INEQUALITIES FOR THE HODGE NUMBERS OF IRREGULAR COMPACT KÄHLER MANIFOLDS

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Abstract. Based on work of Lazarsfeld and Popa, we use the derivative complex associated to the bundle of holomorphic p-forms to provide inequalities for all the Hodge numbers of a special class of irregular compact Kähler manifolds. For 3-folds and 4-folds we give an asymptotic bound for all the Hodge numbers in terms of the irregularity. As a byproduct, via the BGG correspondence, we also bound the regularity of the exterior cohomology modules of bundles of holomorphic p-forms.

1. Introduction

Given an irregular compact Kähler manifold X, a problem one tries to understand is under which hypotheses there exist relations between its various Hodge numbers $h^{p,q}(X)$. Along these lines, one can ask whether there exist formulas for the $h^{p,q}(X)$'s in terms of the fundamental invariant $q(X) = h^{1,0}(X)$, the *irregularity* of X. A classical result in this direction is the well-known Castelnuovo-De Franchis inequality $h^{0,2}(X) \geq 2q(X) - 3$, which holds for surfaces that do not carry any fibrations onto a smooth curve of genus $g \geq 2$ (see [BHPV] IV.5.2). This was generalized by Catanese to higher dimensional manifolds as follows: if a manifold X does not admit any *higher irrational pencils*[1] then $h^{0,k}(X) \geq k(q(X)-k)+1$ for all k (*cf.* [Cat]), by means of sophisticated arguments involving the exterior algebra of holomorphic forms. Another generalization of the Castelnuovo-De Franchis inequality is provided by the work of Pareschi and Popa. Theorem A in [PP2] states that if X is of maximal Albanese dimension and does not carry any higher irrational pencils then $\chi(\omega_X) \geq q(X) - \dim \mathrm{X}$. Their inequality is deduced using Generic Vanishing Theory for irregular varieties and the Evans-Griffith Syzygy Theorem.

New techniques for the study of this problem were introduced recently by Lazarsfeld and Popa in [LP]. Their approach relies on the study of a global version of the *derivative complex associated to the structure sheaf* $\mathcal{O}_X$ and on the theory of vector bundles on projective spaces. Their inequalities mainly involve Hodge numbers of type $h^{0,k}(X)$.

In this paper we extend the methods of [LP] to the bundles of holomorphic p-forms Ω^p_X. In this way we get inequalities for all the Hodge numbers of a special class of irregular compact Kähler manifolds. The main idea behind the inequalities of [LP] and the ones of this paper goes as follows. Let d be the dimension of X and $0 \leq p \leq d$ be an integer. Via cup product, any element $0 \neq v \in H^1(X, \mathcal{O}_X)$ defines a complex of vector spaces

$$0 \longrightarrow H^0(X, \Omega^p_X) \xrightarrow{\cup v} H^1(X, \Omega^p_X) \xrightarrow{\cup v} \ldots \xrightarrow{\cup v} H^d(X, \Omega^p_X) \longrightarrow 0. \quad (1)$$

[1]A *higher irrational pencil* is a surjective morphism with connected fibers $f : X \longrightarrow Y$ having the property that a smooth model of Y is of maximal Albanese dimension and with non-surjective Albanese map.

Denoting by $\mathbf{P} = \mathbf{P}_{\mathrm{sub}}(H^1(X, \mathcal{O}_X))$ the projective space of one dimensional linear subspaces of $H^1(X, \mathcal{O}_X)$, we can arrange all of these, as v varies, into a complex $\underline{\mathbf{L}}_X^p$ of locally free sheaves on $\mathbf{P}$:

$$(2) \qquad \underline{\mathbf{L}}_X^p : \qquad 0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-d) \otimes H^0(X, \Omega_X^p) \longrightarrow \mathcal{O}_{\mathbf{P}}(-d+1) \otimes H^1(X, \Omega_X^p) \longrightarrow \dots$$
$$\dots \longrightarrow \mathcal{O}_{\mathbf{P}}(-1) \otimes H^{d-1}(X, \Omega_X^p) \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^d(X, \Omega_X^p) \longrightarrow 0$$

whose fiber at a point $[v] \in \mathbf{P}$ is the complex (1). The complex (1) is known as the *derivative complex associated to the bundle* Ω_X^p *with respect to the vector* v. It was introduced for the first time by Green and Lazarsfeld in [GL1] and [GL2] in their study of Generic Vanishing Theorems for irregular compact Kähler manifolds. In order to get inequalities involving the Hodge numbers of X we need to study the exactness of $\underline{\mathbf{L}}_X^p$. In [LP] the case $p = 0$ is analyzed. For this case we have that if X does not carry any *irregular fibrations*, *i.e.* surjective morphisms $f : X \longrightarrow Y$ with connected fibers having the property that a smooth model of Y is of maximal Albanese dimension, then $\underline{\mathbf{L}}_X^p$ is exact at the first d-steps from the left and the first d morphisms are of constant rank. In general, for $p > 0$, we show that the exactness of the complex $\underline{\mathbf{L}}_X^p$ depends on the non-negative integer $m(X)$, the least codimension of the zero-locus of a non-zero holomorphic one-form, *i.e.*

$$(3) \qquad m(X) = \min\{\operatorname{codim} Z(\omega) \mid 0 \neq \omega \in H^0(X, \Omega_X^1)\},$$

with the convention $m(X) = +\infty$ if every non-zero holomorphic one-form is everywhere non-vanishing. For instance, examples of varieties with $m(X) = \dim \mathrm{X}$ are smooth varieties X whose Albanese map is an embedding with ample normal bundle. More generally, any smooth variety X embedded in an abelian variety A with ample normal bundle such that every holomorphic one-form of X is the restriction of a form of A satisfies $m(X) = \dim \mathrm{X}$ (see [La] Proposition 6.3.10). In Proposition 2.1 we show that if $m(X) > p$ then the complex (2) is exact at the first $(m(X)-p)$-terms from the left where the first $m(X)-p$ maps are of constant rank. This is enough to ensure that the cokernel of the map

$$\mathcal{O}_{\mathbf{P}}(m(X) - d - p - 1) \otimes H^{m(X)-p-1}(X, \Omega_X^p) \longrightarrow \mathcal{O}_{\mathbf{P}}(m(X) - d - p) \otimes H^{m(X)-p}(X, \Omega_X^p)$$

is a locally free sheaf. This in turn leads to inequalities for the Hodge numbers by the Evans-Griffith Theorem and by the fact that the Chern classes of a globally generated locally free sheaf are non-negative.

We turn to a more detailed presentation of our results. To simplify notation we only present the case $m(X) = \dim \mathrm{X}$ and we refer to Theorem 3.1 and Theorem 3.2 for general statements where all possible values of $m(X)$ are considered. Let $1 \leq p \leq d$ be an integer, for any $1 \leq i \leq q-1$ we define $\gamma_i(X, \Omega_X^p)$ to be the coefficient of t^i in the formal power series

$$\gamma(X, \Omega_X^p; t) \stackrel{\text{def}}{=} \prod_{j=1}^{p} (1 - jt)^{(-1)^j h^{p,d-p+j}} \in \mathbf{Z}[[t]]$$

where $h^{i,j} = h^{i,j}(X)$ are the Hodge numbers of X.

Theorem 1.1. *Let X be a compact Kähler manifold of dimension d and irregularity $q \geq 2$. If $m(X) = d$ then*

(i) *Any Schur polynomial of weight $\leq q-1$ in the $\gamma_i(X, \Omega_X^p)$ is non-negative. In particular*

$$\gamma_i(X, \Omega_X^p) \geq 0.$$

(ii) *If* $q \geq \max\{2, p, d-p\}$ *then*

$$\sum_{j=d-p}^{d} (-1)^{d-p+j} h^{p,j} \geq q - p.$$

For instance, the $\gamma_1(X, \Omega_X^p)$'s are non-negative linear polynomials in the variables $h^{p,j}$. For manifolds of dimension two we recover the inequalities $h^{0,2} \geq 2q - 3$ and $h^{1,1} \geq 2q - 1$ (*cf.* [BHPV] IV.5.4). In higher dimensions, the degree i polynomials $\gamma_i(X, \Omega_X^p)$ give new inequalities involving Hodge numbers of X. In addition to the methods for $\mathcal{O}_X$, for $p > 0$, the Serre Duality offers a way to see until which step $\underline{\mathbf{L}}_X^p$ is exact *counting from the right*. This trick leads to further inequalities for the Hodge numbers and, in the special case when $m(X) = \dim \mathrm{X}$ and $q(X) > \dim \mathrm{X}$, also to get a bound for the Euler characteristic of Ω_X^p:

$$\left|\chi(\Omega_X^1)\right| \geq 2 \quad \text{and} \quad \left|\chi(\Omega_X^p)\right| \geq 1 \quad \text{for} \quad p = 2, \ldots, \dim \mathrm{X} - 2$$

(*cf.* Corollary 2.3). In section 4 we list the inequalities coming from Theorem 1.1 for manifolds of dimension three, four and five. Finally for threefolds and fourfolds with $m(X) = \dim \mathrm{X}$ we are able to give asymptotic bounds for all the Hodge numbers in terms of the irregularity q. In the case of threefolds we obtain

$$h^{0,2} \succeq 4q, \quad h^{0,3} \succeq 4q, \quad h^{1,1} \succeq 2q + \sqrt{2q}, \quad h^{1,2} \succeq 5q + \sqrt{2q}$$

and for the case of fourfolds we get

$$h^{0,2} \succeq 4q, \quad h^{0,3} \succeq 5q + \sqrt{2q}, \quad h^{0,4} \succeq 4q$$
$$h^{1,1} \succeq 2q, \quad h^{1,2} \succeq 8q + 2\sqrt{2q}, \quad h^{1,3} \succeq 12q + 3\sqrt{2q}, \quad h^{2,2} \succeq 8q + 4\sqrt{2q}.$$

Asymptotic inequalities for Hodge numbers of type $h^{0,j}$ were already established in [LP] for manifolds which do not carry any irregular fibrations.

In the last section we study the *regularity* of the cohomology modules for a complex smooth irregular projective variety X. Setting $E = \bigwedge^* H^1(X, \mathcal{O}_X)$ for the graded exterior algebra over $H^1(X, \mathcal{O}_X)$, via cup product we consider the E-modules $\bigoplus_i H^i(X, \Omega_X^p)$. Using the Bernstein-Gel'fand-Gel'fand (BGG) correspondence and Generic Vanishing Theorems for bundles of holomorphic p-forms, we give a bound on their regularity. We refer to Section 5 for the definition of regularity for finitely generated graded modules over an exterior algebra and for references about the BGG correspondence and the Generic Vanishing Theorems used. The case $p = \dim \mathrm{X}$ has been studied in [LP] Theorem B. If we denote by k for the dimension of the general fiber of the Albanese map $\mathrm{alb}_X : X \longrightarrow \mathrm{Alb}(X)$, then the E-module $\bigoplus_i H^i(X, \omega_X)$ is k-regular but not $(k-1)$-regular. Moreover, in the paper [LPS], the authors show that the minimal E-resolution of the module $\bigoplus_i H^i(X, \omega_X)$ is a direct sum of shifts of linear resolutions. Our result about the regularity of the cohomology modules $\bigoplus_i H^i(X, \Omega_X^p)$ for $p < \dim \mathrm{X}$ is the following:

Theorem 1.2. *Let X be a complex smooth irregular projective variety of dimension d. Let k be the dimension of the general fiber of* $\mathrm{alb}_X : X \longrightarrow \mathrm{Alb}(X)$ *and f be the maximal dimension of a fiber of* alb_X*. Let $0 \leq p \leq d$ be an integer and set $l = \max\{k, f-1\}$. If $p > l$ then the E-module $\bigoplus_i H^i(X, \Omega_X^p)$ is $(d - p + l)$-regular.*

2. Exactness of the complex $\underline{\mathbf{L}}^p_X$

Let X be a compact Kähler manifold of dimension d. The *irregularity* of X is the non-negative integer $q(X) \stackrel{\text{def}}{=} h^1(X, \mathcal{O}_X)$. The manifold X is *irregular* if $q(X) > 0$. We aim to study the exactness of the complex (2) in terms of the non-negative integer $m = m(X)$ (see (3)). Refer also to Proposition 5.2 for another study of the exactness of $\underline{\mathbf{L}}^p_X$ in terms of different invariants.

Proposition 2.1. *Let X be an irregular compact Kähler manifold of dimension d and $0 \leq p \leq d$ be an integer.*

(i) *If $p < m \leq d$ then the complex $\underline{\mathbf{L}}^p_X$ is exact at the first $(m-p)$-steps from the left, and the first $m-p$ maps are of constant rank.*

(ii) *If $d-p < m \leq d$ then the complex $\underline{\mathbf{L}}^p_X$ is exact at the first $(m-d+p)$-steps from the right, and the last $m-d+p$ maps are of constant rank.*

(iii) *If $m = +\infty$ then the whole complex $\underline{\mathbf{L}}^p_X$ is exact and all the involved maps are of constant rank.*

Proof. Under the Hodge conjugate-linear isomorphism $H^i(X, \Omega^j_X) \cong H^j(X, \Omega^i_X)$, the fiber at a point $[v] \in \mathbf{P} \stackrel{\text{def}}{=} \mathbf{P}_{\text{sub}}(H^1(X, \mathcal{O}_X))$ of $\underline{\mathbf{L}}^p_X$ is identified with the complex of vector spaces

$$0 \longrightarrow H^p(X, \mathcal{O}_X) \xrightarrow{\wedge\omega} H^p(X, \Omega^1_X) \xrightarrow{\wedge\omega} \dots \xrightarrow{\wedge\omega} H^p(X, \Omega^d_X) \longrightarrow 0, \tag{4}$$

where $\omega \in H^0(X, \Omega^1_X)$ is the holomorphic one-form conjugate to $v \in H^1(X, \mathcal{O}_X)$. For every non-zero holomorphic one-form ω the complex (4) is exact at the first $(m-p)$-steps from the left by [GL1] Proposition 3.4. Hence the complex $\underline{\mathbf{L}}^p_X$ is itself exact at the first $(m-p)$-steps since exactness can be checked at the level of fibers. This also shows that the first $m-p$ maps of $\underline{\mathbf{L}}^p_X$ are of constant rank.

For point (ii), using Serre Duality and thinking of the spaces $H^p(X, \Omega^q_X)$ as the (p,q)-Dolbeault cohomology, we have a diagram

$$\begin{array}{ccccccccc} \dots & \xrightarrow{\wedge\omega} & H^{d-p}(X, \Omega^{i-1}_X) & \xrightarrow{\wedge\omega} & H^{d-p}(X, \Omega^{i}_X) & \xrightarrow{\wedge\omega} & H^{d-p}(X, \Omega^{i+1}_X) & \xrightarrow{\wedge\omega} & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & H^{p}(X, \Omega^{d-i+1}_X)^\vee & \longrightarrow & H^{p}(X, \Omega^{d-i}_X)^\vee & \longrightarrow & H^{p}(X, \Omega^{d-i-1}_X)^\vee & \longrightarrow & \dots \end{array}$$

where the bottom complex is the dual complex of (4). This diagram commutes up to sign and hence, if $m > d-p$, the upper complex (and therefore the bottom one as well) is exact at the first $(m-d+p)$-steps from the left. Finally dualizing again the bottom complex we have that (4) is exact at the first $(m-d+p)$-steps from the right.

The case $m = +\infty$ follows as the complexes (4) are now everywhere exact. □

In the special case when the zero-set of every non-zero holomorphic one-form consists of a finite number of points, *i.e.* when $m(X) = \dim \mathrm{X}$, Proposition 2.1 implies that the complex $\underline{\mathbf{L}}^p_X$ is everywhere exact except at most at one term. This allows us to give a bound on the

Euler characteristics $\chi(\Omega^p_X)$ in the case $q(X) > \dim \mathrm{X}$. Before stating the bounds, we prove a simple Lemma which will be useful in the sequel.

Lemma 2.2. *Let $e \geq 2$, $t \geq 1$, $q \geq 2$ and a be integers. For $i = 1, \dots, e+1$ and $s = 1, \dots, t$ let V_i and Z_s be complex vector spaces of positive dimension.*

(i) *If a complex of locally free sheaves on $\mathbf{P} = \mathbf{P}^{q-1}$ of length $e+1$ of the form*

$$0 \longrightarrow V_{e+1} \otimes \mathcal{O}_{\mathbf{P}}(-a) \longrightarrow V_e \otimes \mathcal{O}_{\mathbf{P}}(-a+1) \longrightarrow \dots \longrightarrow V_1 \otimes \mathcal{O}_{\mathbf{P}}(-a+e) \longrightarrow 0 \tag{5}$$

is exact, then $q \leq e$.

(ii) *Let $k_s \geq -a+e$ be integers. If a complex of locally free sheaves on $\mathbf{P} = \mathbf{P}^{q-1}$ of length $e+2$ of the form*

$$0 \longrightarrow V_{e+1} \otimes \mathcal{O}_{\mathbf{P}}(-a) \longrightarrow V_e \otimes \mathcal{O}_{\mathbf{P}}(-a+1) \longrightarrow \dots \longrightarrow V_1 \otimes \mathcal{O}_{\mathbf{P}}(-a+e) \longrightarrow \bigoplus_{s=1}^{t} (Z_s \otimes \mathcal{O}_{\mathbf{P}}(k_s)) \longrightarrow 0$$

is exact, then $q \leq e+1$.

Proof. If $e = 2$ then $q = 2$ since line bundles on projective spaces have no intermediate cohomology. We can then suppose that $e > 2$. After having twisted the complex (5) by $\mathcal{O}_{\mathbf{P}}(-e+a)$ we get the exact complex

$$0 \longrightarrow V_{e+1} \otimes \mathcal{O}_{\mathbf{P}}(-e) \xrightarrow{f_1} V_e \otimes \mathcal{O}_{\mathbf{P}}(-e+1) \longrightarrow \dots$$
$$\dots \longrightarrow V_4 \otimes \mathcal{O}_{\mathbf{P}}(-3) \xrightarrow{f_{e-2}} V_3 \otimes \mathcal{O}_{\mathbf{P}}(-2) \longrightarrow V_2 \otimes \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow V_1 \otimes \mathcal{O}_{\mathbf{P}} \longrightarrow 0.$$

Set $W_j = \operatorname{coker} f_j$ for $j = 1, \dots, e-2$. If by contradiction $q > e$ we would have that $H^{e-1-j}(\mathbf{P}, W_j) \neq 0$ for every $j = 1, \dots, e-2$ and hence that $H^{e-1}(\mathbf{P}, V_{e+1} \otimes \mathcal{O}_{\mathbf{P}}(-e)) \neq 0$. This yields a contradiction and therefore $q \leq e$. To prove (ii) we can use the same argument used to prove (i). ☐

Corollary 2.3. *Let X be a compact Kähler manifold of dimension d and irregularity $q(X) > d$. If $m(X) = d$ then*

$$(-1)^{d-1}\chi(\Omega^1_X) \geq 2,$$

and

$$(-1)^{d-p}\chi(\Omega^p_X) \geq 1$$

for any $p = 2, \dots, d-2$.

Proof. The Corollary is clear for $d = 1$, so we assume $d \geq 2$. We first prove that $h^d(X, \Omega^p_X) \neq 0$ so that the complex $\underline{\mathbf{L}}^p_X$ is non-zero as well. To see this, note that by Proposition 2.1 (ii) the assumption $m(X) = d$ implies that the non-zero complex $\underline{\mathbf{L}}^d_X$ is exact at the first d-steps from the right. If we had $h^d(X, \Omega^p_X) = h^p(X, \omega_X) = 0$, then $\underline{\mathbf{L}}^d_X$ would induce an exact complex of length $\leq d$ whose terms are sums of line bundles all of the same degree, and by Lemma 2.2 we would have a contradiction.

By Proposition 2.1 $\underline{\mathbf{L}}^p_X$ is exact at the first $(d-p)$-steps from the left. Therefore we get an exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-d) \otimes H^0(X, \Omega^p_X) \longrightarrow \dots \longrightarrow \mathcal{O}_{\mathbf{P}}(-p-1) \otimes H^{d-p-1}(X, \Omega^p_X) \xrightarrow{f}$$
$$\mathcal{O}_{\mathbf{P}}(-p) \otimes H^{d-p}(X, \Omega^p_X) \longrightarrow F \longrightarrow 0,$$

where the locally free sheaf F is the cokernel of the map f. We also get an induced map of locally free sheaves $h : F \longrightarrow \mathcal{O}_{\mathbf{P}}(-p+1) \otimes H^{d-p+1}(X, \Omega_X^p)$, which is of constant rank. Denoting by E the kernel of h, we obtain another exact sequence of locally free sheaves

$$0 \longrightarrow E \longrightarrow F \longrightarrow \mathcal{O}_{\mathbf{P}}(-p+1) \otimes H^{d-p+1}(X, \Omega_X^p) \longrightarrow \ldots \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^d(X, \Omega_X^p) \longrightarrow 0,$$

from which we deduce $\operatorname{rank} E = (-1)^{d-p}\chi(\Omega_X^p)$. If E were the zero sheaf, then the complex $\underline{\mathbf{L}}_X^p$ would be an exact sequence of length $\leq d+1$ of the form (5) which is not possible by our hypothesis $q(X) > d$. Thus

$$\text{rank } E = (-1)^{d-p}\chi(\Omega_X^p) \geq 1.$$

For $p = d-1$ we can slightly improve our bound. In this case $\underline{\mathbf{L}}_X^{d-1}$ is exact at the first $(d-1)$-steps from the right, and hence we get an exact sequence

$$0 \longrightarrow G \longrightarrow \mathcal{O}_{\mathbf{P}}(-d+1) \otimes H^1(X, \Omega_X^{d-1}) \xrightarrow{g}$$
$$\mathcal{O}_{\mathbf{P}}(-d+2) \otimes H^2(X, \Omega_X^{d-1}) \longrightarrow \ldots \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^d(X, \Omega_X^{d-1}) \longrightarrow 0,$$

where the locally free sheaf G is the kernel of the map g. Thus there is a natural map $h' : \mathcal{O}_{\mathbf{P}}(-d) \otimes H^0(X, \Omega_X^{d-1}) \longrightarrow G$ and a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-d) \otimes H^0(X, \Omega_X^{d-1}) \xrightarrow{h'} G \longrightarrow E' \longrightarrow 0,$$

where E' is the cokernel of the map h'. The locally free sheaf E' is non-zero again by Lemma 2.2. If the rank of E' were one, then E' would be a line bundle, *i. e.* $E' = \mathcal{O}_{\mathbf{P}}(j)$ for some integer j, and

$$G \in \operatorname{Ext}^1\big(\mathcal{O}_{\mathbf{P}}(j), \mathcal{O}_{\mathbf{P}}(-d) \otimes H^0(X, \Omega_X^{d-1})\big) = H^1\big(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(d+j) \otimes H^0(X, \Omega_X^{d-1})^\vee\big) = 0.$$

Hence G would split as a sum of line bundles and by Lemma 2.2 (ii) this is again not possible. Therefore

$$\text{rank } E' = (-1)^{d-1}\chi(\Omega_X^1) \geq 2.$$

□

3. Inequalities for the Hodge numbers

After having studied the exactness of $\underline{\mathbf{L}}_X^p$, we can derive inequalities for the Hodge numbers by using well-known results for locally free sheaves on projective spaces: the Evans-Griffith Theorem and the non-negativity of the Chern classes for globally generated locally free sheaves.

In this section X denotes a compact Kähler manifold of dimension d and irregularity $q = q(X) \geq 2$. Let $m = m(X)$ be as in (3) and $h^{p,q} = h^{p,q}(X) = \dim \mathrm{H}^{\mathrm{q}}(\mathrm{X}, \Omega_{\mathrm{X}}^{\mathrm{p}})$ be the Hodge numbers of X.

Before stating the results we need to introduce some notation. Fix an integer $0 \leq p \leq d$. If $d-p < m \leq d$, for $1 \leq i \leq q-1$ we define $\gamma_i(X, \Omega_X^p)$ to be the coefficient of t^i in the formal power series

$$\gamma(X, \Omega_X^p; t) \stackrel{\text{def}}{=} \prod_{j=1}^{m-d+p} (1-jt)^{(-1)^j h^{p,2d-m-p+j}} \in \mathbf{Z}[[t]].$$

If $p < m \leq d$, for $1 \leq i \leq q-1$ we define $\delta_i(X, \Omega^p_X)$ to be the coefficient of t^i in the formal power series

$$\delta(X, \Omega^p_X; t) \stackrel{\text{def}}{=} \prod_{j=1}^{m-p} (1-jt)^{(-1)^j h^{p,m-p-j}} \in \mathbf{Z}[[t]].$$

If $m = +\infty$, for $i = 1, \ldots, q-1$ we define $\varepsilon_i(X, \Omega^p_X)$ to be the coefficient of t^i in the formal power series

$$\varepsilon(X, \Omega^p_X; t) \stackrel{\text{def}}{=} \prod_{j=1}^{d} (1-jt)^{(-1)^j h^{p,d-j}} \in \mathbf{Z}[[t]].$$

Also consider the following pieces of the Euler characteristic of the bundle Ω^p_X. If $d-p < m \leq d$ define

$$\chi^{\geq 2d-m-p}(\Omega^p_X) \stackrel{\text{def}}{=} \sum_{j=2d-m-p}^{d} (-1)^{2d-m-p+j} h^{p,j}$$

and if $p < m \leq d$ define

$$\chi^{\leq m-p}(\Omega^p_X) \stackrel{\text{def}}{=} \sum_{j=0}^{m-p} (-1)^{m-p+j} h^{p,j}.$$

Theorem 3.1. *Let X be a compact Kähler manifold of dimension d and irregularity $q \geq 2$. Let $m = m(X) = \min\{\operatorname{codim} \mathrm{Z}(\omega) \mid 0 \neq \omega \in \mathrm{H}^0(\mathrm{X}, \Omega^1_{\mathrm{X}})\}$ and let $0 \leq p \leq d$ be an integer.*

(i) *If $d-p < m \leq d$ then any Schur polynomial of weight $\leq q-1$ in the $\gamma_i(X, \Omega^p_X)$ is non-negative. In particular*

$$\gamma_i(X, \Omega^p_X) \geq 0$$

for every $1 \leq i \leq q-1$. Moreover, if i is such that $\chi^{\geq 2d-m-p}(\Omega^p_X) < i < q$, then $\gamma_i(X, \Omega^p_X) = 0$.

(ii) *If $p < m \leq d$ then any Schur polynomial of weight $\leq q-1$ in the $\delta_i(X, \Omega^p_X)$ is non-negative. In particular*

$$\delta_i(X, \Omega^p_X) \geq 0$$

for every $1 \leq i \leq q-1$. Moreover, if i is such that $\chi^{\leq m-p}(\Omega^p_X) < i < q$, then $\delta_i(X, \Omega^p_X) = 0$.

(iii) *If $m = +\infty$ then*

$$\varepsilon_i(X, \Omega^p_X) = 0$$

for every $i = 1, \ldots, q-1$.

Proof. If $m > d-p$ then by Proposition 2.1 (ii) $\underline{\mathbf{L}}^p_X$ is exact at the first $(m-d+p)$-steps from the right, and hence we get an exact sequence whose maps are of constant rank

$$0 \longrightarrow G \longrightarrow \mathcal{O}_{\mathbf{P}}(d-m-p) \otimes H^{2d-m-p}(X, \Omega^p_X) \xrightarrow{g} \tag{6}$$
$$\mathcal{O}_{\mathbf{P}}(d-m-p+1) \otimes H^{2d-m-p+1}(X, \Omega^p_X) \longrightarrow \ldots \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^d(X, \Omega^p_X) \longrightarrow 0,$$

and where G is the kernel of the map g. Dualizing (6) and then tensorizing it by $\mathcal{O}_{\mathbf{P}}(d-m-p)$, we see that the polynomial $\gamma(X, \Omega^p_X; t)$ is the Chern polynomial of the locally free sheaf $G^\vee(d-m-p)$ and that its Chern classes are identified with the $\gamma_i(X, \Omega^p_X)$'s. To conclude we note that $G^\vee(d-m-p)$ is globally generated and therefore its Chern classes, and the Schur polynomials in these, are non-negative. The last statement of (i) follows from the fact that $c_i(G)=0$ for $i>\text{rank}\, G=\chi^{\geq 2d-m-p}(\Omega^p_X)$.

The proof of (ii) is analogous to the proof of the previous point. If $m>p$ then by Proposition 2.1 (i) $\underline{\mathbf{L}}^p_X$ is exact at the first $(m-p)$-steps from the left and induces the following exact sequence

$$(7)\qquad 0\longrightarrow \mathcal{O}_{\mathbf{P}}(-d)\otimes H^0(X,\Omega^p_X)\longrightarrow\ldots\longrightarrow \mathcal{O}_{\mathbf{P}}(-d+m-p-1)\otimes H^{m-p-1}(X,\Omega^p_X)\xrightarrow{f}$$
$$\mathcal{O}_{\mathbf{P}}(-d+m-p)\otimes H^{m-p}(X,\Omega^p_X)\longrightarrow F\longrightarrow 0$$

where F is the cokernel of the map f. Tensoring (7) by $\mathcal{O}_{\mathbf{P}}(d-m+p)$ we get that $F(d-m+p)$ is globally generated and that its Chern polynomial is $\delta(X,\Omega^p_X;t)$. At this point we conclude as in (i).

If $m=+\infty$ then $\underline{\mathbf{L}}^p_X$ is everywhere exact and $\varepsilon(X,\Omega^p_X;t)$ is just the Chern polynomial of the zero sheaf. Thus its Chern classes satisfy $\varepsilon_i(X,\Omega^p_X)=0$, for every $i=1,\ldots,q-1$. □

Under the assumptions of Theorem 3.1 we also have

Theorem 3.2. (i) *Suppose $d-p<m\leq d$. If $q\geq d-p$ then*

$$h^{d-p,1}\geq h^{d-p,0}+q-1.$$

If $q\geq \max\{2,m-d+p,d-p\}$ then

$$\chi^{\geq 2d-m-p}(\Omega^p_X)\geq q+d-m-p.$$

(ii) *Suppose $p<m\leq d$. If $q\geq p$ then*

$$h^{p,1}\geq h^{p,0}+q-1.$$

If $q\geq\max\{2,m-p,p\}$ then

$$\chi^{\leq m-p}(\Omega^p_X)\geq q-m+p.$$

Proof. (i). By Proposition 2.1 $\underline{\mathbf{L}}^p_X$ is exact at the first $(m-d+p)$-steps from the right. Since $q\geq d-p$ we can prove, with a similar argument to the one used in Corollary 2.3, that $h^d(X,\Omega^p_X)\neq 0$ and hence that the complex $\underline{\mathbf{L}}^p_X$ is non-zero as well. By (6) we obtain a surjection $H^{d-1}(X,\Omega^p_X)\otimes\mathcal{O}_{\mathbf{P}}\longrightarrow H^d(X,\Omega^p_X)\otimes\mathcal{O}_{\mathbf{P}}(1)$ and the inequality $h^{d-p,1}\geq h^{d-p,0}+q-1$ is then an application of Example 7.2.2 in [La] which we recall here for reader's ease. Let E be an ample locally free sheaf on a projective variety Y of rank e and F be another locally free sheaf on Y of rank f such that its dual $F^\vee$ is nef. If E is a quotient of F then $f\geq e+\dim \mathrm{Y}$.

Now we prove the second inequality. By looking again at (6) we note that $\chi^{\geq 2d-m-p}(\Omega^p_X)=$ rank $G\geq 0$. If $q=m-d+p$ then the inequality is trivially satisfied. If $q=m-d+p+1$ (*resp.* $q=m-d+p+2$) then by Lemma 2.2 (i) rank $G\geq 1$ (*resp.* rank $G\geq 2$) and the inequality follows. So we can suppose $q\geq m-d+p+3$. By chasing the sequence (6) we see that $H^k(\mathbf{P},G^\vee(j))=0$ for all $j\in\mathbf{Z}$ and $k=1,\ldots,q+d-m-p-2$. By Lemma 2.2 $G^\vee$ is neither the zero sheaf nor splits as a sum of line bundles and thus the Evans-Griffith Theorem (see [La] p. 92) yields

$$\text{rank}\, G=\chi^{\geq 2d-m-p}(\Omega^p_X)\geq q+d-m-p.$$

(ii). The hypothesis $p < m \leq d$ implies that $\underline{\mathbf{L}}^p_X$ is exact at the first $(m-p)$-steps from the left. Since $q \geq p$ we have that $h^0(X, \Omega^p_X) = h^d(X, \Omega^{d-p}_X) \neq 0$ as in (i), and therefore the complex $\underline{\mathbf{L}}^p_X$ is non-zero as well. After having noted that rank $F = \chi^{\leq m-p}(\Omega^p_X)$ we can argue as in the previous point. □

3.1. **The case** $m(X) = \dim \mathrm{X}$**.** When $m(X) = \dim \mathrm{X}$ further inequalities hold thanks to Catanese's work [Cat]. Let $\mathrm{alb}_X : X \longrightarrow \mathrm{Alb}(X)$ be the Albanese map of X. We say that X is of *maximal Albanese dimension* if $\dim \mathrm{alb_X(X)} = \dim \mathrm{X}$. Following Catanese's terminology we say that X is of *Albanese general type* if it is of maximal Albanese dimension and $q(X) > \dim \mathrm{X}$. An *irregular fibration* (*resp. higher irrational pencil*) is a surjective morphism with connected fibers $f : X \longrightarrow Y$ onto a normal variety Y with $0 < \dim \mathrm{Y} < \dim \mathrm{X}$ and such that a smooth model of Y is of maximal Albanese dimension (*resp.* Albanese general type).

Lemma 3.3. *If X is an irregular compact Kähler manifold with $m(X) = \dim \mathrm{X}$, then X does not carry any higher irrational pencils.*

Proof. We proceed by contradiction. Suppose a higher irrational pencil $f : X \longrightarrow Y$ exists and let $\mathrm{alb}_Y : Y \longrightarrow \mathrm{Alb}(Y)$ be the Albanese map of Y, which is well defined since Y is normal. The map alb_Y is not surjective, hence following an idea contained in the proof of [EL] Proposition 2.2, one can show that given a general point $y \in Y$ there exists a holomorphic 1-form ω of $\mathrm{Alb}(Y)$ whose restriction to $\mathrm{alb}_Y(Y)$ vanishes at the point $\mathrm{alb}_Y(y)$. Pulling back ω to X, we get a holomorphic 1-form which vanishes along some fibers of f which are of positive dimension, this contradicting the hypothesis $m(X) = \dim \mathrm{X}$. The form ω can be constructed as follows. Let z be a smooth point of the Albanese image $\mathrm{alb}_Y(Y) \subset \mathrm{Alb}(Y)$. The coderivative map $T^*_z \mathrm{Alb}(Y) \longrightarrow T^*_z \mathrm{alb_Y(Y)}$ is surjective with non trivial kernel. Then take ω to be the extension to a holomorphic 1-form on $\mathrm{Alb}(Y)$ of any non-zero form belonging to this kernel. □

The previous Lemma allow us to use results of [LP] Remark 4.3 and the ones in the references therein, so when $m(X) = \dim \mathrm{X}$ we obtain other inequalities:

$$h^{0,k} \geq k(q(X) - k) + 1 \qquad k = 0, \ldots, \dim \mathrm{X}, \tag{8}$$

and if $\dim \mathrm{X} \geq 3$

$$h^{0,2} \geq 4q(X) - 10. \tag{9}$$

These inequalities will be used to give asymptotic bounds for the Hodge numbers in terms of $q(X)$ for manifolds of dimension three and four (*cf.* Corollary 4.1 and Corollary 4.2).

3.2. **The case** $m(X) = +\infty$**.** The proof of Lemma 3.3 also shows that compact Kähler manifolds X with $m(X) = +\infty$ have surjective Albanese map and consequently $q(X) \leq \dim \mathrm{X}$ (this inequality also follows by Proposition 2.1 and Lemma 2.2). Furthermore, since for this case the complexes $\underline{\mathbf{L}}^p_X$ are everywhere exact, we automatically get that $\chi(\Omega^p_X) = 0$ for all p. Complex smooth projective surfaces with $m(X) = +\infty$ are completely classified. They are either abelian surfaces, bielliptic surfaces or geometrically ruled surfaces over an elliptic curve. This can been seen by computing their Hodge numbers with Theorem 3.1 (iii) and by the classification of complex smooth projective surfaces. In particular we note that their Kodaira dimension is non-positive.

In higher dimension, according to Conjecture 2 of [LZ], we expect that the Kodaira dimension of such varieties is $\leq \dim \mathrm{X} - \mathrm{q(X)}$. This conjecture has been verified for the case of threefolds (*cf. loc. cit.* Theorem 2). Lastly we observe that if X is a smooth projective variety with $q(X) = \dim \mathrm{X}$ and $m(X) = +\infty$ then X is an abelian variety. First we note that ω_X is trivial since Ω^1_X has $\dim \mathrm{X}$ linearly independent sections which never vanish. Also, the cohomological support loci $V^i(\omega_X) \stackrel{\text{def}}{=} \{\alpha \in \mathrm{Pic}^0(X) \mid h^i(X, \omega_X \otimes \alpha) > 0\}$ consists of at most a finite set of points for all $i \geq 0$ by [GL1] Remark on p. 405, and therefore X is of maximal Albanese dimension thanks to the converse of the Generic Vanishing Theorem (*cf.* [LP] Remark 2.4 or [BLNP] Proposition 2.7). Now, a Theorem of Ein and Lazarsfeld (see [ChH] Theorem 1.8) says that if a smooth projective variety X is of maximal Albanese dimension with $\dim \mathrm{V}^0(\omega_{\mathrm{X}}) = 0$ then X is birational to an abelian variety. Since $\omega_X \cong \mathcal{O}_X$, X is in fact an abelian variety.

Finally, we note that Theorem 3.1 (iii) determines the Hodge numbers of smooth projective threefolds X with irregularity $q = q(X) \geq 2$ and $m(X) = +\infty$ in terms of $q = q(X)$:

$$h^{0,2} = 2q - 3,\ h^{0,3} = q - 2,\ h^{1,1} = 4q - 3,\ h^{1,2} = 5q - 6.$$

4. Examples and asymptotic bounds for threefolds and fourfolds

In this section we list concrete inequalities coming from Theorem 3.1 and Theorem 3.2 in the most interesting case $m(X) = \dim \mathrm{X}$, $q(X) \geq \dim \mathrm{X}$, and for $\dim \mathrm{X} = 3, 4, 5$. Moreover for threefolds and fourfolds we list asymptotic bounds in terms of the irregularity $q(X)$ for all the Hodge numbers. We also point out that some of the inequalities are still valid for some values of $q(X)$ smaller than $\dim \mathrm{X}$ and that other inequalities hold for different values of $m(X)$. Set $q = q(X)$ for the irregularity and $h^{p,q} = h^{p,q}(X)$ for the Hodge numbers of X.

Let us start with Theorem 3.1. We get a first set of inequalities by imposing the conditions $\gamma_1(X, \Omega^p_X) \geq 0$ for $p = 0, 1, 2$. Hence

$$h^{0,2} \geq 2q - 3, \quad h^{1,1} \geq 2q \qquad \text{for } \dim \mathrm{X} = 3$$

$$h^{1,2} \geq 2h^{1,1} - 3q, \quad h^{1,2} \geq 2h^{0,2}, \quad h^{0,3} \geq 2h^{0,2} - 3q + 4 \qquad \text{for } \dim \mathrm{X} = 4$$

$$h^{0,4} \geq 4q - 3h^{0,2} + 2h^{0,3} - 5, \quad h^{1,4} \geq 4h^{1,1} - 3h^{1,2} + 2h^{1,3}, \quad h^{2,2} \geq 2h^{1,2} - 3h^{0,2} \qquad \text{for } \dim \mathrm{X} = 5$$

Finer inequalities are obtained by solving $\gamma_2(X, \Omega^p_X) \geq 0$. For $\dim \mathrm{X} = 3$ we have

$$h^{0,2} \geq 2q - \frac{7}{2} + \frac{\sqrt{8q - 23}}{2}, \qquad h^{1,1} \geq 2q - \frac{1}{2} + \frac{\sqrt{8q + 1}}{2} \tag{10}$$

and for $\dim \mathrm{X} = 4$ we get

$$h^{0,3} \geq 2h^{0,2} - 3q + \frac{7}{2} + \frac{\sqrt{8h^{0,2} - 24q + 49}}{2} \tag{11}$$

$$h^{1,2} \geq 2h^{0,2} - \frac{1}{2} + \frac{\sqrt{8h^{0,2} + 1}}{2} \tag{12}$$

$$h^{1,2} \geq 2h^{1,1} - 3q + \sqrt{4h^{1,1} - 9q}$$

where the quantity $4h^{1,1}-9q$ is non-negative by the first inequality of Theorem 3.2 (i) and the quantity $8h^{0,2}-24q+49$ is non-negative by inequality (9). Finally for $\dim \mathrm{X}=5$ we get

$$h^{0,4} \geq 4q-3h^{0,2}+2h^{0,3}-\frac{11}{2}+\frac{\sqrt{48q-24h^{0,2}+8h^{0,3}-79}}{2}$$
$$h^{1,4} \geq 2h^{1,3}+4h^{1,1}-3h^{1,2}-\frac{1}{2}+\frac{\sqrt{48h^{1,1}-24h^{1,2}+8h^{1,3}+1}}{2}$$
$$h^{2,2} \geq 2h^{1,2}-3h^{0,2}-\frac{1}{2}+\frac{\sqrt{8h^{1,2}-24h^{0,2}+1}}{2}$$

which hold as long as the quantities under the square roots are non-negative.

Applying Theorem 3.2 with $m(X)=\dim \mathrm{X}$ and $q(X)\geq \dim \mathrm{X}$, we get for $\dim \mathrm{X}=3$

$$\chi(\omega_X)\geq q-3, \quad h^{1,1}\geq 2q-1, \quad h^{1,2}\geq h^{1,1}-2, \quad h^{1,2}\geq h^{0,2}+q-1,$$

for $\dim \mathrm{X}=4$

$$\chi(\omega_X)\geq q-4, \qquad h^{2,2}\geq h^{1,2}-h^{0,2}+q-2, \qquad h^{1,3}\geq h^{1,2}-h^{1,1}+2q-3$$
$$h^{1,1}\geq 2q-1, \qquad h^{1,2}\geq h^{2,0}+q-1, \qquad h^{1,3}\geq h^{0,3}+q-1$$

and for $\dim \mathrm{X}=5$

$$\chi(\omega_X)\geq q-5, \qquad h^{1,4}\geq h^{1,3}-h^{1,2}+h^{1,1}-4, \qquad h^{1,1}\geq 2q-1,$$
$$2h^{1,2}\geq h^{2,2}+h^{0,2}+q-3, \qquad h^{1,2}\geq h^{0,2}+q-1, \qquad h^{1,2}\geq h^{1,3}-h^{0,3}+q-2,$$
$$h^{1,3}\geq h^{0,3}+q-1, \qquad h^{1,4}\geq h^{0,4}+q-1.$$

We select the strongest of the inequalities above in dimension three and four and the ones in Section 3.1 in statements formulated asymptotically with respect to $q(X)$ for simplicity:

Corollary 4.1. *Let X be an irregular compact Kähler threefold with $m(X)=3$. Then asymptotically*

$$h^{0,2}\succeq 4q, \quad h^{0,3}\succeq 4q, \quad h^{1,1}\succeq 2q+\sqrt{2q}, \quad h^{1,2}\succeq 5q+\sqrt{2q}.$$

Proof. The inequality (9) implies the asymptotic bound $h^{0,2}\succeq 4q$. The inequality $\chi(\omega_X)\geq q-3$ of Theorem 3.2 implies the inequality $h^{0,3}\geq h^{0,2}-2$ and therefore $h^{0,3}\succeq 4q$. The asymptotic bound for $h^{1,1}$ follows by (10). Finally, since by Corollary 2.3 $\chi(\Omega^1_X)\geq 2$, we also get the bound for $h^{1,2}$. □

Corollary 4.2. *Let X be an irregular compact Kähler fourfold with $m(X)=4$. Then asymptotically*

$$h^{0,2}\succeq 4q, \qquad h^{0,3}\succeq 5q+\sqrt{2q}, \qquad h^{0,4}\succeq 4q$$
$$h^{1,1}\succeq 2q, \qquad h^{1,2}\succeq 8q+2\sqrt{2q}, \qquad h^{1,3}\succeq 12q+3\sqrt{2q}, \quad h^{2,2}\succeq 8q+4\sqrt{2q}.$$

Proof. The asymptotic bounds for $h^{0,2},h^{0,3}$ and $h^{0,4}$ follow by (9), (11) and (8) respectively. Using the first inequality of Theorem 3.2 (i) we get $h^{1,1}\succeq 2q$, and by (12) we also get the bound for $h^{1,2}$. Lastly, by Corollary 2.3 we have $\chi(\Omega^1_X)\leq 2$ and $\chi(\Omega^2_X)\geq 1$ which provide the bounds for $h^{1,3}$ and $h^{2,2}$. □

5. Regularity of the cohomology modules

In this section we give the proof of Theorem 1.2 from the Introduction.

5.1. **Regularity and BGG correspondence.**

We start by giving the definition of *regularity* for a finitely generated graded module over an exterior algebra and recalling its relationship with the BGG correspondence.

Let V be a complex vector space of dimension q and $W = V^\vee$ be its dual space. We assume that elements of V are of degree -1 and elements of W of degree 1. Let $E = \bigwedge^* V$ be the graded exterior algebra over V and $S = \mathrm{Sym}^* W$ be the symmetric algebra over W. A finitely generated graded E-module $Q = \bigoplus_{j=0}^d Q_{-j}$ with graded components Q_{-j} in degrees $-j$, is called *c-regular* if it is generated in degrees 0 up to $-c$ and if its minimal free resolution has at most $c+1$ linear strands. Equivalently, Q is c-regular if and only if $\mathrm{Tor}_i^E(Q, \mathbf{C})_{-i-j} = 0$ for all $i \geq 0$ and all $j \geq c+1$. The *dual* over E of a finitely generated graded module $P = \bigoplus_{j=0}^d P_j$ with graded components P_j in degrees j, is defined to be the E-module $\hat{P} = \bigoplus_{j=0}^d P_{-j}^\vee$ with graded components $P_{-j}^\vee$ in degrees $-j$, so that $\hat{P}$ is a module with no component of positive degree (*cf.* [Eis], [EFS], [LP]).

Let $e_1, \ldots, e_q$ be a basis of V and $x_1, \ldots, x_q$ be the dual basis. The BGG correspondence associated to a finitely generated graded E-module $P = \bigoplus_{j=0}^d P_j$ is the linear complex $\mathbf{L}(P)$ of free S-modules

$$\mathbf{L}(P): \quad \ldots \longrightarrow S \otimes_{\mathbf{C}} P_{j+1} \longrightarrow S \otimes_{\mathbf{C}} P_j \longrightarrow S \otimes_{\mathbf{C}} P_{j-1} \longrightarrow \ldots$$

with differentials given by

$$s \otimes p \mapsto \sum_{i=1}^q x_i s \otimes e_i p.$$

For references about the BGG correspondence see [BGG], [EFS] and Chapter 7B of [Eis]. The following Proposition is a criterion to bound the regularity of an E-module $P = \bigoplus_{j=0}^d P_j$; the proof can be found in [LP].

Proposition 5.1 ([LP] Proposition 3.2)**.** *Let $P = \bigoplus_{j=0}^d P_j$ be a finitely generated graded E-module with no component of negative degree. The dual module $\hat{P}$ of P over E is c-regular if and only if the complex $\mathbf{L}(P)$*

$$0 \longrightarrow S \otimes_{\mathbf{C}} P_d \longrightarrow S \otimes_{\mathbf{C}} P_{d-1} \longrightarrow \ldots \longrightarrow S \otimes_{\mathbf{C}} P_{c+1} \longrightarrow S \otimes_{\mathbf{C}} P_c$$

is exact at the first $(d-c)$-steps from the left.

5.2. **Proof of Theorem 1.2.**

This time X denotes a complex smooth irregular projective variety of dimension d. Set $V = H^1(X, \mathcal{O}_X)$, $W = V^\vee$, $E = \bigwedge^* V$ and $S = \mathrm{Sym}^* W$. Fix an integer $p = 0, \ldots, d$. Via cup product we consider the graded E-module

$$P_X^p = \bigoplus_i H^i(X, \Omega_X^{d-p})$$

where the graded component $H^i(X,\Omega_X^{d-p})$ is in degree $d-i$. By Serre Duality $H^i(X,\Omega_X^p)\cong H^{d-i}(X,\Omega_X^{d-p})^\vee$ and therefore the dual module of P_X^p over E is the module

$$Q_X^p=\bigoplus_i H^i(X,\Omega_X^p)$$

where the graded component $H^i(X,\Omega_X^p)$ is in degree $-i$. We apply Proposition 5.1 in order to study the regularity of Q_X^p and therefore we only need to count the number of exact terms of $\mathbf{L}(P_X^p)$ from the left. It is not difficult to show that $\mathbf{L}(P_X^p)$ is in fact isomorphic to a complex $\mathbf{L}_X^{d-p}$ of S-graded modules defined as $\mathbf{L}_X^{d-p}\stackrel{\text{def}}{=}\Gamma_*(\underline{\mathbf{L}}_X^{d-p})$

$$\mathbf{L}_X^{d-p}:\quad 0\longrightarrow S\otimes_{\mathbf{C}}H^0(X,\Omega_X^{d-p})\longrightarrow S\otimes_{\mathbf{C}}H^1(X,\Omega_X^{d-p})\longrightarrow\ldots\longrightarrow S\otimes_{\mathbf{C}}H^d(X,\Omega_X^{d-p})\longrightarrow 0$$

(see [Ha] p. 118 for the definition of Γ_*; the proof of the isomorphism between $\mathbf{L}(P_X^p)$ and $\mathbf{L}_X^{d-p}$ is analogous to the proof of [LP] Lemma 3.3 for the case $p=\dim \mathrm{X}$). At this point Theorem 1.2 follows by the previous discussion and the following

Proposition 5.2. *Let X be a complex smooth projective irregular variety of dimension d. Let k be the dimension of the generic fiber of the Albanese map $\mathrm{alb}_X : X\longrightarrow \mathrm{Alb}(X)$ and let f be the maximal dimension of a fiber of alb_X. Set $l=\max\{k,f-1\}$. If $p>l$ then the complexes $\mathbf{L}_X^{d-p}$ and $\underline{\mathbf{L}}_X^{d-p}$ are exact at the first $(p-l)$-steps from the left.*

Proof. We follow [LP] Proposition 2.1. We start with the study of the exactness of $\mathbf{L}_X^{d-p}$. Let $\mathbf{A}=\mathrm{Spec}(\mathrm{Sym}^*W)$ be the affine space over V viewed as an affine algebraic variety and let $\mathcal{K}_{d-p}$ be the complex

$$\mathcal{K}_{d-p}:\quad 0\longrightarrow \mathcal{O}_{\mathbf{A}}\otimes H^0(X,\Omega_X^{d-p})\longrightarrow \mathcal{O}_{\mathbf{A}}\otimes H^1(X,\Omega_X^{d-p})\longrightarrow\ldots\longrightarrow \mathcal{O}_{\mathbf{A}}\otimes H^d(X,\Omega_X^{d-p})\longrightarrow 0$$

of free sheaves whose maps are given at each point of $\mathbf{A}$ by cupping with the corresponding element of V. Since $\Gamma(\mathbf{A},\mathcal{O}_{\mathbf{A}})=\mathrm{Sym}^*W$, there exists an isomorphism of complexes

$$\mathbf{L}_X^{d-p}\cong\Gamma(\mathbf{A},\mathcal{K}_{d-p})$$

where Γ is the global section functor. Therefore the study of the exactness of $\mathbf{L}_X^{d-p}$ is equivalent to the study of the exactness of $\mathcal{K}_{d-p}$ since Γ is an exact functor on affine varieties. Moreover, defining $\mathbf{V}$ to be the vector space V viewed as a complex manifold, by GAGA, the exactness of $\mathcal{K}_{d-p}$ is in turn equivalent to the exactness of $\mathcal{K}_{d-p}^{\mathrm{an}}$ which is the complex

$$\mathcal{K}_{d-p}^{\mathrm{an}}:\quad 0\longrightarrow \mathcal{O}_{\mathbf{V}}\otimes H^0(X,\Omega_X^{d-p})\longrightarrow \mathcal{O}_{\mathbf{V}}\otimes H^1(X,\Omega_X^{d-p})\longrightarrow\ldots\longrightarrow \mathcal{O}_{\mathbf{V}}\otimes H^d(X,\Omega_X^{d-p})\longrightarrow 0$$

of analytic sheaves where the differentials at a point of $\mathbf{V}$ are again given by cupping with the corresponding element. If $p>l$ then it is enough to check the exactness of $\mathcal{K}_{d-p}^{\mathrm{an}}$ at the first $(p-l)$-steps from the left, namely the vanishing of the cohomologies $\mathcal{H}^i(\mathcal{K}_{d-p}^{\mathrm{an}})$ for any $i<p-l$. Since the differentials of $\mathcal{K}_{d-p}^{\mathrm{an}}$ scale linearly along lines through the origin, it is then enough to check the vanishing of the stalks at the origin 0, *i.e.*

$$\mathcal{H}^i(\mathcal{K}_{d-p}^{\mathrm{an}})_0=0$$

for $i<p-l$.

Let $p_1 : X\times\mathrm{Pic}^0(X)\longrightarrow X$ and $p_2 : X\times\mathrm{Pic}^0(X)\longrightarrow\mathrm{Pic}^0(X)$ be the projections from $X\times\mathrm{Pic}^0(X)$ onto the first and second factor respectively and let $\mathcal{P}$ be a normalized Poincaré line bundle on $X\times\mathrm{Pic}^0(X)$. Then Theorem 6.2 in [ClH] gives an isomorphism

$$\mathcal{H}^i(\mathcal{K}_{d-p}^{\mathrm{an}})_0\cong R^ip_{2*}(p_1^*\Omega_X^{d-p}\otimes\mathcal{P})_0, \tag{13}$$

via the exponential map $\exp : V \longrightarrow \mathrm{Pic}^0(X)$. To prove the vanishing of the stalks in (13), we use Generic Vanishing Theorems for bundles of holomorphic p-forms established by Pareschi and Popa. Denote by $V^i(\Omega^j_X) \stackrel{\text{def}}{=} \{\alpha \in \mathrm{Pic}^0(X) \mid h^i(X, \Omega^j_X \otimes \alpha) > 0\}$ the cohomological support loci for Ω^j_X. The vanishing of the stalks $R^i p_{2*}(p_1^* \Omega^{d-p}_X \otimes \mathcal{P})_0$ is closely related to the varieties $V^i(\Omega^p_X)_0$. In fact by Theorem 2.2 in [PP2] we have that

$$R^i p_{2*}(p_1^* \Omega^{d-p}_X \otimes \mathcal{P})_0 = 0 \text{ for all } i < p - l \Longleftrightarrow \mathrm{codim}_{\mathrm{Pic}^0(X),0}\, V^i(\Omega^p_X) \geq i - d + p - l \text{ for all } i > 0$$

and we conclude then thanks to the Generic Nakano-Type Vanishing ([PP1] Theorem 5.11 (2)) which asserts that

$$\mathrm{codim}_{\mathrm{Pic}^0(X)} V^i(\Omega^p_X) \geq \max\{i + p - d - l, d - i - p - l\}$$

for any i. We remark that, for the case $p = \dim \mathrm{X}$, Lazarsfeld and Popa used [PP2] Theorem C to get the vanishing of (13) for $i < d - k$, and consequently to get a bound for the regularity of $\bigoplus_i H^i(X, \omega_X)$ depending only on the dimension of the general fiber of the Albanese map.

Finally we observe that $\underline{\mathbf{L}}^{d-p}_X$ is exact at the first $(p - l)$-steps from the left as well, since it is the sheafification of $\mathbf{L}^{d-p}_X$ which is an exact functor. □

By using Serre Duality and the trick used to prove point (ii) of Proposition 2.1, we also obtain that the complexes $\mathbf{L}^{d-p}_X$ and $\underline{\mathbf{L}}^{d-p}_X$ are exact at the first $(d-p-l)$-steps from the right as long as $d - p > l$.

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