Y_\beta} & & \parallel \\
 X & \xleftarrow{s_1} & X \times Y_\beta & \xrightarrow{s_2} & Y_\beta \\
 \parallel & & \downarrow 1_X \times \pi_\beta & & \downarrow \pi_\beta \\
 X & \xleftarrow{p_1} & X \times Y & \xrightarrow{p_2} & Y.
 \end{array}$$

Let E be an object in $\mathbf{D}(Y_\beta)$. Then

$$\begin{aligned}
 \pi_{\alpha*} \Psi_{\widetilde{\mathcal{E}}_R}(E) &\simeq \pi_{\alpha*} t_{1*} (t_2^* E \otimes \widetilde{\mathcal{E}}_R) \simeq s_{1*} (\pi_\alpha \times 1_{Y_\beta})_* ((\pi_\alpha \times 1_{Y_\beta})^* E \otimes \widetilde{\mathcal{E}}_R) \\
 &\simeq s_{1*} (s_2^* E \otimes (\pi_\alpha \times 1_{Y_\beta})_* \widetilde{\mathcal{E}}_R) \simeq s_{1*} (s_2^* E \otimes (1_X \times \pi_\beta)^* \mathcal{E}_R) \\
 &\simeq p_{1*} (1_X \times \pi_\beta)_* (s_2^* E \otimes (1_X \times \pi_\beta)^* \mathcal{E}_R) \simeq p_{1*} ((1_X \times \pi_\beta)_* s_2^* E \otimes \mathcal{E}_R) \\
 &\simeq p_{1*} (p_2^* \pi_{\beta*} E \otimes \mathcal{E}_R) \simeq \Psi_{\mathcal{E}_R} (\pi_{\beta*} E).
 \end{aligned}$$

This concludes the proof of (6.8).

Now, since $\Psi_{\mathcal{E}_R} \circ \Phi_{\mathcal{E}} \simeq 1_{\mathbf{D}(X)}$, by using (6.7) and (6.8) we get

$$\pi_{\alpha*} \circ \Psi_{\widetilde{\mathcal{E}}_R} \circ \Phi_{\widetilde{\mathcal{E}}} \simeq \Psi_{\mathcal{E}_R} \circ \pi_{\beta*} \circ \Phi_{\widetilde{\mathcal{E}}} \simeq \Psi_{\mathcal{E}_R} \circ \Phi_{\mathcal{E}} \circ \pi_{\alpha*} \simeq \pi_{\alpha*}. \quad (6.9)$$

Hence, by Lemma 6.1.5, we get $\Psi_{\widetilde{\mathcal{E}}_R} \circ \Phi_{\widetilde{\mathcal{E}}} \simeq g_*$. Similarly, we can show that $\Phi_{\widetilde{\mathcal{E}}} \circ \Psi_{\widetilde{\mathcal{E}}_R} \simeq h_*$ for some $h \in G$, and hence that $g^* \circ \Psi_{\widetilde{\mathcal{E}}_R}$, or equivalently $\Psi_{\widetilde{\mathcal{E}}_R} \circ h^*$, is a quasi-inverse to $\Phi_{\widetilde{\mathcal{E}}}$. Finally, Remark 6.1.4 implies that $\Phi_{\widetilde{\mathcal{E}}}$ is a lift of $\Phi_{\mathcal{E}}$.

Now we show that $\Phi_{\widetilde{\mathcal{E}}}$ is equivariant. Let $g \in G$ be an arbitrary element. Then we have isomorphisms

$$\pi_{\alpha*}(\Psi_{\widetilde{\mathcal{E}}} h^*)(g^* \Phi_{\widetilde{\mathcal{E}}}) \simeq \pi_{\alpha*} \Psi_{\widetilde{\mathcal{E}}}(gh)^* \Phi_{\widetilde{\mathcal{E}}} \simeq \Psi_{\mathcal{E}'\pi_{\beta*}}(gh)^* \Phi_{\widetilde{\mathcal{E}}} \simeq \Psi_{\mathcal{E}'\pi_{\beta*}} \Phi_{\widetilde{\mathcal{E}}} \simeq \Psi_{\mathcal{E}'} \Phi_{\mathcal{E}} \pi_{\alpha*} \simeq \pi_{\alpha*}.$$

By Lemma 6.1.5, there exists an element $\mu(g)$ such that $g^* \circ \Phi_{\widetilde{\mathcal{E}}} \simeq \Phi_{\widetilde{\mathcal{E}}} \circ \mu(g)^*$. From this it is easy to see that μ defines an automorphism of G .

We now prove (ii). Let $\Phi_{\widetilde{\mathcal{F}}} : \mathbf{D}(X_\alpha) \rightarrow \mathbf{D}(Y_\beta)$ be an equivariant equivalence. By (19) Ex. 5.12 and by

$$g^* \circ \Phi_{\widetilde{\mathcal{F}}} \simeq \Phi_{\widetilde{\mathcal{F}}} \circ \mu(g)^*, \quad (6.10)$$

we get isomorphisms $(1_{X_\alpha} \times g)^* \widetilde{\mathcal{F}} \simeq (\mu(g) \times 1_{Y_\beta})_* \widetilde{\mathcal{F}}$ for any $g \in G$. Therefore $(1_{X_\alpha} \times \pi_\beta)_* \widetilde{\mathcal{F}} \simeq (\mu(g) \times 1_Y)_*(1_{X_\alpha} \times \pi_\beta)_* \widetilde{\mathcal{F}}$, and by Lemma 6.1.1 there exists an object \mathcal{F} in $\mathbf{D}(X \times Y)$ such that $(\pi_\alpha \times 1_Y)^* \mathcal{F} \simeq (1_{X_\alpha} \times \pi_\beta)_* \widetilde{\mathcal{F}}$. Moreover, as we have done for the isomorphism (6.7), we get

$$\pi_{\beta*} \circ \Phi_{\widetilde{\mathcal{F}}} \simeq \Phi_{\mathcal{F}} \circ \pi_{\alpha*}. \quad (6.11)$$

Now we show that $\Phi_{\mathcal{F}}$ is an equivalence. Let $\Psi_{\widetilde{\mathcal{F}}_R}$ be the right adjoint to $\Phi_{\widetilde{\mathcal{F}}}$. From (6.10), we deduce that $\mu(g)^* \circ \Psi_{\widetilde{\mathcal{F}}_R} \simeq \Psi_{\widetilde{\mathcal{F}}_R} \circ g^*$ for all $g \in G$. Then $\Psi_{\widetilde{\mathcal{F}}_R}$ is equivariant as well and

moreover $(\mu(g) \times 1_{Y_\beta})^* \tilde{\mathcal{F}}_R \simeq (1_{X_\alpha} \times g)_* \tilde{\mathcal{F}}_R$. By applying the functor $(\pi_\alpha \times 1_{Y_\beta})_*$, we obtain $(\pi_\alpha \times 1_{Y_\beta})_* \tilde{\mathcal{F}}_R \simeq (1_X \times g)_* (\pi_\alpha \times 1_{Y_\beta})_* \tilde{\mathcal{F}}_R$. Hence, by Lemma 6.1.1, there exists an object \mathcal{F}' such that $(1_X \times \pi_\beta)^* \mathcal{F}' \simeq (\pi_\alpha \times 1_{Y_\beta})_* \tilde{\mathcal{F}}_R$ and consequently an isomorphism (cf. the computations for (6.8))

$$\pi_{\alpha*} \circ \Psi_{\tilde{\mathcal{F}}_R} \simeq \Psi_{\mathcal{F}'} \circ \pi_{\beta*}. \quad (6.12)$$

At this point, by (6.11) and (6.12), it follows that the composition $\Psi_{\mathcal{F}'} \circ \Phi_{\mathcal{F}}$ satisfies the hypotheses of Lemma 6.1.5. Therefore $\Psi_{\mathcal{F}'} \circ \Phi_{\mathcal{F}} \simeq (\alpha^i \otimes -)$ for some i , and similarly $\Phi_{\mathcal{F}} \circ \Psi_{\mathcal{F}'} \simeq (\beta^j \otimes -)$ for some j . This says that $(\alpha^{-i} \otimes -) \circ \Psi_{\mathcal{F}'}$, or equivalently $\Psi_{\mathcal{F}'} \circ (\beta^{-j} \otimes -)$, is a quasi-inverse to $\Phi_{\mathcal{F}}$.

□

6.2 Comparison of cohomological support loci

The following theorem is the main result of this chapter.

Theorem 6.2.1. *Let X and Y be smooth projective varieties of dimension n and $\Phi_{\mathcal{E}} : \mathbf{D}(X) \simeq \mathbf{D}(Y)$ be an equivalence. Suppose that for some integer $i \geq 0$ the Hodge number $h^{0,n-i}(-)$ is invariant under an arbitrary derived equivalence of smooth projective varieties of dimension n . Then the induced Rouquier isomorphism $F_{\mathcal{E}} = F$ maps*

$$F(1_X, V_r^i(\omega_X)_0) = (1_Y, V_r^i(\omega_Y)_0)$$

for all integers $r \geq 1$. In particular, F induces isomorphisms of algebraic sets $V_r^i(\omega_X)_0 \simeq V_r^i(\omega_Y)_0$, and if $F(1_X, \alpha) = (1_Y, \beta)$ with $\alpha \in V^i(\omega_X)_0$, then $h^i(X, \omega_X \otimes \alpha) = h^i(Y, \omega_Y \otimes \beta)$.

Proof. We aim to prove the isomorphisms $F(1_X, V_r^i(\omega_X)_0) = (1_Y, V_r^i(\omega_Y)_0)$ for all integers $r \geq 1$, as the other assertions easily follow from this one. To this end we let $\alpha \in V_r^i(\omega_X)_0$ so that, by Lemma 4.3.3, $F(1_X, \alpha) = (1_Y, \beta)$ for some $\beta \in \text{Pic}^0(Y)$. We divide the proof in two steps.

Step 1. Since F^{-1} is the Rouquier isomorphism induced by a quasi-inverse to $\Phi_{\mathcal{E}}$ (cf. Lemma 2.1.9), it is enough to show that $\beta \in V_r^i(\omega_Y)_0$. In this step we show that it is enough to prove this assertion in the case when α is a torsion point of (special) prime order. First, since F is a group isomorphism, if α is torsion of some order if and only if β is torsion of the same order.

According to a well-known theorem of Simpson (cf. Theorem 2.2.2 (i)), every irreducible component Z of $V_r^i(\omega_Y)$ is a torsion translate $\tau_Z + A_Z$ of an abelian subvariety of $\text{Pic}^0(Y)$. We consider the set P_i of all prime numbers that do not divide $\text{ord}(\tau_Z)$ for any such component Z . As $V_r^i(\omega_Y)$ is an algebraic set by the semicontinuity theorem, we are only throwing away a finite set of primes. We will show that it is enough to prove the assertion above when α is torsion with order in P_i . First note that it is a standard fact that torsion points of prime order are Zariski dense in a complex abelian variety (this follows for instance from the fact that real numbers can be approximated with rational numbers with prime denominators). Consequently, torsion points with order in the set P_i are dense as well.

Let now W be a component of $V_r^i(\omega_X)_0$. It suffices to show that

$$Z := p_2(F(1_X, W)) \subset V_r^i(\omega_Y)_0,$$

where p_2 is the projection onto the second component of $\text{Aut}^0(Y) \times \text{Pic}^0(Y)$. Indeed, since one can repeat the same argument for the inverse homomorphism F^{-1} , this implies that Z has to be a component of $V_r^i(\omega_Y)_0$, isomorphic to W via F . Now Z is an abelian variety, and therefore by the discussion above torsion points β of order in P_i are dense in Z . By semicontinuity, it suffices to show that $\beta \in V_r^i(\omega_Y)_0$. These β 's are precisely the images of $\alpha \in W$ of order in P_i , which concludes our reduction step.

Step 2. Let now $\alpha \in V_r^i(\omega_X)_0$ be a torsion point of order belonging to the set P_i , and $F(1_X, \alpha) = (1_Y, \beta)$. Denote

$$p = \text{ord}(\alpha) = \text{ord}(\beta).$$

Consider the cyclic covers $\pi_\alpha : X_\alpha \rightarrow X$ and $\pi_\beta : Y_\beta \rightarrow Y$ associated to α and β respectively.

We can apply Theorem 6.1.7 to conclude that there exists an equivalence $\Phi_{\mathcal{E}} : \mathbf{D}(X_\alpha) \rightarrow \mathbf{D}(Y_\beta)$

lifting $\Phi_{\mathcal{E}}$. Assuming that $h^{0,n-i}(-)$ is invariant, we have in particular that

$$h^{0,n-i}(X) = h^{0,n-i}(Y) \quad \text{and} \quad h^{0,n-i}(X_\alpha) = h^{0,n-i}(Y_\beta).$$

On the other hand, using (6.1), we have

$$H^{n-i}(X_\alpha, \mathcal{O}_{X_\alpha}) \simeq \bigoplus_{j=0}^{p-1} H^{n-i}(X, \alpha^{-j}) \quad \text{and} \quad H^{n-i}(Y_\beta, \mathcal{O}_{Y_\beta}) \simeq \bigoplus_{j=0}^{p-1} H^{n-i}(Y, \beta^{-j}).$$

The terms on the left hand side and the terms corresponding to $j = 0$ on the right hand side have the same dimension. On the other hand, since every component of $V_r^i(\omega_X)_0$ is an abelian subvariety of $\text{Pic}^0(X)$, we have that $\alpha^j \in V_r^i(\omega_X)_0$ for all j , so $h^{n-i}(X, \alpha^{-j}) \geq r$ for all j . We conclude that

$$h^{n-i}(Y, \beta^{-k}) \geq r \quad \text{for some} \quad 1 \leq k \leq p-1.$$

This says that $\beta^k \in V_r^i(\omega_Y)$. We claim that in fact $\beta^k \in V_r^i(\omega_Y)_0$. Assuming that this is the case, we can conclude the argument. Indeed, pick a component $T \subset V_r^i(\omega_Y)_0$ such that $\beta^k \in T$. But β^k generates the cyclic group of prime order $\{\mathcal{O}_Y, \beta, \dots, \beta^{p-1}\}$, so $\beta \in T$ as well, since T is an abelian variety.

We are left with proving that $\beta^k \in V_r^i(\omega_Y)_0$. Pick any component S in $V_r^i(\omega_Y)$ containing β^k . By the Simpson theorem mentioned above, we have that $S = \tau + B$, where τ is a torsion point and B is an abelian subvariety of $\text{Pic}^0(Y)$. We claim that we must have $\tau \in B$, so that $S = B$, confirming our statement (note that in fact we are proving something stronger: β^k belongs *only* to components of $V_r^i(\omega_Y)$ passing through the origin). To this end, switching abusively to additive notation, say $k\beta = \tau + b$ with $b \in B$. Since the order p of β is assumed to be in the set P_i , we have that $\text{ord}(\tau)$ and p are coprime. Now on one hand $(\text{ord}(\tau))\tau = 0 \in B$,

while on the other hand $p\tau + pb = kp\beta = 0$, so $p\tau \in B$ as well. Since $\text{ord}(\tau)$ and p are coprime, one easily concludes that $\tau \in B$. \square

In view of the previous theorem, it is natural to conjecture a stronger version of Variant 4.1.2 (cf. (2) Conjecture 3.1):

Conjecture 6.2.2. *Let X and Y be smooth projective D -equivalent varieties and let F be the induced Rouquier isomorphism. Then $F(1_X, V^i(\omega_X)_0) = (1_Y, V^i(\omega_Y)_0)$, so that $V^i(\omega_X)_0 \simeq V^i(\omega_Y)_0$. Moreover, if $\alpha \in V^i(\omega_X)_0$ and $F(1_X, \alpha) = (1_Y, \beta)$, then $h^i(X, \omega_X \otimes \alpha) = h^i(Y, \omega_Y \otimes \beta)$.*

In the following two corollaries we apply Theorem 6.2.1 in the cases where the derived invariance of Hodge numbers is known in order to obtain the derived invariance of further cohomological support loci.

Corollary 6.2.3. *Let X and Y be smooth projective D -equivalent varieties of dimension n and let F be the induced Rouquier isomorphism. Then $F(1_X, V_r^{n-1}(\omega_X)_0) = (1_Y, V_r^{n-1}(\omega_Y)_0)$ for all integers $r \geq 1$. Consequently, F induces isomorphisms of algebraic sets $V_r^{n-1}(\omega_X)_0 \simeq V_r^{n-1}(\omega_Y)_0$. Furthermore, if $F(1_X, \alpha) = (1_Y, \beta)$ with $\alpha \in V_r^{n-1}(\omega_X)_0$, then $h^{n-1}(X, \omega_X \otimes \alpha) = h^{n-1}(Y, \omega_Y \otimes \beta)$.*

Proof. By Theorem 6.2.1 we only need to verify the derived invariance of $h^{0,1}(X)$ which is ensured by Theorem 2.1.10. \square

Corollary 6.2.4. *Let X and Y be smooth projective D -equivalent fourfolds and let F be the induced Rouquier isomorphism. Assume that either (i) X is of maximal Albanese dimension; or*

(ii) $\dim \text{alb}_X(X) = 3$ and $\kappa(X) \geq 0$; or (iii) $\dim \text{alb}_X(X) = 3$ and ω_X^{-1} is nef; or (iv) $\text{Aut}^0(X)$ is not affine. Then $F(1_X, V_r^2(\omega_X)_0) = (1_Y, V_r^2(\omega_Y)_0)$ for all integers $r \geq 1$. Consequently, F induces isomorphisms of algebraic sets $V_r^2(\omega_X)_0 \simeq V_r^2(\omega_Y)_0$. Furthermore, if $F(1_X, \alpha) = (1_Y, \beta)$ with $\alpha \in V^2(\omega_X)_0$, then $h^2(X, \omega_X \otimes \alpha) = h^2(Y, \omega_Y \otimes \beta)$.

Proof. According to Theorem 6.2.1 it suffices to have $h^{0,2}(X) = h^{0,2}(Y)$, which is ensured by Corollary 4.6.3 and (9) Corollary 3.4. \square

6.3 Behavior of fibrations under derived equivalence

We study the behavior of special types of fibrations onto irregular varieties under derived equivalence.

6.3.1 Fibrations onto curves

Theorem 6.3.1. *Let X and Y be smooth projective D -equivalent varieties such that X admits a surjective morphism to a smooth projective curve C of genus $g \geq 2$. Then Y admits a surjective morphism with connected fibers to a curve of genus $\geq g$.*

Proof. Let $f : X \rightarrow C$ be a surjective morphism as in the statement. By taking the Stein factorization we can assume that f has connected fibers. We have that $f^*\text{Pic}^0(C) \subset V^{n-1}(\omega_X)_0$. By Corollary 6.2.3 we have $V^{n-1}(\omega_X)_0 \simeq V^{n-1}(\omega_Y)_0$, and therefore there exists a component T of $V^{n-1}(\omega_Y)_0$ of dimension at least g . Finally, by Theorem 2.2.4, there exists a smooth projective curve D and a surjective morphism with connected fibers $g : Y \rightarrow D$ such that $T = g^*\text{Pic}^0(D)$. Note that $g(D) = \dim T \geq g$. \square

We remark that it is known from results of Beauville and Siu that X admits a surjective morphism to a curve of genus $\geq g$ if and only if $\pi_1(X)$ has a surjective homomorphism onto Γ_g , the fundamental group of a Riemann surface of genus g (see the Appendix to (52)). On the other hand, it is also known that derived equivalent varieties do not necessarily have isomorphic fundamental groups (cf. (53) and (54)), so this would not suffice in order to deduce Theorem 6.3.1.

Remark 6.3.2. *The discussion above shows in fact the following more refined statement. For a smooth projective variety Z , define*

$$A_Z := \{g \in \mathbf{N} \mid g = \dim T \text{ for some irreducible component } T \subset V^{n-1}(\omega_Z)_0\}.$$

Then if $\mathbf{D}(X) \simeq \mathbf{D}(Y)$, we have $A_X = A_Y$. Denoting this set by A , for each $g \in A$ both X and Y have surjective maps onto curves of genus g . The maximal genus of a curve admitting a surjective map from X (or Y) is $\max(A)$.

Question 6.3.3. *If $\mathbf{D}(X) \simeq \mathbf{D}(Y)$, is the set of curves of genus at least 2 admitting non-constant maps from X the same as that for Y ? Or at least the set of curves corresponding to irreducible components of $V^{n-1}(\omega_X)_0$?*

6.3.2 Fano fibrations over curves

We study Fourier-Mukai partners of Fano fibrations over curves of genus ≥ 2 .

Theorem 6.3.4. *Let X and Y be smooth projective D -equivalent varieties. Assume that there is an algebraic fiber space $f : X \rightarrow C$ such that C is a smooth projective curve of genus ≥ 2 and the general fiber of f is Fano. Then:*

(i). *X and Y are K -equivalent.*

(ii). *There is an algebraic fiber space $g : Y \rightarrow C$ such that for $c \in C$ where the fibers X_c and Y_c are smooth, with X_c Fano, one has $Y_c \simeq X_c$.*

(iii). *If ω_X^{-1} is f -ample (e.g. if f is a Mori fiber space), then $X \simeq Y$.*

Proof. Let p and q be the projections of $X \times Y$ onto the first and second factor respectively. Consider the unique up to isomorphism $\mathcal{E} \in \mathbf{D}(X \times Y)$ such that the given equivalence is the Fourier-Mukai functor $\Phi_{\mathcal{E}}$. Then, by (19) Corollary 6.5, there exists a component Z of $\text{Supp}(\mathcal{E})$ such that $p|_Z : Z \rightarrow X$ is surjective. We first claim that $\dim Z = \dim X$.

Assuming by contradiction that $\dim Z > \dim X$, we show that ω_X^{-1} is nef. We denote by F the general fiber of f , which is Fano. We also define $Z_F := p_Z^{-1}(F) \subset Z$, while $q_F : Z_F \rightarrow Y$ is the projection obtained by restricting q to Z_F . Since ω_F^{-1} is ample, we obtain that q_F is finite onto its image; see (19) Corollary 6.8. On the other hand, the assumption that $\dim Z > \dim X$ implies that $\dim Z_F \geq \dim X = \dim Y$, so q_F must be surjective (and consequently $\dim Z_F = \dim X$).

By passing to its normalization if necessary, we can assume without loss of generality that Z_F is normal. Denoting by p_F the projection of Z_F to X , by (19) Corollary 6.9 we have that there exists $r > 0$ such that

$$p_F^* \omega_X^{-r} \simeq q_F^* \omega_Y^{-r}.$$

Now since p_F factors through F and ω_F^{-1} is ample, we have that $p_F^*\omega_X^{-1}$ is nef, hence by the isomorphism above so is $q_F^*\omega_Y^{-1}$. Finally, since q_F is finite and surjective, we obtain that ω_Y^{-1} is nef, so by Theorem 2.1.12 (iii), ω_X^{-1} is nef as well.

We can now conclude the proof of the claim using the main result of Zhang (43) saying that a smooth projective variety with nef anticanonical bundle has surjective Albanese map. In our case, since the general fiber of f is Fano, the Albanese map of X is obtained by composing f with the Abel-Jacobi embedding of C . But this implies that C has genus at most one, a contradiction. The claim is proved, so $\dim Z = \dim X = \dim Y$. At this stage, the K -equivalence statement follows from Lemma 6.3.5 below.

For statements (ii) and (iii) we emphasize that, once we know that X and Y are K -equivalent, the argument is standard and independent of derived equivalence (we thank Professor Alessio Corti for having pointed this out to us). Note first that smooth birational varieties have the same Albanese variety and Albanese image. Since f is the Albanese map of X , it follows that the Albanese map of Y is a surjective morphism $g : Y \rightarrow C$. Furthermore, C is the

Albanese image of any other birational model as well, hence any smooth model Z inducing a K -equivalence between X and Y sits in a commutative diagram

$$\begin{array}{ccc}
 & Z & \\
 p \swarrow & & \searrow q \\
 X & & Y \\
 f \searrow & h \downarrow & \swarrow g \\
 & C &
 \end{array}$$

Note that in particular g has connected fibers since f does.

For a point $c \in C$, denote by X_c , Y_c and Z_c the fibers of f , g and h over c . By adjunction, Z_c realizes a K -equivalence between X_c and Y_c . First, assuming that c is chosen such that X_c and Y_c are smooth, with X_c Fano, we show that $X_c \simeq Y_c$.

To this end, if we assume that the induced rational map $\varphi_c : Y_c \rightarrow X_c$ is not a morphism, there must be a curve $B \subset Z_c$ which is contracted by q_c but not by p_c . Then $q_c^* \omega_{Y_c} \cdot B = 0$, and so $p_c^* \omega_{X_c} \cdot B = 0$ as well. On the other hand, $\omega_{X_c}^{-1} \cdot p_c(B) < 0$ which is a contradiction. Therefore we obtain that φ_c is a birational morphism with the property that $\varphi_c^* \omega_{X_c} \simeq \omega_{Y_c}$, which implies that φ_c is an isomorphism.

If in fact ω_X^{-1} is f -ample, this argument can be globalized: indeed, assuming that the rational map $\varphi : Y \rightarrow X$ is not a morphism, there exists a curve $B \subset Z$ which is contracted by g and hence h , but not by p . Since B lives in a fiber of f (by the commutativity of the diagram),

we again obtain a contradiction. Once we know that φ is a morphism, the same argument as above implies that it is an isomorphism. \square

The following lemma used in the proof above is due to Kawamata, and can be extracted from his argument leading to the fact that D -equivalent varieties of general type are K -equivalent (*cf.* (25)); we sketch the argument for convenience.

Lemma 6.3.5. *Let X and Y be smooth projective varieties and let $\Phi_{\mathcal{E}} : \mathbf{D}(X) \rightarrow \mathbf{D}(Y)$ be an equivalence. Assume that there exists a component Z of the support of \mathcal{E} such that $\dim Z = \dim X$ and Z dominates X . Then X and Y are K -equivalent.*

Proof. Denote by p and q the projections of Z to X and Y . Since p is surjective, (19) Corollary 6.12 tells us that p is birational, and Z is the unique component of $\text{Supp}(\mathcal{E})$ dominating X . We claim that q is also surjective, in which case by the same reasoning q is birational as well. Since (on the normalization of Z) we have $p^*\omega_X^r \simeq q^*\omega_Y^r$ for some $r \geq 1$, this suffices to conclude that X and Y are K -equivalent as in (25) Theorem 2.3 (*cf.* also (19) p. 149).

Assuming that q is not surjective, we can find general points x_1 and x_2 in X such that $p^{-1}(x_1)$ and $p^{-1}(x_2)$ consist of one point, and $q(p^{-1}(x_1)) = q(p^{-1}(x_2)) = y$ for some $y \in Y$. One then sees that

$$\text{Supp } \Phi_{\mathcal{E}}(\mathcal{O}_{x_1}) = \text{Supp } \Phi_{\mathcal{E}}(\mathcal{O}_{x_2}) = \{y\}.$$

This implies in standard fashion that

$$\text{Hom}_{\mathbf{D}(X)}^{\bullet}(\mathcal{O}_{x_1}, \mathcal{O}_{x_2}) \simeq \text{Hom}_{\mathbf{D}(Y)}^{\bullet}(\Phi_{\mathcal{E}}(\mathcal{O}_{x_1}), \Phi_{\mathcal{E}}(\mathcal{O}_{x_2})) \neq 0,$$

a contradiction. □

6.3.3 Fibrations onto higher-dimensional varieties

In (4) Corollary 3.4, Popa notices that a consequence of Conjecture 4.1.1 is that if X admits a fibration onto a variety with non-surjective Albanese map, then any Fourier-Mukai partner of X admits an irregular fibration. With our results at hand, we can verify this statement under an additional hypothesis on X . We begin by recalling some terminology. Let X be a smooth projective variety. Following Catanese's paper (52), we say that X is of *Albanese general type* if it has non-surjective Albanese map and it is of maximal Albanese dimension. An *irregular fibration* (resp. *higher irrational pencil*) is a surjective morphism with connected fibers $f : X \rightarrow Z$ onto a normal variety Z with $0 < \dim Z < \dim X$ and such that a smooth model of Z is of maximal Albanese dimension (resp. Albanese general type).

Proposition 6.3.6. *Let X and Y be smooth projective D -equivalent varieties with $\dim \text{alb}_X(X) \geq \dim X - 1$. If X admits a surjective morphism $f : X \rightarrow Z$ with connected fibers onto a normal variety Z having non-surjective Albanese morphism and $\dim X > \dim Z$, then Y admits an irregular fibration.*

Proof. Let $Z \xrightarrow{f'} Z' \rightarrow \text{alb}_Z(Z)$ be the Stein factorization of the Albanese map of Z . One checks that Z' is a normal variety, of maximal Albanese dimension (so that $\mathcal{O}_{Z'} \in V^0(\omega_{Z'})$), and it has non-surjective Albanese map. Hence, by (21) Proposition 2.2, there exists a positive-dimensional irreducible component V of $V^0(\omega_{Z'})$ passing through $\mathcal{O}_{Z'}$. Moreover, by Lemma 5.2.8, we have $(f \circ f')^*V \subset V^k(\omega_X)_0$ where $k = \dim X - \dim Z'$, and by (2.9) we get inclusions

$(f \circ f')^*V \subset V^k(\omega_X)_0 \subset V^1(\omega_X)_0$. Finally, by Proposition 4.4.1, there exists a positive-dimensional irreducible component $V' \subset V^1(\omega_Y)_0$ and we conclude then by applying Theorem 2.2.4. \square

Thanks to Theorem 5.2.1, in dimension three the previous theorem holds without the hypothesis “ $\dim \text{alb}_X(X) \geq \dim X - 1$ ”.

Finally, we study fibrations onto surfaces.

Proposition 6.3.7. *Let X and Y be smooth projective D -equivalent threefolds. If X admits a higher irrational pencil $f : X \rightarrow Z$ with $\dim Z = 2$, then Y admits a higher irrational pencil $g : Y \rightarrow W$ with $0 < \dim W \leq 2$.*

Proof. It is a general fact that, by possibly replacing Z with a smaller dimensional variety, one can assume $\chi(\omega_{\tilde{Z}}) > 0$ for any smooth model \tilde{Z} of Z (cf. (34) p. 271). If $\dim Z = 1$, then we apply Theorem 6.3.1. Suppose then $\dim Z = 2$. Then $q(Z) \geq 3$ and by Lemma 5.2.8 we have $f^*V^0(\omega_Z) = f^*\text{Pic}^0(Z) \subset V^1(\omega_X)_0$. Moreover, by Theorem 5.2.1, there exists a component $T \subset V^1(\omega_Y)_0$ such that $\dim T \geq q(Z') \geq 3$, and by Theorem 2.2.4 there exists an irregular fibration $g : Y \rightarrow W$ such that $T \subset g^*\text{Pic}^0(W) + \gamma$. We conclude that $q(W) \geq \dim T \geq 3$ and that g is in fact an higher irrational pencil. \square

We can slightly improve Proposition 6.3.7 by keeping track of the irregularities of the fibrations. By the classification theory of projective surfaces, any smooth model of a normal surface of Albanese general type is either birational to a surface of general type or to an elliptic surface of Kodaira dimension one and of maximal Albanese dimension. Thanks to this little

fact, we can prove the following two statements. Fix $q \geq 3$. If X admits a higher irrational pencil $f : X \rightarrow Z$ such that a smooth model \tilde{Z} of Z is a surface of general type and with $q(\tilde{Z}) \geq q$, then Y admits a higher irrational pencil $g : Y \rightarrow W$ such that a smooth model \tilde{W} of W has $q(\tilde{W}) \geq q$. On the other hand, if X admits a higher irrational pencil $f : X \rightarrow Z$ such that a smooth model \tilde{Z} of Z is birational to an elliptic surface of Kodaira dimension one and of maximal Albanese dimension and with $q(\tilde{Z}) \geq q$, then Y admits an fibration onto a smooth curve $g : Y \rightarrow W$ such that a smooth model of \tilde{W} of W has $q(\tilde{W}) \geq q - 1$.

CHAPTER 7

CONSTRAINTS ON HODGE NUMBERS

We aim to study constraints on Hodge numbers of irregular compact Kähler manifolds as in (14). More specifically, by analyzing the exactness of global versions of derivative complexes associated to bundles of holomorphic p -forms Ω_X^p (cf. Proposition 7.1.2), we derive inequalities among Hodge numbers thanks to the fact that Chern classes of globally generated vector bundles are non-negative and the Evans-Griffith theorem. Our inequalities hold for particular classes of irregular varieties, for instance for smooth subvarieties of abelian varieties having ample normal bundle.

In the last section we study the regularity of cohomology modules $\bigoplus_i H^i(X, \Omega_X^p)$ over the exterior algebra $\bigwedge^* H^1(X, \mathcal{O}_X)$ of a smooth projective variety X . We write down bounds for their regularity in terms of the defect of semismallness of the Albanese map (cf. Theorem 7.4.1). This is achieved by using the BGG correspondence together with generic vanishing theorems for Ω_X^p .

7.1 Exactness of BGG complexes

Let X be a compact Kähler manifold of dimension n and irregularity $q(X) := \dim_{\mathbf{C}} H^0(X, \Omega_X^1) > 0$. We set $V = H^1(X, \mathcal{O}_X)$, $W = V^\vee$ and fix an integer $0 \leq p \leq n$. Via cup product, any element $0 \neq v \in V$ defines a complex of vector spaces called *derivative complex associated to*

Ω_X^p in the direction v (we have already encountered these complexes in Theorems 2.2.2 (iv) and 2.2.3 (ii)):

$$0 \longrightarrow H^0(X, \Omega_X^p) \xrightarrow{\cup v} H^1(X, \Omega_X^p) \xrightarrow{\cup v} \dots \xrightarrow{\cup v} H^n(X, \Omega_X^p) \longrightarrow 0. \quad (7.1)$$

We define $\mathbf{A} := \text{Spec}(\text{Sym}(W))$ so that V is viewed as an affine variety. In this way, we can arrange all the above complexes as v varies into a complex of locally free sheaves \mathcal{K}_p on \mathbf{A} :

$$\begin{aligned} \mathcal{K}_p : \quad 0 \longrightarrow \mathcal{O}_{\mathbf{A}} \otimes H^0(X, \Omega_X^p) \longrightarrow \mathcal{O}_{\mathbf{A}} \otimes H^1(X, \Omega_X^p) \longrightarrow \dots \\ \dots \longrightarrow \mathcal{O}_{\mathbf{A}} \otimes H^{n-1}(X, \Omega_X^p) \longrightarrow \mathcal{O}_{\mathbf{A}} \otimes H^n(X, \Omega_X^p) \longrightarrow 0 \end{aligned} \quad (7.2)$$

whose fiber at a point $v \in \mathbf{A}$ is the complex (7.1).

We denote by $\mathbf{P} = \mathbf{P}_{\text{sub}}(V)$ the projective space of one-dimensional linear subspaces of V . Since the differentials of \mathcal{K}_p scale linearly through radial directions, \mathcal{K}_p descends to a complex on \mathbf{P} :

$$\begin{aligned} \underline{\mathbf{L}}_X^p : \quad 0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-n) \otimes H^0(X, \Omega_X^p) \longrightarrow \mathcal{O}_{\mathbf{P}}(-n+1) \otimes H^1(X, \Omega_X^p) \longrightarrow \dots \\ \dots \longrightarrow \mathcal{O}_{\mathbf{P}}(-1) \otimes H^{n-1}(X, \Omega_X^p) \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^n(X, \Omega_X^p) \longrightarrow 0. \end{aligned} \quad (7.3)$$

Following (16) we refer to $\underline{\mathbf{L}}_X^p$ as the *BGG complex associated to Ω_X^p* (the choice of this terminology will become clear later). The exactness of $\underline{\mathbf{L}}_X^p$ has been analyzed in (16) Proposition 2.1 in the case $p = 0$ where it is shown to depend on the Albanese dimension of X .

Proposition 7.1.1. *Let $\dim \text{alb}_X(X) = n - k$. Then $\underline{\mathbf{L}}_X^0$ is exact at the first $n - k$ terms from the left.*

One of the goal of this section is to study the exactness of $\underline{\mathbf{L}}_X^p$ for $p > 0$. This exactness turns out to depend on the non-negative integer:

$$m(X) = \min \{ \text{codim } Z(\omega) \mid 0 \neq \omega \in H^0(X, \Omega_X^1) \}, \quad (7.4)$$

representing the least codimension of the zero-locus of a non-zero holomorphic one-form. We use the convention $m(X) = +\infty$ if every non-zero holomorphic one-form is nowhere vanishing. Examples of varieties with $m(X) = \dim X$ (i.e. the ones for which the zero-locus of every non-zero holomorphic one-form consists of at most a finite set of points) are the smooth varieties for which the Albanese map is an embedding with ample normal bundle. More in general, smooth varieties X embedded in an abelian variety A with ample normal bundle and such that every holomorphic one-form of X is the restriction of a form from A satisfy $m(X) = \dim X$ (cf. (39) Proposition 6.3.10).

Proposition 7.1.2. *Let X be a compact Kähler manifold of dimension n . Moreover let $0 \leq p \leq n$ be an integer and set $m = m(X)$.*

(i). *If $p < m \leq n$ then $\underline{\mathbf{L}}_X^p$ is exact at the first $(m - p)$ -steps from the left. Furthermore, the first $m - p$ maps of $\underline{\mathbf{L}}_X^p$ are of constant rank.*

(ii). *If $n - p < m \leq n$ then $\underline{\mathbf{L}}_X^p$ is exact at the first $(m - n + p)$ -steps from the right. Furthermore, the last $m - n + p$ maps of $\underline{\mathbf{L}}_X^p$ are of constant rank.*

(iii). If $m = +\infty$ then the whole complex $\underline{\mathbf{L}}_X^p$ is exact and all the involved maps are of constant rank.

Proof. It is enough to show the exactness of \mathcal{K}_p at any point $0 \neq v \in \mathbf{A}$. Under the Hodge linear-conjugate isomorphism, the fiber at a point $v \in \mathbf{A}$ of \mathcal{K}_p is identified to the complex of vector spaces

$$0 \longrightarrow H^p(X, \mathcal{O}_X) \xrightarrow{\wedge \omega} H^p(X, \Omega_X^1) \xrightarrow{\wedge \omega} \dots \xrightarrow{\wedge \omega} H^p(X, \Omega_X^n) \longrightarrow 0, \quad (7.5)$$

where $\omega \in H^0(X, \Omega_X^1)$ is the holomorphic one-form conjugate to $v \in H^1(X, \mathcal{O}_X)$. By Theorem 2.2.3 (ii) (which holds in the Kähler case as well), the complex (7.5) is exact at the first $(m-p)$ -steps from the left for every $0 \neq \omega \in H^0(X, \Omega_X^1)$. Hence the complex \mathcal{K}_p is itself exact at the first $(m-p)$ -steps since exactness can be checked at the level of fibers. This also shows that the first $m-p$ maps of \mathcal{K}_p , and therefore of $\underline{\mathbf{L}}_X^p$, are of constant rank.

For (ii), using Serre duality and thinking of the spaces $H^p(X, \Omega_X^q)$ as the (p, q) -Dolbeault cohomology, we have a diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & H^{n-p}(X, \Omega_X^{i-1}) & \xrightarrow{\wedge \omega} & H^{n-p}(X, \Omega_X^i) & \xrightarrow{\wedge \omega} & H^{n-p}(X, \Omega_X^{i+1}) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & H^p(X, \Omega_X^{n-i+1})^\vee & \xrightarrow{\wedge \omega} & H^p(X, \Omega_X^{n-i})^\vee & \xrightarrow{\wedge \omega} & H^p(X, \Omega_X^{n-i-1})^\vee & \longrightarrow & \dots \end{array}$$

where the bottom complex is the dual complex of (7.5). This diagram commutes up to sign and hence, if $m > n - p$, the upper complex (and therefore also the bottom one) is exact at the first $(m - n + p)$ -steps from the left. Finally, by dualizing again the bottom complex, we have that (7.5) is exact at the first $(m - n + p)$ -steps from the right.

The case $m = +\infty$ follows as the complexes (7.5) are now everywhere exact. \square

In the special case $m(X) = \dim X$, Proposition 7.1.2 implies that $\underline{\mathbf{L}}_X^p$ is everywhere exact except at most at one term. This allows us to give a bound on $\chi(\Omega_X^p) := \sum (-1)^i h^i(X, \Omega_X^p)$ in the case $q(X) > \dim X$. Before stating the bounds, we prove a simple lemma which will be useful in the sequel.

Lemma 7.1.3. *Let $e \geq 2$, $t \geq 1$, $q \geq 2$ and a be integers. For $i = 1, \dots, e + 1$ and $s = 1, \dots, t$ let V_i and Z_s be complex vector spaces of positive dimension.*

(i). *If a complex of locally free sheaves on $\mathbf{P} = \mathbf{P}^{q-1}$ of length $e + 1$ of the form*

$$0 \longrightarrow V_{e+1} \otimes \mathcal{O}_{\mathbf{P}}(-a) \longrightarrow V_e \otimes \mathcal{O}_{\mathbf{P}}(-a + 1) \longrightarrow \dots \longrightarrow V_1 \otimes \mathcal{O}_{\mathbf{P}}(-a + e) \longrightarrow 0 \quad (7.6)$$

is exact, then $q \leq e$.

(ii). *Let $k_s \geq -a + e$ be integers. If a complex of locally free sheaves on $\mathbf{P} = \mathbf{P}^{q-1}$ of length $e + 2$ of the form*

$$0 \longrightarrow V_{e+1} \otimes \mathcal{O}_{\mathbf{P}}(-a) \longrightarrow V_e \otimes \mathcal{O}_{\mathbf{P}}(-a + 1) \longrightarrow \dots \longrightarrow V_1 \otimes \mathcal{O}_{\mathbf{P}}(-a + e) \longrightarrow \bigoplus_{s=1}^t (Z_s \otimes \mathcal{O}_{\mathbf{P}}(k_s)) \longrightarrow 0$$

is exact, then $q \leq e + 1$.

Proof. It is easy to see that if $e = 2$, then $q = 2$ since line bundles on projective spaces have no intermediate cohomology. We can then suppose $e > 2$. After having twisted the complex (7.6) by $\mathcal{O}_{\mathbf{P}}(-e + a)$, we get the exact complex

$$\begin{aligned} 0 \longrightarrow V_{e+1} \otimes \mathcal{O}_{\mathbf{P}}(-e) \xrightarrow{f_1} V_e \otimes \mathcal{O}_{\mathbf{P}}(-e+1) \longrightarrow \dots \\ \dots \longrightarrow V_4 \otimes \mathcal{O}_{\mathbf{P}}(-3) \xrightarrow{f_{e-2}} V_3 \otimes \mathcal{O}_{\mathbf{P}}(-2) \longrightarrow V_2 \otimes \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow V_1 \otimes \mathcal{O}_{\mathbf{P}} \longrightarrow 0. \end{aligned}$$

Set $W_j = \text{Coker } f_j$ for $j = 1, \dots, e - 2$. If by contradiction $q > e$, then we would have $H^{e-1-j}(\mathbf{P}, W_j) \neq 0$ for every $j = 1, \dots, e - 2$, and hence $H^{e-1}(\mathbf{P}, V_{e+1} \otimes \mathcal{O}_{\mathbf{P}}(-e)) \neq 0$. This yields a contradiction and therefore $q \leq e$. To prove point (ii), we proceed analogously as we just did. \square

Corollary 7.1.4. *Let X be a compact Kähler manifold of dimension n and irregularity $q(X) > n$. If $m(X) = n$, then we have $(-1)^{n-1}\chi(\Omega_X^1) \geq 2$ and $(-1)^{n-p}\chi(\Omega_X^p) \geq 1$ for any $p = 2, \dots, n - 2$.*

Proof. The corollary is clear for $n = 1$, so we assume $n \geq 2$. We first prove that $h^n(X, \Omega_X^p) \neq 0$ so that the complex $\underline{\mathbf{L}}_X^p$ is non-zero as well. To see this, note that by Proposition 7.1.2 (ii) the assumption $m(X) = n$ implies that the non-zero complex $\underline{\mathbf{L}}_X^n$ is exact at the first n -steps from the right. If we had $h^n(X, \Omega_X^p) = h^p(X, \omega_X) = 0$, then $\underline{\mathbf{L}}_X^n$ would induce an exact complex of length $\leq n$ whose terms are sums of line bundles all of the same degree, and by Lemma 7.1.3 we would have a contradiction.

By Proposition 7.1.2, $\underline{\mathbf{L}}_X^p$ is exact at the first $(n-p)$ -steps from the left. Therefore we get an exact sequence:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-n) \otimes H^0(X, \Omega_X^p) \longrightarrow \dots \longrightarrow \mathcal{O}_{\mathbf{P}}(-p-1) \otimes H^{n-p-1}(X, \Omega_X^p) \xrightarrow{f} \\ \mathcal{O}_{\mathbf{P}}(-p) \otimes H^{n-p}(X, \Omega_X^p) \longrightarrow F \longrightarrow 0, \end{aligned}$$

where the locally free sheaf F is the cokernel of the map f . We also get an induced map of locally free sheaves $h : F \longrightarrow \mathcal{O}_{\mathbf{P}}(-p+1) \otimes H^{n-p+1}(X, \Omega_X^p)$, which is of constant rank. Denoting by E the kernel of h , we obtain another exact sequence of locally free sheaves

$$0 \longrightarrow E \longrightarrow F \longrightarrow \mathcal{O}_{\mathbf{P}}(-p+1) \otimes H^{n-p+1}(X, \Omega_X^p) \longrightarrow \dots \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^n(X, \Omega_X^p) \longrightarrow 0,$$

from which we deduce $\text{rank } E = (-1)^{n-p} \chi(\Omega_X^p)$. If E were the zero sheaf, then the complex $\underline{\mathbf{L}}_X^p$ would be an exact sequence of length $\leq n+1$ of the form (7.6), which is not possible by our hypothesis $q(X) > n$. Thus $\text{rank } E = (-1)^{n-p} \chi(\Omega_X^p) \geq 1$. For $p = n-1$ we can slightly improve our bound. In this case $\underline{\mathbf{L}}_X^{n-1}$ is exact at the first $(n-1)$ -steps from the right, and hence we get an exact sequence

$$\begin{aligned} 0 \longrightarrow G \longrightarrow \mathcal{O}_{\mathbf{P}}(-n+1) \otimes H^1(X, \Omega_X^{n-1}) \xrightarrow{g} \\ \mathcal{O}_{\mathbf{P}}(-n+2) \otimes H^2(X, \Omega_X^{n-1}) \longrightarrow \dots \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^n(X, \Omega_X^{n-1}) \longrightarrow 0, \end{aligned}$$

where the locally free sheaf G is the kernel of the map g . Thus there is a natural map $h' : \mathcal{O}_{\mathbf{P}}(-n) \otimes H^0(X, \Omega_X^{n-1}) \rightarrow G$ and a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-n) \otimes H^0(X, \Omega_X^{n-1}) \xrightarrow{h'} G \longrightarrow E' \longrightarrow 0,$$

where E' is the cokernel of the map h' . The locally free sheaf E' is non-zero again by Lemma 7.1.3. If the rank of E' were one, then E' would be a line bundle, *i. e.* $E' = \mathcal{O}_{\mathbf{P}}(j)$ for some integer j , and $G \in \text{Ext}^1(\mathcal{O}_{\mathbf{P}}(j), \mathcal{O}_{\mathbf{P}}(-n) \otimes H^0(X, \Omega_X^{n-1})) = H^1(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(n+j) \otimes H^0(X, \Omega_X^{n-1})^\vee) = 0$. Hence G would split as a sum of line bundles and by Lemma 7.1.3 (ii) this is again not possible. Therefore $\text{rank } E' = (-1)^{n-1} \chi(\Omega_X^1) \geq 2$. \square

7.2 Inequalities for the Hodge numbers

After having studied the exactness of $\underline{\mathbf{L}}_X^p$, we can derive inequalities for the Hodge numbers by using well-known results for locally free sheaves on projective spaces: the Evans-Griffith Theorem and the non-negativity of the Chern classes for globally generated locally free sheaves.

In this section X denotes a compact Kähler manifold of dimension n and irregularity $q = q(X) \geq 2$. Let $m = m(X)$ be as in (7.4) and $h^{p,q} = h^{p,q}(X) = \dim H^q(X, \Omega_X^p)$ be the Hodge numbers of X .

Before stating the results, we introduce some notation. Fix an integer $0 \leq p \leq n$. If $n - p < m \leq n$, for $1 \leq i \leq q - 1$ we define $\gamma_i(X, \Omega_X^p)$ to be the coefficient of t^i in the formal power series:

$$\gamma(X, \Omega_X^p; t) \stackrel{\text{def}}{=} \prod_{j=1}^{m-n+p} (1 - jt)^{(-1)^j h^{p, 2n-m-p+j}} \in \mathbf{Z}[[t]].$$

If $p < m \leq n$, for $1 \leq i \leq q - 1$ we define $\delta_i(X, \Omega_X^p)$ to be the coefficient of t^i in the formal power series:

$$\delta(X, \Omega_X^p; t) \stackrel{\text{def}}{=} \prod_{j=1}^{m-p} (1 - jt)^{(-1)^j h^{p, m-p-j}} \in \mathbf{Z}[[t]].$$

If $m = +\infty$, for $i = 1, \dots, q - 1$ we define $\varepsilon_i(X, \Omega_X^p)$ to be the coefficient of t^i in the formal power series:

$$\varepsilon(X, \Omega_X^p; t) \stackrel{\text{def}}{=} \prod_{j=1}^n (1 - jt)^{(-1)^j h^{p, n-j}} \in \mathbf{Z}[[t]].$$

Also consider the following pieces of the Euler characteristic of the bundle Ω_X^p . If $n - p < m \leq n$ define

$$\chi^{\geq 2n-m-p}(\Omega_X^p) \stackrel{\text{def}}{=} \sum_{j=2n-m-p}^n (-1)^{2n-m-p+j} h^{p, j}$$

and if $p < m \leq n$ define

$$\chi^{\leq m-p}(\Omega_X^p) \stackrel{\text{def}}{=} \sum_{j=0}^{m-p} (-1)^{m-p+j} h^{p, j}.$$

Theorem 7.2.1. *Let X be a compact Kähler manifold of dimension n and irregularity $q \geq 2$.*

Let $m = m(X) = \min \{\text{codim } Z(\omega) \mid 0 \neq \omega \in H^0(X, \Omega_X^1)\}$ and let $0 \leq p \leq n$ be an integer.

(i). *If $n - p < m \leq n$, then any Schur polynomial of weight $\leq q - 1$ in the $\gamma_i(X, \Omega_X^p)$ is non-negative. In particular $\gamma_i(X, \Omega_X^p) \geq 0$ for every $1 \leq i \leq q - 1$. Moreover, if i is such that $\chi^{\geq 2n-m-p}(\Omega_X^p) < i < q$, then $\gamma_i(X, \Omega_X^p) = 0$.*

(ii). *If $p < m \leq n$, then any Schur polynomial of weight $\leq q - 1$ in the $\delta_i(X, \Omega_X^p)$ is non-negative. In particular $\delta_i(X, \Omega_X^p) \geq 0$ for every $1 \leq i \leq q - 1$. Moreover, if i is such that $\chi^{\leq m-p}(\Omega_X^p) < i < q$, then $\delta_i(X, \Omega_X^p) = 0$.*

(iii). If $m = +\infty$, then $\varepsilon_i(X, \Omega_X^p) = 0$ for every $i = 1, \dots, q - 1$.

Proof. If $m > n - p$, then by Proposition 7.1.2 (ii) $\underline{\mathbf{L}}_X^p$ is exact at the first $(m - n + p)$ -steps from the right, and hence we get an exact sequence whose maps are of constant rank

$$\begin{aligned} 0 \longrightarrow G \longrightarrow \mathcal{O}_{\mathbf{P}}(n - m - p) \otimes H^{2n-m-p}(X, \Omega_X^p) \xrightarrow{g} & (7.7) \\ \mathcal{O}_{\mathbf{P}}(n - m - p + 1) \otimes H^{2n-m-p+1}(X, \Omega_X^p) \longrightarrow \dots \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^n(X, \Omega_X^p) \longrightarrow 0, \end{aligned}$$

and where G is the kernel of the map g . Dualizing (7.7) and then tensorizing it by $\mathcal{O}_{\mathbf{P}}(n - m - p)$, we see that the polynomial $\gamma(X, \Omega_X^p; t)$ is the Chern polynomial of the locally free sheaf $G^\vee(n - m - p)$ and moreover that its Chern classes are identified with the $\gamma_i(X, \Omega_X^p)$'s. To conclude we note that $G^\vee(n - m - p)$ is globally generated and therefore its Chern classes, as well as the Schur polynomials in these, are non-negative. The last statement of (i) follows from the fact that $c_i(G) = 0$ for $i > \text{rank } G = \chi^{\geq 2n-m-p}(\Omega_X^p)$.

The proof of (ii) is analogous to the proof of the previous point. If $m > p$ then by Proposition 7.1.2 (i) $\underline{\mathbf{L}}_X^p$ is exact at the first $(m - p)$ -steps from the left and induces the following exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-n) \otimes H^0(X, \Omega_X^p) \longrightarrow \dots \longrightarrow \mathcal{O}_{\mathbf{P}}(-n + m - p - 1) \otimes H^{m-p-1}(X, \Omega_X^p) \xrightarrow{f} & (7.8) \\ \mathcal{O}_{\mathbf{P}}(-n + m - p) \otimes H^{m-p}(X, \Omega_X^p) \longrightarrow F \longrightarrow 0 \end{aligned}$$

where F is the cokernel of the map f . Tensoring (7.8) by $\mathcal{O}_{\mathbf{P}}(n-m+p)$ we get that $F(n-m+p)$ is globally generated and moreover that its Chern polynomial is $\delta(X, \Omega_X^p; t)$. At this point we conclude as in (i).

If $m = +\infty$ then $\underline{\mathbf{L}}_X^p$ is everywhere exact and $\varepsilon(X, \Omega_X^p; t)$ is just the Chern polynomial of the zero sheaf. Thus its Chern classes satisfy $\varepsilon_i(X, \Omega_X^p) = 0$, for every $i = 1, \dots, q-1$. \square

Under the assumption of Theorem 7.2.1 we also have

Theorem 7.2.2. (i). *Suppose $n-p < m \leq n$. If $q \geq n-p$, then $h^{n-p,1} \geq h^{n-p,0} + q - 1$. If moreover $q \geq \max\{2, m-n+p, n-p\}$, then $\chi^{\geq 2n-m-p}(\Omega_X^p) \geq q + n - m - p$.*

(ii). *Suppose $p < m \leq n$. If $q \geq p$, then $h^{p,1} \geq h^{p,0} + q - 1$. If moreover $q \geq \max\{2, m-p, p\}$, then $\chi^{\leq m-p}(\Omega_X^p) \geq q - m + p$.*

Proof. (i). By Proposition 7.1.2, $\underline{\mathbf{L}}_X^p$ is exact at the first $(m-n+p)$ -steps from the right. Since $q \geq n-p$ we can prove, with a similar argument to the one used in Corollary 7.1.4, that $h^n(X, \Omega_X^p) \neq 0$ and hence that the complex $\underline{\mathbf{L}}_X^p$ is non-zero as well. By (7.7) we obtain a surjection $H^{n-1}(X, \Omega_X^p) \otimes \mathcal{O}_{\mathbf{P}} \rightarrow H^n(X, \Omega_X^p) \otimes \mathcal{O}_{\mathbf{P}}(1)$, so that the inequality $h^{n-p,1} \geq h^{n-p,0} + q - 1$ is an application of Example 7.2.2 in (39) which we recall here for reader's ease. Let E be an ample locally free sheaf on a projective variety Y of rank e , and F be another locally free sheaf on Y of rank f such that its dual F^\vee is nef. If E is a quotient of F , then we have $f \geq e + \dim Y$.

Now we prove the second inequality. By looking again at (7.7), we note that $\chi^{\geq 2n-m-p}(\Omega_X^p) = \text{rank } G \geq 0$. If $q = m - n + p$ then the inequality is trivially satisfied. If $q = m - n + p + 1$

(resp. $q = m - n + p + 2$) then by Lemma 7.1.3 (i) $\text{rank } G \geq 1$ (resp. $\text{rank } G \geq 2$) and the inequality follows. So we can suppose $q \geq m - n + p + 3$. By chasing the sequence (7.7), we see that $H^k(\mathbf{P}, G^\vee(j)) = 0$ for all $j \in \mathbf{Z}$ and $k = 1, \dots, q + n - m - p - 2$. By Lemma 7.1.3 we have that G^\vee is neither the zero sheaf nor splits as a sum of line bundles. Thus the Evans-Griffith Theorem (cf. (39) p. 92) yields $\text{rank } G = \chi^{\geq 2n-m-p}(\Omega_X^p) \geq q + n - m - p$.

(ii). The hypothesis $p < m \leq n$ implies that $\underline{\mathbf{L}}_X^p$ is exact at the first $(m - p)$ -steps from the left. Since $q \geq p$ we have that $h^0(X, \Omega_X^p) = h^n(X, \Omega_X^{n-p}) \neq 0$ as in (i), and therefore the complex $\underline{\mathbf{L}}_X^p$ is non-zero as well. After having noted that $\text{rank } F = \chi^{\leq m-p}(\Omega_X^p)$, we argue as in the previous point. \square

7.2.1 The case $m(X) = \dim X$

When $m(X) = \dim X$ further inequalities hold thanks to Catanese's work (52).

Lemma 7.2.3. *If X is an irregular compact Kähler manifold with $m(X) = \dim X$, then X does not carry any higher irrational pencils (cf. §6.3.3).*

Proof. We proceed by contradiction. Suppose a higher irrational pencil $f : X \rightarrow Y$ exists and let $\text{alb}_Y : Y \rightarrow \text{Alb}(Y)$ be the Albanese map of Y , which is well defined since Y is normal. The map alb_Y is not surjective, hence following an idea contained in the proof of (21) Proposition 2.2, one can show that given a general point $y \in Y$ there exists a holomorphic 1-form ω of $\text{Alb}(Y)$ whose restriction to $\text{alb}_Y(Y)$ vanishes at the point $\text{alb}_Y(y)$. Pulling back ω to X , we get a holomorphic 1-form which vanishes along some fibers of f which are of positive dimension, this contradicting the hypothesis $m(X) = \dim X$. The form ω can be constructed as follows.

Let z be a smooth point of the Albanese image $\text{alb}_Y(Y) \subset \text{Alb}(Y)$. The coderivative map $T_z^* \text{Alb}(Y) \rightarrow T_z^* \text{alb}_Y(Y)$ is surjective with non trivial kernel. Then take ω to be the extension to a holomorphic one-form on $\text{Alb}(Y)$ of any non-zero form belonging to this kernel. \square

The previous lemma allow us to use results of (16) Remark 4.3 and the ones in the references therein, so when $m(X) = \dim X$ we obtain other inequalities:

$$h^{0,k} \geq k(q(X) - k) + 1 \quad k = 0, \dots, \dim X, \quad (7.9)$$

and if $\dim X \geq 3$

$$h^{0,2} \geq 4q(X) - 10. \quad (7.10)$$

These inequalities will be used to give asymptotic bounds for the Hodge numbers in terms of $q(X)$ in dimension three and four (*cf.* Corollaries 7.3.1 and 7.3.2).

7.2.2 The case $m(X) = +\infty$

The proof of Lemma 7.2.3 also shows that compact Kähler manifolds X with $m(X) = +\infty$ have surjective Albanese map and consequently $q(X) \leq \dim X$ (this inequality also follows from Proposition 7.1.2 and Lemma 7.1.3). Furthermore, since for this case the complexes $\underline{\mathbf{L}}_X^p$ are everywhere exact, we automatically get $\chi(\Omega_X^p) = 0$ for all p . Complex smooth projective irregular surfaces with $m(X) = +\infty$ are completely classified. They are either abelian surfaces, or bielliptic surfaces or geometrically ruled surfaces over an elliptic curve. This can be seen

by computing their Hodge numbers with Theorem 7.2.1 (iii) and by the classification of complex smooth projective surfaces.

In higher dimension, according to a theorem of Popa and Schnell answering to a conjecture of Luo and Zhang (*cf.* (55) Theorem 2.1), the Kodaira dimension of such varieties is $\leq \dim X - q(X)$. Lastly, we observe that if X is a complex smooth projective variety with $q(X) = \dim X$ and $m(X) = +\infty$, then X is an abelian variety. To see this we first note that ω_X is trivial since Ω_X^1 has $\dim X$ linearly independent sections which never vanish. Also, the cohomological support loci $V^i(\omega_X)$ consists of at most a finite set of points for all $i \geq 0$ (*cf.* (10) Remark on p. 405) and therefore X is of maximal Albanese dimension by (2.8). Now, a theorem of Ein and Lazarsfeld (*cf.* (56) Theorem 1.8) says that if a smooth projective variety X is of maximal Albanese dimension with $\dim V^0(\omega_X) = 0$, then X is birational to an abelian variety. Since $\omega_X \simeq \mathcal{O}_X$, X is in fact an abelian variety.

Finally, we note that Theorem 7.2.1 (iii) determines the Hodge numbers of compact Kähler threefolds X with irregularity $q(X) = 2$ and $m(X) = +\infty$:

$$h^{0,2} = 1, h^{0,3} = 0, h^{1,1} = 5, h^{1,2} = 4.$$

7.3 Asymptotic bounds for threefolds and fourfolds

In this section we list concrete inequalities coming from Theorems 7.2.1 and 7.2.2 in the most interesting case $m(X) = \dim X$, $q(X) \geq \dim X$, and for $\dim X = 3, 4, 5$. Moreover for threefolds and fourfolds we list asymptotic bounds in terms of the irregularity $q(X)$ for all the

Hodge numbers. We also point out that some of the inequalities are still valid for some values of $q(X)$ smaller than $\dim X$ and that other inequalities hold for different values of $m(X)$. We set $q = q(X)$, $n = \dim X$ and $h^{p,q} = h^{p,q}(X)$.

Let us start with Theorem 7.2.1. We get a first set of inequalities by imposing the conditions $\gamma_1(X, \Omega_X^p) \geq 0$ for $p = 0, 1, 2$. Hence

$$h^{0,2} \geq 2q - 3, \quad h^{1,1} \geq 2q \quad \text{for } n = 3$$

$$h^{1,2} \geq 2h^{1,1} - 3q, \quad h^{1,2} \geq 2h^{0,2}, \quad h^{0,3} \geq 2h^{0,2} - 3q + 4 \quad \text{for } n = 4$$

$$h^{0,4} \geq 4q - 3h^{0,2} + 2h^{0,3} - 5, \quad h^{1,4} \geq 4h^{1,1} - 3h^{1,2} + 2h^{1,3}, \quad h^{2,2} \geq 2h^{1,2} - 3h^{0,2} \quad \text{for } n = 5$$

Finer inequalities are obtained by solving $\gamma_2(X, \Omega_X^p) \geq 0$. For $n = 3$ we have

$$h^{0,2} \geq 2q - \frac{7}{2} + \frac{\sqrt{8q - 23}}{2}, \quad h^{1,1} \geq 2q - \frac{1}{2} + \frac{\sqrt{8q + 1}}{2} \quad (7.11)$$

and for $n = 4$ we get

$$h^{0,3} \geq 2h^{0,2} - 3q + \frac{7}{2} + \frac{\sqrt{8h^{0,2} - 24q + 49}}{2} \quad (7.12)$$

$$h^{1,2} \geq 2h^{0,2} - \frac{1}{2} + \frac{\sqrt{8h^{0,2} + 1}}{2} \quad (7.13)$$

$$h^{1,2} \geq 2h^{1,1} - 3q + \sqrt{4h^{1,1} - 9q}$$

where the quantity $4h^{1,1} - 9q$ is non-negative by the first inequality of Theorem 7.2.2 (i) and the quantity $8h^{0,2} - 24q + 49$ is non-negative by inequality (7.10). Finally for $n = 5$ we get

$$\begin{aligned} h^{0,4} &\geq 4q - 3h^{0,2} + 2h^{0,3} - \frac{11}{2} + \frac{\sqrt{48q - 24h^{0,2} + 8h^{0,3} - 79}}{2} \\ h^{1,4} &\geq 2h^{1,3} + 4h^{1,1} - 3h^{1,2} - \frac{1}{2} + \frac{\sqrt{48h^{1,1} - 24h^{1,2} + 8h^{1,3} + 1}}{2} \\ h^{2,2} &\geq 2h^{1,2} - 3h^{0,2} - \frac{1}{2} + \frac{\sqrt{8h^{1,2} - 24h^{0,2} + 1}}{2} \end{aligned}$$

which hold as long as the quantities under the square roots are non-negative.

Applying Theorem 7.2.2 with $m(X) = n$ and $q(X) \geq n$, we get for $n = 3$

$$\chi(\omega_X) \geq q - 3, \quad h^{1,1} \geq 2q - 1, \quad h^{1,2} \geq h^{1,1} - 2, \quad h^{1,2} \geq h^{0,2} + q - 1,$$

for $n = 4$

$$\begin{aligned} \chi(\omega_X) &\geq q - 4, & h^{2,2} &\geq h^{1,2} - h^{0,2} + q - 2, & h^{1,3} &\geq h^{1,2} - h^{1,1} + 2q - 3 \\ h^{1,1} &\geq 2q - 1, & h^{1,2} &\geq h^{2,0} + q - 1, & h^{1,3} &\geq h^{0,3} + q - 1 \end{aligned}$$

and for $n = 5$

$$\begin{aligned} \chi(\omega_X) &\geq q - 5, & h^{1,4} &\geq h^{1,3} - h^{1,2} + h^{1,1} - 4, & h^{1,1} &\geq 2q - 1, \\ 2h^{1,2} &\geq h^{2,2} + h^{0,2} + q - 3, & h^{1,2} &\geq h^{0,2} + q - 1, & h^{1,2} &\geq h^{1,3} - h^{0,3} + q - 2, \\ h^{1,3} &\geq h^{0,3} + q - 1, & h^{1,4} &\geq h^{0,4} + q - 1. \end{aligned}$$

We select the strongest of the inequalities above in dimension three and four and the ones in §7.2 in statements formulated asymptotically with respect to q for simplicity.

Corollary 7.3.1. *Assume $n = m(X) = 3$. Then asymptotically $h^{0,2} \succeq 4q$, $h^{0,3} \succeq 4q$, $h^{1,1} \succeq 2q + \sqrt{2q}$ and $h^{1,2} \succeq 5q + \sqrt{2q}$.*

Proof. The inequality (7.10) implies the asymptotic bound $h^{0,2} \succeq 4q$. The inequality $\chi(\omega_X) \geq q - 3$ of Theorem 7.2.2 implies the inequality $h^{0,3} \geq h^{0,2} - 2$ and therefore $h^{0,3} \succeq 4q$. The asymptotic bound for $h^{1,1}$ follows from (7.11). Finally, since by Corollary 7.1.4 $\chi(\Omega_X^1) \geq 2$, we also get the bound for $h^{1,2}$. \square

Corollary 7.3.2. *Assume $n = m(X) = 4$. Then asymptotically $h^{0,2} \succeq 4q$, $h^{0,3} \succeq 5q + \sqrt{2q}$, $h^{0,4} \succeq 4q$, $h^{1,1} \succeq 2q$, $h^{1,2} \succeq 8q + 2\sqrt{2q}$, $h^{1,3} \succeq 12q + 3\sqrt{2q}$ and $h^{2,2} \succeq 8q + 4\sqrt{2q}$.*

Proof. The asymptotic bounds for $h^{0,2}$, $h^{0,3}$ and $h^{0,4}$ follow from (7.10), (7.12) and (7.9) respectively. Using the first inequality of Theorem 7.2.2 (i) we get $h^{1,1} \succeq 2q$, and by (7.13) we also get the bound for $h^{1,2}$. Lastly, by Corollary 7.1.4 we have $\chi(\Omega_X^1) \leq 2$ and $\chi(\Omega_X^2) \geq 1$ which provide the bounds for $h^{1,3}$ and $h^{2,2}$. \square

7.4 Regularity of cohomology modules

Setting $E = \bigwedge^* H^1(X, \mathcal{O}_X)$ for the graded exterior algebra over $H^1(X, \mathcal{O}_X)$, via cup product we consider the E -modules $\bigoplus_i H^i(X, \Omega_X^p)$ for $p = 0, \dots, n$. Using the Bernstein-Gel'fand-Gel'fand (BGG) correspondence and generic vanishing theorems for bundles of holomorphic p -forms, we give a bound on their regularity. The case $p = n$ has been studied in (16) Theorem B. If we denote by k the dimension of the general fiber of the Albanese map $\text{alb}_X : X \rightarrow \text{Alb}(X)$, then the E -module $\bigoplus_i H^i(X, \omega_X)$ is k -regular but not $(k - 1)$ -regular. Moreover, in the paper (57), the authors show that the minimal E -resolution of the module $\bigoplus_i H^i(X, \omega_X)$ is a direct sum of shifts of linear resolutions. Here we study the remaining cases $p < n$. Before to show our result, we recall the *defect of semismallness* of the Albanese map

$$\delta(\text{alb}_X) := \max_{l \in \mathbb{N}} \{2l - \dim X + \dim \text{Alb}(X)_l\}$$

where $\text{Alb}(X)_l := \{y \in \text{Alb}(X) \mid \dim \text{alb}_X^{-1}(y) \geq l\}$.

Theorem 7.4.1. *Let X be a complex smooth irregular projective variety of dimension n . If $p > \delta(\text{alb}_X)$, then the E -module $\bigoplus_i H^i(X, \Omega_X^p)$ is $(n - p + \delta(\text{alb}_X))$ -regular.*

Before giving the proof of the previous theorem, we recall the BGG correspondence and its relationship to the regularity of a module.

7.4.1 BGG correspondence and regularity

Let V be a complex vector space of dimension q , $W = V^\vee$ its dual space, $E = \bigwedge^* V$ the graded exterior algebra over V and $S = \text{Sym}^* W$ the symmetric algebra over W .

Definition 7.4.2. A finitely generated graded E -module $Q = \bigoplus_{j=0}^n Q_{-j}$ with graded components Q_{-j} in degrees $-j$ is said c -regular if it is generated in degrees 0 up to $-c$ and if its minimal free resolution has at most $c+1$ linear strands. Equivalently, Q is c -regular if and only if $\mathrm{Tor}_i^E(Q, \mathbf{C})_{-i-j} = 0$ for all $i \geq 0$ and all $j \geq c+1$.

The dual over E of a finitely generated graded module $P = \bigoplus_{j=0}^n P_j$ with graded components P_j in degrees j is defined to be the E -module $\widehat{P} = \bigoplus_{j=0}^n P_{-j}^\vee$ with graded components P_{-j}^\vee in degrees $-j$, so that \widehat{P} is a module with no component of positive degree (cf. (58), (59), (16)).

Let e_1, \dots, e_q be a basis of V and x_1, \dots, x_q be the dual basis. The BGG correspondence associated to a finitely generated graded E -module $P = \bigoplus_{j=0}^n P_j$ is the linear complex $\mathbf{L}(P)$ of free S -modules

$$\mathbf{L}(P) : \quad \dots \longrightarrow S \otimes_{\mathbf{C}} P_{j+1} \longrightarrow S \otimes_{\mathbf{C}} P_j \longrightarrow S \otimes_{\mathbf{C}} P_{j-1} \longrightarrow \dots$$

with differentials given by

$$s \otimes p \mapsto \sum_{i=1}^q x_i s \otimes e_i p.$$

For references about the BGG correspondence cf. (60), (59) and Chapter 7B of (58).

The following proposition is a criterion to bound the regularity of an E -module $P = \bigoplus_{j=0}^n P_j$; the proof can be found in (16) Proposition 3.2.

Proposition 7.4.3. *Let $P = \bigoplus_{j=0}^n P_j$ be a finitely generated graded E -module with no components in negative degree. The dual module \widehat{P} of P over E is c -regular if and only if the complex $\mathbf{L}(P)$*

$$0 \longrightarrow S \otimes_{\mathbf{C}} P_n \longrightarrow S \otimes_{\mathbf{C}} P_{n-1} \longrightarrow \dots \longrightarrow S \otimes_{\mathbf{C}} P_{c+1} \longrightarrow S \otimes_{\mathbf{C}} P_c$$

is exact at the first $(n - c)$ -steps from the left.

7.4.2 Proof of Theorem 7.4.1

Let X be a complex irregular smooth projective variety of dimension n . Set $V = H^1(X, \mathcal{O}_X)$, $W = V^\vee$, $E = \bigwedge^* V$ and $S = \text{Sym}^* W$. Fix an integer $p = 0, \dots, n$. Via cup product we consider the graded E -module

$$P_X^p = \bigoplus_i H^i(X, \Omega_X^{n-p})$$

where the graded component $H^i(X, \Omega_X^{n-p})$ is in degree $n - i$. By Serre duality the dual module of P_X^p over E is the module

$$Q_X^p = \bigoplus_i H^i(X, \Omega_X^p)$$

where the graded component $H^i(X, \Omega_X^p)$ is in degree $-i$. We apply Proposition 7.4.3 in order to study the regularity of Q_X^p . It is not difficult to show that $\mathbf{L}(P_X^p)$ is isomorphic to a complex \mathbf{L}_X^{n-p} of S -graded modules defined as $\mathbf{L}_X^{n-p} \stackrel{\text{def}}{=} \Gamma_*(\underline{\mathbf{L}}_X^{n-p})$

$$\mathbf{L}_X^{n-p} : \quad 0 \longrightarrow S \otimes_{\mathbf{C}} H^0(X, \Omega_X^{n-p}) \longrightarrow S \otimes_{\mathbf{C}} H^1(X, \Omega_X^{n-p}) \longrightarrow \dots \longrightarrow S \otimes_{\mathbf{C}} H^n(X, \Omega_X^{n-p}) \longrightarrow 0$$

(cf. (61) p. 118 for the definition of Γ_* and (16) Lemma 3.3 for a proof of the isomorphism between $\mathbf{L}(P_X^p)$ and \mathbf{L}_X^{n-p} in the case $p = n$). At this point Theorem 7.4.1 follows from the previous discussion and the following

Proposition 7.4.4. *Under the assumptions of Theorem 7.4.1, the complexes \mathbf{L}_X^{n-p} and $\underline{\mathbf{L}}_X^{n-p}$ are exact at the first $(p - \delta(\text{alb}_X))$ -steps from the left provided that $p > \delta(\text{alb}_X)$.*

Proof. We follow (16) Proposition 2.1. We start with the study of the exactness of \mathbf{L}_X^{n-p} . Let $\mathbf{A} = \text{Spec}(\text{Sym}^*W)$ and recall the complex

$$\mathcal{K}_{n-p} : 0 \longrightarrow \mathcal{O}_{\mathbf{A}} \otimes H^0(X, \Omega_X^{n-p}) \longrightarrow \mathcal{O}_{\mathbf{A}} \otimes H^1(X, \Omega_X^{n-p}) \longrightarrow \dots \longrightarrow \mathcal{O}_{\mathbf{A}} \otimes H^n(X, \Omega_X^{n-p}) \longrightarrow 0$$

defined in (7.2). Since $\Gamma(\mathbf{A}, \mathcal{O}_{\mathbf{A}}) = \text{Sym}^*W$, there exists an isomorphism of complexes $\mathbf{L}_X^{n-p} \simeq \Gamma(\mathbf{A}, \mathcal{K}_{n-p})$ where Γ is the global section functor. Therefore the study of the exactness of \mathbf{L}_X^{n-p} is equivalent to the study of the exactness of \mathcal{K}_{n-p} since Γ is an exact functor on affine varieties. Moreover, if we denote by \mathbf{V} for the vector space V viewed as a complex manifold, then by GAGA the exactness of \mathcal{K}_{n-p} is in turn equivalent to the exactness of the following complex

$$\mathcal{K}_{n-p}^{an} : 0 \longrightarrow \mathcal{O}_{\mathbf{V}} \otimes H^0(X, \Omega_X^{n-p}) \longrightarrow \mathcal{O}_{\mathbf{V}} \otimes H^1(X, \Omega_X^{n-p}) \longrightarrow \dots \longrightarrow \mathcal{O}_{\mathbf{V}} \otimes H^n(X, \Omega_X^{n-p}) \longrightarrow 0.$$

We note that \mathcal{K}_{n-p}^{an} is a complex of analytic sheaves where the differentials at a point of \mathbf{V} are again given by cupping with the corresponding element. If $p > \delta(\text{alb}_X)$ then it is enough to check the exactness of \mathcal{K}_{n-p}^{an} at the first $(p - \delta(\text{alb}_X))$ -steps from the left, namely the vanishing

of the cohomologies $\mathcal{H}^i(\mathcal{K}_{n-p}^{an})$ for any $i < p - \delta(\text{alb}_X)$. Since the differentials of \mathcal{K}_{n-p}^{an} scale linearly in radial directions through the origin, it is then enough to check the vanishing of the stalks at the origin 0, *i.e.*

$$\mathcal{H}^i(\mathcal{K}_{n-p}^{an})_0 = 0$$

for $i < p - \delta(\text{alb}_X)$.

Let $p_1 : X \times \text{Pic}^0(X) \rightarrow X$ and $p_2 : X \times \text{Pic}^0(X) \rightarrow \text{Pic}^0(X)$ be the projections from $X \times \text{Pic}^0(X)$ onto the first and second factor respectively, and let \mathcal{P} be a normalized Poincaré line bundle on $X \times \text{Pic}^0(X)$. Then Theorem 6.2 in (62) gives an isomorphism

$$\mathcal{H}^i(\mathcal{K}_{n-p}^{an})_0 \simeq R^i p_{2*}(p_1^* \Omega_X^{n-p} \otimes \mathcal{P})_0 \quad (7.14)$$

via the exponential map $\exp : V \rightarrow \text{Pic}^0(X)$. To prove the vanishing of the stalks in (7.14), we use generic vanishing theorems for bundles of holomorphic p -forms. Denote by $V^i(\Omega_X^j) = \{\alpha \in \text{Pic}^0(X) \mid h^i(X, \Omega_X^j \otimes \alpha) > 0\}$ the cohomological support loci of Ω_X^j . The vanishing of the stalks $R^i p_{2*}(p_1^* \Omega_X^{n-p} \otimes \mathcal{P})_0$ is closely related to the algebraic closed sets $V^i(\Omega_X^p)_0$ as shown in Theorem 2.2.3 (v)

$$\begin{aligned} R^i p_{2*}(p_1^* \Omega_X^{n-p} \otimes \mathcal{P})_0 = 0 \text{ for all } i < p - \delta(\text{alb}_X) &\iff \\ \text{codim}_{\text{Pic}^0(X), 0} V^i(\Omega_X^p) \geq i - n + p - \delta(\text{alb}_X) \text{ for all } i > 0, \end{aligned}$$

and we conclude then thanks to Theorem 2.2.3 (i). We remark that, for the case $p = n$, Lazarsfeld and Popa use Theorem 2.2.2 (v) and (ii) to get the vanishing of (7.14) for $i < n - k$, and consequently to get a bound for the regularity of $\bigoplus_i H^i(X, \omega_X)$ depending only on the dimension of the general fiber of the Albanese map.

Finally we observe that $\underline{\mathbf{L}}_X^{n-p}$ is exact at the first $(p - \delta(\text{alb}_X))$ -steps from the left as well, since it is the sheafification of \mathbf{L}_X^{n-p} which is an exact functor. \square

Finally we observe that by Serre duality and the trick we used to prove (ii) of Proposition 7.1.2, we also obtain that the complexes \mathbf{L}_X^{n-p} and $\underline{\mathbf{L}}_X^{n-p}$ are exact at the first $(n - p - \delta(\text{alb}_X))$ -steps from the right as long as $n - p > \delta(\text{alb}_X)$.

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