DERIVED EQUIVALENCE AND FIBRATIONS OVER CURVES AND SURFACES

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Abstract. We prove that the bounded derived category of coherent sheaves $\mathbf{D}(X)$ reconstructs the isomorphism classes of fibrations of a smooth projective complex variety X onto smooth curves of genus $g \geq 2$. Moreover, if the Hodge number $h^{0,2}$ is derived invariant, then $\mathbf{D}(X)$ reconstruct the isomorphism classes of fibrations of X onto normal projective surfaces with finite Albanese map and positive holomorphic Euler characteristic.

1. Introduction

In [Kaw02, Theorem 1.4] Kawamata proves that if X and Y are smooth projective derived equivalent varieties such that the (anti)canonical bundle of X is big, then the varieties are K-equivalent. Namely there exists a third smooth projective variety U together with two birational morphisms $X \overset{p}{\leftarrow} U \overset{q}{\rightarrow} Y$ such that $p^*\omega_X \simeq q^*\omega_Y$. It follows in particular that if X admits a fibration onto a smooth curve of genus $g \geq 1$, then Y is fibered over the same curve (*cf.* Proposition 27). In this paper we show that this property is invariant under equivalence of derived categories as soon as $g \geq 2$, regardless the positivity of the (anti)canonical bundle.

To begin with, we recall first some terminology in order to present our results. An *irrational pencil of genus* $g \geq 1$ of a smooth projective variety X is a surjective morphism with connected fibers from X to a smooth curve of genus g. Two irrational pencils $f_1 \colon X \to C_1$ and $f_2 \colon X \to C_2$ of genus g are isomorphic if there exists an isomorphism of curves $C_1 \simeq C_2$ that commutes with the fibrations. For any integer $g \geq 1$ we define the following sets of pencils attached to the variety X:

$$F^g_X \;=\; F^{1,g}_X \;\overset{\text{def}}{=}\; \big\{ \text{isomorphism classes of irrational pencils } f \colon X \to C \text{ of genus } g \big\}.$$

Finally, we recall that two smooth projective varieties X and Y are *D-equivalent* if there exists an equivalence of triangulated categories between their bounded derived categories of coherent sheaves:

$$\mathbf{D}(X) \;\overset{\text{def}}{=}\; D^b\big(\mathcal{C}oh(X)\big) \;\simeq\; D^b\big(\mathcal{C}oh(Y)\big) \;\overset{\text{def}}{=}\; \mathbf{D}(Y).$$

Theorem 1. *Suppose that X and Y are smooth projective D-equivalent complex varieties. Then for any integer $g \geq 2$ there exists a bijection of sets $\mu_g \colon F^g_X \to F^g_Y$ preserving the bases of the fibrations. More specifically, if $\mu_g(f \colon X \to C) = (h \colon Y \to D)$, then there exists an isomorphism of curves $C \simeq D$.*

An application of Stein factorization yields the following corollary.

Corollary 2. *If C is a smooth projective curve of genus $g \geq 2$, then X admits a non-constant morphism over C if and only if Y does.*

The proof of Theorem 1 is based on an analysis of the geometric properties of Green–Lazarsfeld's non-vanishing loci attached to the canonical bundle

$$V^i(\omega_X) \stackrel{\text{def}}{=} \{\, \alpha \in \mathrm{Pic}^0(X) \mid h^i(X, \omega_X \otimes \alpha) > 0 \,\}, \qquad i = 0, \ldots, n \tag{1}$$

under equivalence of derived categories (*cf.* [GL87] and [GL91]). We set $n = \dim X$ and denote by $V^i(\omega_X)_0$ the union of the irreducible components of $V^i(\omega_X)$ passing through the origin. More concretely, the definition of the bijection μ_g in Theorem 1 relies on the following two ingredients: the invariance of $V^{n-1}(\omega_X)_0$ under equivalence of derived categories, and the fact that F_X^g is in bijection with the set of g-dimensional irreducible components of $V^{n-1}(\omega_X)_0$ (*cf.* Theorems 6 and 16). On the other hand, the isomorphism $C \simeq D$ between the bases of the fibrations is constructed by manipulating the support of the kernel of an equivalence $\mathbf{D}(X) \simeq \mathbf{D}(Y)$ in the style of Kawamata ([Kaw02]).

As the existence of an irrational pencil of genus $g \geq 2$ is dictated by the topology of a variety (*cf.* [Cat91]), in §5 we will collect an application of Theorem 1 concerning the derived invariance of a topological property of projective complex varieties encoded by the fundamental group. Earlier attempts to proving Theorem 1 can be found in [Lom14, Remark 7.4] and [LP15, Theorems 6(i) and 14]. In addition Theorem 1 answers affirmatively a question posed in [LP15, Question 13] (*cf.* also [Pop13, Corollary 3.4]).

In the second part of this paper we will analyze the behavior, under derived equivalence, of a class of fibrations over normal projective surfaces. A *χ-positive 2-irrational pencil* is a surjective morphism with connected fibers $f\colon X \to S$ such that: (i) S is a normal projective surface, (ii) $\chi(\omega_{\widetilde{S}}) > 0$ where $\widetilde{S}$ is a resolution of singularities of S, and (iii) there exists a finite morphism $S \to \mathrm{Alb}(\widetilde{S})$ to the Albanese variety of $\widetilde{S}$. Moreover, we say that two χ-positive 2-irrational pencils are isomorphic if there exists an isomorphism between the bases of the fibrations that commutes with the structure morphisms, analogously to the case of fibrations onto curves (*cf.* §3). For any integer $q \geq 2$ we consider the following sets of fibrations attached to a variety X:

$$F_X^{2,q} \stackrel{\text{def}}{=} \{\text{isomorphism classes of } \chi\text{-positive 2-irrational pencils } f\colon X \to S \text{ such that } h^0(\widetilde{S}, \Omega^1_{\widetilde{S}}) = q\}.$$

It turns out that the behavior of χ-positive 2-irrational pencils under equivalence of derived categories is related to the conjectured derived invariance of the Hodge number $h^{0,2}(X) = \dim H^2(X, \mathcal{O}_X)$.

Conjecture (H_n^2)**.** *Let $n \geq 1$ be an integer. If Z_1 and Z_2 are smooth projective D-equivalent complex varieties of dimension n, then $h^{0,2}(Z_1) = h^{0,2}(Z_2)$.*

In general all the Hodge numbers of a complex variety are expected to be derived invariant. There are partial results in this direction for which we direct the interested reader to consult for instance [Huy06], [PS11] and [Abu17].

Theorem 3. *Suppose that Conjecture* (H^2_n) *holds for some integer* n*, and let* X *and* Y *be two smooth projective* D*-equivalent complex varieties of dimension* n*. Then for any integer* $q \geq 2$ *there exists a bijection of sets* $\nu_q \colon F^{2,q}_X \to F^{2,q}_Y$ *preserving the bases of the fibrations. More specifically, if* $\nu_q(v \colon X \to S) = (w \colon Y \to T)$*, then there exists an isomorphism of surfaces* $S \simeq T$*. Finally, if* X *admits a surjective morphism onto the basis* S *of some* χ*-positive* 2*-irrational pencil, then also* Y *admits a surjective morphism onto* S*.*

The strategy to proving Theorem 3 is analogous, but much more technical, to that of Theorem 1. We assume the derived invariance of $h^{0,2}(X)$ in order to invoke the derived invariance of the non-vanishing locus $V^{n-2}(\omega_X)_0$, whose components are in turn related to χ-positive 2-irrational pencils (*cf.* Theorems 6 and 16). An application of a recent result of Abuaf shows that Conjecture (H^2_n) holds in dimension $n \leq 4$, thus permitting a detailed study of the invariance of fibrations of D-equivalent varieties in dimensions three and four. We start with the case of threefolds, which is indeed a simple reformulation of Theorems 1 and 3. (More in general R. Abuaf proves that in several interesting cases the whole graded algebra $H^*(X, \mathcal{O}_X)$ is a derived invariant; *cf.* [Abu17, Theorem 1.0.3].)

Corollary 4. *If* X *and* Y *are smooth projective* D*-equivalent complex threefolds, then for every pair of integers* $g \geq 2$ *and* $q \geq 2$ *there exist bijections of sets* $\mu_g \colon F^g_X \to F^g_Y$ *and* $\nu_q \colon F^{2,q}_X \to F^{2,q}_Y$ *preserving the bases of the fibrations.*

In dimension four, in order to study fibrations over threefolds, we exploit additionally the derived invariance of $V^1(\omega_X)_0$ obtained as a consequence of the derived invariance of Hochschild homology. We define the sets $F^{3,q}_X$ of χ-positive 3-irrational pencils analogously to the sets $F^{2,q}_X$. For the proof of Corollary 5 we refer to §6.

Corollary 5. *Suppose that* X *and* Y *are smooth projective* D*-equivalent complex fourfolds. Then for every pair of integers* $g \geq 2$ *and* $q \geq 2$ *there exist bijections of sets* $\mu_g \colon F^g_X \to F^g_Y$ *and* $\nu_q \colon F^{2,q}_X \to F^{2,q}_Y$ *preserving the bases of the fibrations. Moreover, if for every integer* $g \geq 2$ *the variety* X *admits no irrational pencils of genus* g *(namely* $F^g_X = \emptyset$ *for all* $g \geq 2$*), then for any integer* $q \geq 3$ *there exists a bijection of sets* $\eta_q \colon F^{3,q}_X \to F^{3,q}_Y$ *preserving the bases of the fibrations.*

Notation. Throughout the paper all varieties are defined over the field of the complex numbers. By the term *fibration* we mean a surjective morphism of normal algebraic varieties with connected fibers. We denote by $q(X) = h^1(X, \mathcal{O}_X) = \dim \mathrm{Pic}^0(X)$ the *irregularity* of a smooth projective variety, and by $alb_X \colon X \to \mathrm{Alb}(X)$ its Albanese map. Finally, we say that a fibration $\widetilde{f} \colon \widetilde{X} \to \widetilde{V}$ is a *non-singular representative* of a fibration $f \colon X \to V$, if there exist resolutions of singularities $g_1 \colon \widetilde{X} \to X$ and $g_2 \colon \widetilde{V} \to V$ such that $f\, g_1 = g_2\, \widetilde{f}$ (*cf.* [Mor87, (1.10)]). In this paper X and Y denote smooth projective complex varieties.

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2. Background material and preview of Theorem 1

2.1. **Rouquier's isomorphism.** An equivalence of triangulated categories $\mathbf{D}(X) \simeq \mathbf{D}(Y)$ of smooth projective varieties induces an isomorphism of algebraic groups

$$F : \ \mathrm{Aut}^0(X) \times \mathrm{Pic}^0(X) \xrightarrow{\simeq} \mathrm{Aut}^0(Y) \times \mathrm{Pic}^0(Y),$$

called *Rouquier's isomorphism* (*cf.* [Rou11, Théorème 4.18] and (3) for its action). The action of F on $V^i(\omega_X)_0$ is related to the conjectured derived invariance of the Hodge number $h^{0.k}(X)$ (*cf.* [LP15, Theorem 12]).

Conjecture (H^k_n)**.** *Let $n \geq 1$ and $k \geq 0$ be integers. If Z_1 and Z_2 are smooth projective D-equivalent complex varieties of dimension n, then $h^{0,k}(Z_1) = h^{0,k}(Z_2)$.*

Theorem 6 (Lombardi–Popa)**.** *Assume that Conjecture (H^k_n) holds for some integers n and k. If X and Y are smooth projective D-equivalent varieties of dimension n, then*

$$F\big(\mathrm{id}_X, V^{n-k}(\omega_X)_0\big) \ = \ \big(\mathrm{id}_Y, V^{n-k}(\omega_Y)_0\big). \tag{2}$$

In particular there is an isomorphism of algebraic sets $V^{n-k}(\omega_X)_0 \simeq V^{n-k}(\omega_Y)_0$.

Remark 7. We recall that the Hodge numbers $h^{0,k}(X)$ are derived invariant in any dimension n for $k \in \{0, 1, n-1, n\}$. Hence also the loci $V^k(\omega_X)_0$ are invariant for $n \geq 1$ and $k \in \{0, 1, n-1, n\}$.

2.2. **Construction of μ_g.** Theorem 6 plays an important role in the construction of the bijection μ_g of Theorem 1. This goes as follows. If $n = \dim X$, then the positive-dimensional irreducible components of $V^{n-1}(\omega_X)_0 \subset \mathrm{Pic}^0(X)$ are abelian subvarieties giving rise to irrational pencils $f\colon X \to C$ of genus $g \geq 2$ (*cf.* [GL91, Theorem 0.1], [Bea92, Corollaire 2.3], or Lemma 13 here below). For $q(X) = \dim \mathrm{Pic}^0(X) > 0$ the algebraic set $V^{n-1}(\omega_X)_0$ can be described as

$$V^{n-1}(\omega_X)_0 \ = \ \{\mathcal{O}_X\} \cup \bigcup_{g \geq 2} \ \bigcup_{F^g_X} \big(f^* \,\mathrm{Pic}^0(C)\big),$$

whereas $V^{n-1}(\omega_X)_0 = \emptyset$ if $q(X) = 0$. In Theorem 16 we will prove that for any integer $g \geq 2$ there exists a one-to-one correspondence

$$u_{X,g} : \ F^g_X \ \to \ \pi^g_0\big(V^{n-1}(\omega_X)_0\big), \quad \big(f\colon X \to C\big) \ \mapsto \ f^* \,\mathrm{Pic}^0(C)$$

between the set F^g_X of irrational pencils of genus g, and the set $\pi^g_0\big(V^{n-1}(\omega_X)_0\big)$ of g-dimensional irreducible components of $V^{n-1}(\omega_X)_0$. In view of the correspondence $u_{X,g}$ and Remark 7, the bijection $\mu_g\colon F^g_X \to F^g_Y$ of Theorem 1 is defined as $\mu_g \stackrel{\mathrm{def}}{=} u^{-1}_{Y,g} \, F \, u_{X,g}$. The fact that μ_g preserves the bases of the fibrations up to isomorphism will be proved in §4.

3. Non-vanishing loci and fibrations

3.1. **Non-vanishing loci.** Throughout this section we denote by X a smooth projective complex variety of dimension n. Moreover we denote by

$$V^i(\mathcal{F}) \stackrel{\text{def}}{=} \{\alpha \in \operatorname{Pic}^0(X) \mid h^i(X, \mathcal{F} \otimes \alpha) > 0\}$$

the non-vanishing loci attached to a coherent sheaf $\mathcal{F}$ on X, and by $V^i(\mathcal{F})_0$ the union of all their irreducible components passing through the origin. We say that $\mathcal{F}$ satisfies the *generic vanishing* condition (*GV* for short) if $\operatorname{codim} V^i(\mathcal{F}) \geq i$ for all $i > 0$. A fundamental theorem of Green–Lazarsfeld proves that the canonical bundle ω_Z of a smooth projective complex variety Z of maximal Albanese dimension satisfies *GV* (*cf.* [GL87, Theorem 1]). In particular, due to the deformation invariance of the holomorphic Euler characteristic $\chi(\omega_Z) = \sum_{j=0}^{\dim Z} (-1)^j h^j(Z, \omega_Z)$, we have that

$$\chi(\omega_Z) > 0 \qquad \Longleftrightarrow \qquad V^0(\omega_Z) = \operatorname{Pic}^0(Z).$$

The following basic fact will be useful later.

Proposition 8. *Let U be a normal projective variety and let $\widetilde{U} \xrightarrow{\rho} U$ be a resolution of singularities. Moreover suppose that there exists a morphism $a\colon U \to \operatorname{Alb}(\widetilde{U})$ such that the composition $\widetilde{U} \xrightarrow{\rho} U \xrightarrow{a} \operatorname{Alb}(\widetilde{U})$ equals the Albanese morphism of $\widetilde{U}$. Then the functors ρ^* and ρ_* induce an isomorphism of algebraic groups $\operatorname{Pic}^0(\widetilde{U}) \simeq \operatorname{Pic}^0(U)$.*

Proof. The morphism $\rho^*\colon \operatorname{Pic}^0(U) \to \operatorname{Pic}^0(\widetilde{U})$ is surjectve as every line bundle $\alpha \in \operatorname{Pic}^0(\widetilde{U})$ pulls back from a topologically trivial line bundle on $\operatorname{Alb}(\widetilde{U})$. Suppose now that $\rho^*(\alpha_1 \otimes \alpha_2^{-1}) \simeq \mathcal{O}_{\widetilde{U}} \simeq \rho^*\mathcal{O}_U$ where α_1 and α_2 are in $\operatorname{Pic}^0(U)$. As ρ is birational and U is normal (so that $\rho_*\mathcal{O}_{\widetilde{U}} \simeq \mathcal{O}_U$), projection formula yields $\alpha_1 \otimes \alpha_2^{-1} \simeq \mathcal{O}_U$. □

Remark 9. If $\rho\colon \widetilde{U} \to U$ is a birational morphism between smooth projective varieties, then the functors ρ^* and ρ_* induce isomorphisms of algebraic sets $V^i(\omega_{\widetilde{U}}) \simeq V^i(\omega_U)$ for all $i \geq 0$ as well.

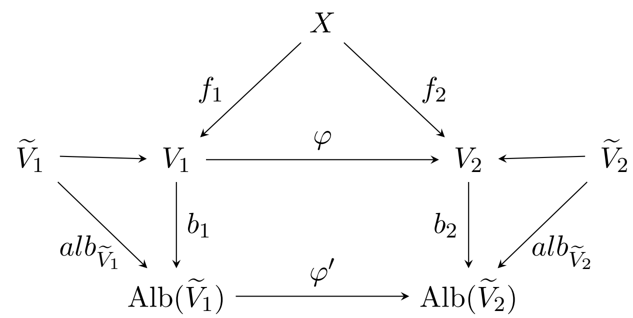
3.2. **Fibrations.** In the following we will prove the existence of the bijections $u_{X,g}$ mentioned in §2. As our techniques generalize to fibrations with higher-dimensional bases, we will work in this greater generality even though we are primarily concerned with fibrations over curves and surfaces.

A triple $\big(f\colon X \to V, \widetilde{V} \to V, b\colon V \to \operatorname{Alb}(\widetilde{V})\big)$ consisting of

(i). a fibration onto a k-dimensional normal projective variety,
(ii). a resolution of singularities,
(iii). and a finite morphism to the Albanese variety $\operatorname{Alb}(\widetilde{V})$

is a *χ-positive k-irrational pencil* if: (i) the composition $\widetilde{V} \to V \xrightarrow{b} \operatorname{Alb}(\widetilde{V})$ equals the Albanese map $alb_{\widetilde{V}}$, (ii) $\widetilde{V}$ is of maximal Albanese dimension, and (iii) $\chi(\omega_{\widetilde{V}}) > 0$.

Two such pencils $X \xrightarrow{f_1} V_1 \xrightarrow{b_1} \mathrm{Alb}(\widetilde{V}_1)$ and $X \xrightarrow{f_2} V_2 \xrightarrow{b_2} \mathrm{Alb}(\widetilde{V}_2)$ are isomorphic if there exist isomorphisms $\varphi\colon V_1 \to V_2$ and $\varphi'\colon \mathrm{Alb}(\widetilde{V}_1) \to \mathrm{Alb}(\widetilde{V}_2)$ such that $f_2 = \varphi\, f_1$ and $b_2\,\varphi = \varphi'\, b_1$. This notion of isomorphism is illustrated in the following commutative diagram:



Remark 10. As b_1 and b_2 are finite morphisms to abelian varieties, any birational map $\varphi\colon V_1 \dashrightarrow V_2$ extends to an isomorphism. Therefore in the definition of a χ-positive k-irrational pencil it is equivalent to require that φ is an isomorphirm or a birational map.

Remark 11. A result of Chen–Hacon shows that the variety $\widetilde{V}$ in the definition of a χ-positive irrational pencil is of general type (*cf.* [CH01, Theorem 1]). However not all varieties of maximal Albanese dimension and maximal Kodaira dimension have positive holomorphic Euler characteristic (*cf.* for instance Ein–Lazarsfeld's threefold constructed in [EL97, Example 1.13]). Moreover, $\widetilde{V}$ does not admit any non-trivial Fourier–Mukai partner.

Remark 12. In [Cat91] the author considers fibrations $f\colon X \to V$ over normal projective varieties of maximal Albanese dimension and with non-sujective Albanese map. We point out that our definition of a χ-positive irrational pencil is not a special case of Catanese's definition. In fact there exist smooth projective varieties with positive holomorphic Euler characteristic, and with finite and surjective Albanese map (*cf.* [BPS17, Example 8.1]).

For any pair of integers $q \geq k \geq 1$ we define the following sets of higher-dimensional pencils attached to the variety X:

$$F_X^{k,q} \stackrel{\text{def}}{=} \{\,\text{isomorphism classes of } \chi\text{-positive } k\text{-irrational pencils } f\colon X \to V \text{ such that } q(V) = q\,\}.$$

The irregularity of V is defined as $q(V) \stackrel{\text{def}}{=} q(\widetilde{V})$. For $k = 1$ and $q = g \geq 2$ the sets $F_X^{k,q}$ coincide with the sets F_X^g defined in the Introduction.

The main result of this section is Theorem 16 which builds upon results of [GL91] and [Par17]. For the proof of the following lemma see [Par17, Lemma 5.1].

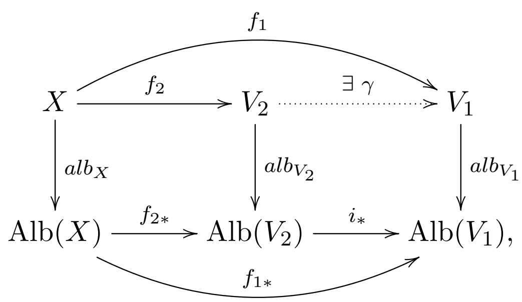
Lemma 13 (Green–Lazarsfeld, Pareschi)**.** *Let $0 < i < n$ be an integer and let $Z \subset V^i(\omega_X)_0$ be an irreducible component of positive dimension. Then there exists a fibration $p\colon X \to V$ to a normal projective variety V of dimension $0 < \dim V \leq n - i$. Moreover, if*

$\widetilde{p}\colon \widetilde{X} \to \widetilde{V}$ is a non-singular representative of p, then $\widetilde{V}$ is of maximal Albanese dimension, $\chi(R^i\widetilde{p}_*\omega_{\widetilde{X}}) > 0$, and $Z = \widetilde{p}^*V^0(R^i\widetilde{p}_*\omega_{\widetilde{X}}) = \widetilde{p}^*\operatorname{Pic}^0(\widetilde{V})$.

Remark 14. The fibration $p\colon X \to V$ of Lemma 13 is obtained as the Stein factorization of the composition $\pi \circ alb_X\colon X \to \widehat{Z}$ where $alb_X\colon X \to \operatorname{Alb}(X)$ is the Albanese map of X, and $\pi\colon \operatorname{Alb}(X) \to \widehat{Z}$ is the dual map of the inclusion $Z \subset \operatorname{Pic}^0(X)$ (recall that Z is an abelian variety by [GL91, Theorem 0.1]). In particular V admits a finite morphism $b\colon V \to \widehat{Z}$ to an abelian variety. In addition, if $\widetilde{V} \to V$ is a resolution of singularities, then $\widehat{Z} = \operatorname{Alb}(\widehat{V})$ and the composition $\widetilde{V} \to V \xrightarrow{b} \widehat{Z}$ equals the Albanese map of $\widetilde{V}$ (*cf.* [Par17, Lemma 5.1]). It follows that $q(V) = \dim Z$.

Lemma 15. *Suppose that $f_1\colon X \to V_1$ and $f_2\colon X \to V_2$ are two fibrations onto smooth projective varieties such that V_1 is of maximal Albanese dimension. If $f_1^*\operatorname{Pic}^0(V_1) \subset f_2^*\operatorname{Pic}^0(V_2)$, then there exists a dominant rational map $\gamma\colon V_2 \dashrightarrow V_1$ such that $f_1 = \gamma\, f_2$. Moreover, if $\dim V_1 = \dim V_2$, then V_1 and V_2 are birational and $f_1^*\operatorname{Pic}^0(V_1) = f_2^*\operatorname{Pic}^0(V_2)$.*

Proof. Denote by alb_{V_1} and alb_{V_2} the Albanese maps of V_1 and V_2, respectively. We consider the following commutative diagram



where f_{1*} and f_{2*} are induced by f_1 and f_2, respectively, and i_* is induced by the inclusion $i\colon f_1^*\operatorname{Pic}^0(V_1) \subset f_2^*\operatorname{Pic}^0(V_2)$. As the general fiber of f_2 is contracted by f_1 (recall that alb_{V_1} is generically finite onto its image), by [Deb01, Lemma 1.15] there exists a dominant rational map $\gamma\colon V_2 \dashrightarrow V_1$ such that $f_1 = \gamma\, f_2$. If in addition $\dim V_1 = \dim V_2$, then γ is is a birational morphism as f_1 has connected fibers. Therefore $q(V_1) = q(V_2)$ which guarantees $f_1^*\operatorname{Pic}^0(V_1) = f_2^*\operatorname{Pic}^0(V_2)$. □

In the following theorem we denote by $\widetilde{f}\colon \widetilde{X} \to \widetilde{V}$ a non-singular representative of a χ-positive irrational pencil $f\colon X \to V$. Moreover we denote by $\pi_0^q\big(V^{n-k}(\omega_X)_0\big)$ the set of all irreducible components of $V^{n-k}(\omega_X)_0$ of dimension q.

Theorem 16. *Let $q \geq k \geq 1$ be integers and suppose that $\dim V^i(\omega_X)_0 \leq 0$ for all $i > n-k$. Then the assignment*

$$u_{X,k,q}\colon F_X^{k,q} \to \pi_0^q\big(V^{n-k}(\omega_X)_0\big), \quad \big(f\colon X \to V\big) \mapsto \widetilde{f}^*\operatorname{Pic}^0(\widetilde{V})$$

is well-defined and defines a bijection of sets.

Remark 17. The bijections $u_{X,g}$ mentioned in §2 are defined as $u_{X,g} := u_{X,1,g}$ for all $g \geq 2$.

Proof of Theorem 16. We begin the proof by showing that an irreducible component $Z \subset V^{n-k}(\omega_X)_0$ of dimension q determines a class of $F_X^{k,q}$. By Lemma 13 the component Z determines a fibration $f\colon X \to V$ to a normal projective variety such that $0 < \dim V \leq k$. Moreover, if $\widetilde{f}\colon \widetilde{X} \to \widetilde{V}$ is a non-singular representative of f, then $\widetilde{V}$ is of maximal Albanese dimension,

$$\chi(R^{n-k}\widetilde{f}_*\omega_{\widetilde{X}}) > 0, \quad \text{and} \quad Z = \widetilde{f}^* V^0(R^{n-k}\widetilde{f}_*\omega_{\widetilde{X}}) = \widetilde{f}^* \operatorname{Pic}^0(\widetilde{V}).$$

Furthermore, by Remark 14 there exist an isomorphism of varieties $\widehat{Z} \simeq \operatorname{Alb}(\widetilde{V})$ together with a finite morphism $V \to \widehat{Z}$ such that the composition $\widetilde{V} \to V \to \widehat{Z}$ equals $alb_{\widetilde{V}}$. Hence, if $\dim V = k$, then $f\colon X \to V$ is a fibration in $F_X^{k,q}$ as $R^{n-k}\widetilde{f}_*\omega_{\widetilde{X}} \simeq \omega_{\widetilde{V}}$ by [Kol86, Proposition 7.6]. Now we prove that $\dim V = k$. We proceed by contradiction and suppose hence that $\dim V = k - s$, for some integer $s > 0$. Suppose first that $\dim V^0(\omega_{\widetilde{V}})_0 > 0$. By pulling-back topological trivial line bundles on $\widetilde{V}$, we have that the locus $V^0(\omega_{\widetilde{V}})_0$ injects in $V^{n-k+s}(\omega_{\widetilde{X}})_0 = V^{n-k+s}(\omega_X)_0$ (*cf.* [Lom14, Lemma 6.3]). This is a contradiction as $\dim V^{n-k+s}(\omega_X)_0 = 0$. Suppose now the second case $V^0(\omega_{\widetilde{V}})_0 = \{\mathcal{O}_{\widetilde{V}}\}$. By [EL97, Proposition 2.2] the Albanese map $alb_{\widetilde{V}}\colon \widetilde{V} \to \operatorname{Alb}(\widetilde{V})$ is surjective. It follows that $q = q(\widetilde{V}) \leq \dim \widetilde{V} = k - s < k$ which yields a contradiction.

For the other direction, let $(f\colon X \to V, \widetilde{V} \to V, b\colon V \to \operatorname{Alb}(\widetilde{V}))$ be a χ-positive k-irrational pencil with $q(V) = q$. We denote by $\widetilde{f}\colon \widetilde{X} \to \widetilde{V}$ a non-singular representative of f over $\widetilde{V}$. It follows that $V^0(\omega_{\widetilde{V}}) = \operatorname{Pic}^0(\widetilde{V})$ and that the pull-back $Z \stackrel{\text{def}}{=} \widetilde{f}^* \operatorname{Pic}^0(\widetilde{V})$ is contained in $V^{n-k}(\omega_{\widetilde{X}})_0 = V^{n-k}(\omega_X)_0$. We show now that Z is an irreducible component of $V^{n-k}(\omega_X)_0$. We argue by contradiction and suppose hence that $Z \subset Z'$ where Z' is an irreducible component of $V^{n-k}(\omega_X)_0 = V^{n-k}(\omega_{\widetilde{X}})_0$. By Lemma 13 the component Z' gives rise to a fibration $p\colon \widetilde{X} \to U$ where U is a normal projective variety with $0 < \dim U \leq k$. Moreover, if $p'\colon X' \to U'$ is a non-singular representative of p, and $\bar{f}$ denotes the composition $X' \to \widetilde{X} \xrightarrow{\widetilde{f}} \widetilde{V}$, then the components Z' and Z are equal to $p'^* \operatorname{Pic}^0(U')$ and $\bar{f}^* \operatorname{Pic}^0(\widetilde{V})$, respectively. By Lemma 15 we conclude that $\dim U' = k$ and hence that Z is an irreducible component. As the loci $V^i(\omega_X)_0$ are birational invariant (*cf.* Remark 9), the component Z neither depends on the particular choice of a non-singular representative of f, nor on a representative of $(f\colon X \to V) \in F_X^{k,q}$ as a class.

In order to conclude the proof, we need to check that the assignments defined above yield bijections of sets. While the composition $\pi_0^q\big(V^{n-k}(\omega_X)_0\big) \to F_X^{k,q} \to \pi_0^q\big(V^{n-k}(\omega_X)_0\big)$ equals the identity by Lemma 13, the other composition $F_X^{k,q} \to \pi_0^q\big(V^{n-k}(\omega_X)_0\big) \to F_X^{k,q}$ equals the identity by Lemma 15 and Remark 10. □

4. Proofs

4.1. **Proof of Theorem 1.** Thanks to Orlov's representability theorem (*cf.* [Orl97, Theorem 2.2]), an equivalence of derived categories $\mathbf{D}(X) \simeq \mathbf{D}(Y)$ is realized by a

Fourier–Mukai functor $\Phi_{\mathcal{E}}(-) \stackrel{\text{def}}{=} \mathbf{R}p_{2*}\big(p_1^*(-) \overset{\mathbf{L}}{\otimes} \mathcal{E}\big)$. Here p_1 and p_2 are the projections from $X \times Y$ onto the first and second factor respectively, and $\mathcal{E}$ is an object in $\mathbf{D}(X \times Y) \stackrel{\text{def}}{=} D^b\big(\mathcal{C}oh(X \times Y)\big)$ uniquely determined up to isomorphism. By [Rou11, Théorème 4.18] the kernel $\mathcal{E}$ induces an isomorphism of algebraic groups

$$F: \ \mathrm{Aut}^0(X) \times \mathrm{Pic}^0(X) \ \to \ \mathrm{Aut}^0(Y) \times \mathrm{Pic}^0(Y),$$

acting as (*cf.* [PS11, Lemma 3.1]):

$$F(\varphi, \alpha) \ = \ (\psi, \beta) \iff p_1^*\alpha \otimes (\varphi \times \mathrm{id}_Y)^*\mathcal{E} \ \simeq \ p_2^*\beta \otimes (\mathrm{id}_X \times \psi)_*\mathcal{E}. \tag{3}$$

Set now $n = \dim X = \dim Y$. By Theorem 6 and Remark 7 F acts on $V^{n-1}(\omega_X)_0$ as

$$F\big(\mathrm{id}_X, V^{n-1}(\omega_X)_0\big) \ = \ \big(\mathrm{id}_Y, V^{n-1}(\omega_Y)_0\big),$$

so that it induces an isomorphism of algebraic sets $V^{n-1}(\omega_X)_0 \simeq V^{n-1}(\omega_Y)_0$. By following §2, the composition

$$\mu_g \ \stackrel{\text{def}}{=} \ u_{Y,g}^{-1}\, F\, u_{X,g} \colon F_X^g \ \to \ F_Y^g$$

defines a bijection for any integer $g \geq 2$ such that if $\mu_g(f\colon X \to C) = (h\colon Y \to D)$, then

$$F\big(\mathrm{id}_X, f^*\,\mathrm{Pic}^0(C)\big) \ = \ \big(\mathrm{id}_Y, h^*\,\mathrm{Pic}^0(D)\big). \tag{4}$$

In the second part of this proof we will show that the curves C and D are isomorphic. This will be achieved by constructing a complex in $\mathbf{D}(C \times D) \stackrel{\text{def}}{=} D^b\big(\mathcal{C}oh(C \times D)\big)$ whose support dominates both C and D via birational morphisms. By (4) and (3) there exists an isomorphism of abelian varieties $\zeta\colon \ \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(D)$ such that

$$p_1^*f^*\alpha \otimes \mathcal{E} \ \simeq \ p_2^*h^*\zeta(\alpha) \otimes \mathcal{E} \quad \text{ for all } \quad \alpha \in \mathrm{Pic}^0(C). \tag{5}$$

We fix now a sufficiently ample line bundle L on $X \times Y$ and define the complex $\mathcal{E}'$ in $\mathbf{D}(C \times D)$ as

$$\mathcal{E}' \ \stackrel{\text{def}}{=} \ \mathbf{R}(f \times h)_*(\mathcal{E} \otimes L).$$

Moreover we denote by t_1 and t_2 the projections from $C \times D$ onto the first and second factor, respectively. As projection formula yields

$$\mathbf{R}(f \times h)_*\big(p_1^*f^*\alpha \otimes \mathcal{E} \otimes L\big) \ \simeq \ \mathbf{R}(f \times h)_*\big(\mathbf{L}(f \times h)^*t_1^*\alpha \otimes \mathcal{E} \otimes L\big) \ \simeq \ t_1^*\alpha \otimes \mathcal{E}',$$

we obtain new isomorphisms of complexes:

$$t_1^*\alpha \otimes \mathcal{E}' \ \simeq \ t_2^*\zeta(\alpha) \otimes \mathcal{E}'. \tag{6}$$

We denote now by $i_b\colon t_2^{-1}(b) \hookrightarrow C \times D$ the closed immersion of the fiber $t_2^{-1}(b) \simeq C$ and set $\mathcal{E}'_b \stackrel{\text{def}}{=} \mathbf{L}i_b^*\mathcal{E}'$. By restricting the isomorphisms in (6) to every fiber of t_2,[1] we obtain new isomorphisms:

$$\mathcal{E}'_b \otimes \alpha \ \simeq \ \mathcal{E}'_b \quad \text{ for all } \quad \alpha \in \mathrm{Pic}^0(C). \tag{7}$$

By taking determinants and considering a line bundle α of infinite order, we deduce that $\dim \mathrm{Supp}\,\big(\mathcal{E}'_b\big) \leq 0$ for all $b \in D$ as described in [Huy06, Lemma 6.9] or [Muk81, Lemma

[1] I thank Christian Schnell for suggesting this strategy.

3.3].[2] Moreover, from the isomorphisms

$$\mathrm{Supp}\big(\mathcal{E}'_b\big) \;=\; \mathrm{Supp}\big(\mathcal{E}'\big) \cap t_2^{-1}(b) \quad \text{for all} \quad b \in D$$

deduced using [Huy06, Lemma 3.29], we conclude that $t_2 \colon \mathrm{Supp}(\mathcal{E}') \to D$ is a finite morphism, and furthermore that $\dim \mathrm{Supp}\big(\mathcal{E}'\big) \leq 1$. In complete analogy we also deduce that $t_1 \colon \mathrm{Supp}(\mathcal{E}') \to C$ is finite.

Now consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Supp}\,\big(\mathcal{E} \otimes L\big) & \xrightarrow{p_1} & X \\ \Big\downarrow{\scriptstyle f\times h} & & \Big\downarrow{\scriptstyle f} \\ \mathrm{Supp}\,\big(\mathcal{E}'\big) & \xrightarrow{t_1} & C. \end{array}$$

The left arrow of the above diagram is well-defined thanks to the following claim.

Claim 18. *The restriction of* $f \times h$ *to* $\mathrm{Supp}\big(\mathcal{E} \otimes L\big) = \mathrm{Supp}\big(\mathcal{E}\big)$ *defines a morphism* $f \times h \colon \mathrm{Supp}\big(\mathcal{E} \otimes L\big) \to \mathrm{Supp}\big(\mathcal{E}'\big)$ *of algebraic sets.*

Proof. Since L is sufficiently ample, we get vanishings

$$R^p(f \times h)_*(H^q(\mathcal{E}) \otimes L) \;=\; 0 \quad \text{for all} \quad p > 0,$$

and surjections

$$(f \times h)^*(f \times h)_*(H^q(\mathcal{E}) \otimes L) \;\twoheadrightarrow\; H^q(\mathcal{E}) \otimes L \quad \text{for all} \quad q.$$

Therefore by the degeneration of the spectral sequence

$$E^1_{p,q} \;:=\; R^p(f \times h)_*(H^q(\mathcal{E}) \otimes L) \;\Rightarrow\; R^{p+q}(f \times h)_*(\mathcal{E} \otimes L),$$

we get the following isomorphisms

$$H^q(\mathcal{E}') \;=\; R^q(f \times h)_*(\mathcal{E} \otimes L) \;\simeq\; (f \times h)_*(H^q(\mathcal{E}) \otimes L) \quad \text{for all} \quad q,$$

$$\mathrm{Supp}\,\big((f \times h)_*\big(H^q(\mathcal{E}) \otimes L\big)\big) \;=\; (f \times h)\big(\,\mathrm{Supp}\,\big(H^q(\mathcal{E}) \otimes L\big)\big) \quad \text{for all} \quad q.$$

We conclude that

$$\mathrm{Supp}\,\big(H^q(\mathcal{E}')\big) \;\simeq\; (f \times h)(\mathrm{Supp}\,\big(\,H^q(\mathcal{E}) \otimes L\big)\big) \quad \text{for all} \quad q,$$

from which we deduce

$$\mathrm{Supp}\,\big(\mathcal{E}'\big) \;=\; \bigcup_q \mathrm{Supp}\,\big(H^q(\mathcal{E}')\big) \;\simeq\; (f \times h)\Big(\bigcup_q \mathrm{Supp}\,\big(H^q(\mathcal{E}) \otimes L\big)\Big) \;=$$
$$= (f \times h)\big(\mathrm{Supp}\,\big(\mathcal{E} \otimes L\big)\big).$$

□

Going back to the commutative square before Claim 18, we observe that by [Huy06, Lemmas 6.4 and 6.11] the projection $\mathrm{Supp}\,\big(\mathcal{E} \otimes L\big) \to X$ is a fibration. Therefore

[2]We denote by $H^q(\mathcal{F})$ the q-th cohomology sheaf of a complex $\mathcal{F}$. Moreover we define the support of $\mathcal{F}$ as $\mathrm{Supp}(\mathcal{F}) = \bigcup_q \mathrm{Supp}\,\big(H^q(\mathcal{F})\big)$. We endow $\mathrm{Supp}(\mathcal{F})$ with the reduced structure.

$t_1\colon \operatorname{Supp}\big(\mathcal{E}'\big) \to C$ is a fibration as well and hence $\operatorname{Supp}\big(\mathcal{E}'\big)$ is irreducible. We conclude that $\operatorname{Supp}\big(\mathcal{E}'\big) \simeq C$. Now consider the analogue commutative diagram:

$$\begin{array}{ccc} \operatorname{Supp}\big(\mathcal{E}\otimes L\big) & \xrightarrow{\;p_2\;} & Y \\ \big\downarrow{\scriptstyle f\times h} & & \big\downarrow{\scriptstyle h} \\ \operatorname{Supp}\big(\mathcal{E}'\big) & \xrightarrow{\;t_2\;} & D. \end{array}$$

As before we can deduce that t_2 is a fibration, and hence that $\operatorname{Supp}\big(\mathcal{E}'\big) \simeq D$. This concludes the proof.

Remark 19. In [LP15, Theorem 14] the authors study special cases where it is possible to say something about the behavior fibers of the fibrations. In particular they prove that if $f\colon X \to C$ is a Fano fibration over a smooth curve of genus $g \geq 2$ and Y is D-equivalent to X, then they are K-equivalent and the generic fibers of f and $h \stackrel{\text{def}}{=} \mu_g(f)\colon Y \to C$ are isomorphic. Moreover, if X is a Mori dream space, then there exists an isomorphism between X and Y that respects the fibers of the fibrations.

4.2. **Proof of Theorem 3.** Let

$$\big(v\colon X \to S,\ \rho_S\colon \widetilde{S} \to S,\ a\colon S \to \operatorname{Alb}(\widetilde{S})\big)$$

be a fibration in $F_X^{2,q}$, and let $Z \stackrel{\text{def}}{=} \widetilde{v}^* \operatorname{Pic}^0(\widetilde{S}) \subset V^{n-2}(\omega_X)_0$ be an irreducible component induced by a non-singular representative $\widetilde{v}\colon \widetilde{X} \to \widetilde{S}$ of v over $\widetilde{S}$. By Theorem 6 the Rouquier isomorphism F acts on $V^{n-2}(\omega_X)_0$ as

$$F\big(\mathrm{id}_X, V^{n-2}(\omega_X)_0\big) \;=\; \big(\mathrm{id}_Y, V^{n-2}(\omega_Y)_0\big),$$

since the Hodge number $h^{0,2}(X)$ is assumed to be invariant. Therefore we obtain an irreducible component $Z' \stackrel{\text{def}}{=} F(\mathrm{id}_X, Z) \subset V^{n-2}(\omega_X)_0$ inducing a fibration $w\colon Y \to T$ onto a normal projective variety of dimension either one or two. Moreover, if $\rho_T\colon \widetilde{T} \to T$ is a resolution of singularities and $\widetilde{w}\colon \widetilde{Y} \to \widetilde{T}$ is a non-singular representative of w over $\widetilde{T}$, then $\widetilde{T}$ is of maximal Albanese dimension, there exists a finite morphism $b\colon T \to \operatorname{Alb}(\widetilde{T})$ that factors through $alb_{\widetilde{T}}$, $\chi(R^{n-2}\widetilde{w}_*\omega_{\widetilde{T}}) > 0$, and $Z' = \widetilde{w}^* \operatorname{Pic}^0(\widetilde{T}) = \widetilde{w}^* V^0(R^{n-2}\widetilde{w}_*\omega_{\widetilde{T}})$. We claim that $\dim T = 2$. If by contradiction $\dim T = 1$, then the following facts would be true: $\widetilde{T} \simeq T$, $V^0(\omega_T) = \operatorname{Pic}^0(T)$,[3] and $w^* \operatorname{Pic}^0(T) = Z' \subset V^{n-1}(\omega_Y)_0$ (*cf.* [Lom14, Lemma 6.3]). Hence by Remark 7 the component Z would be contained in $V^{n-1}(\omega_X)_0$, and it would induce an irrational pencil $u\colon X \to B$ of genus q such that $Z = u^* \operatorname{Pic}^0(B)$. By Lemma 15 we would get a dominant rational map $B \dashrightarrow \widetilde{S}$ which is impossible. We conclude that T is a surface with $\chi(\omega_{\widetilde{T}}) > 0$ defining thus a fibration

$$\big(w\colon Y \to T, \rho_T\colon \widetilde{T} \to T, b\colon T \to \operatorname{Alb}(\widetilde{T})\big)$$

[3] A smooth projective curve of genus $g \geq 2$ satisfies $V^0(\omega_C) = \operatorname{Pic}^0(C)$. This fact is not longer true for smooth projective varieties of dimension $n \geq 2$ and maximal Albanese dimension. For instance isotrivial elliptic surfaces S of Kodaira dimension one provide counterexamples for any $q(S) \geq 3$. This is the reason why the arguments of this proof does not extend to fibrations in $F_X^{k,q}$ with $k \geq 3$.

in $F_Y^{2,q}$. Denote now by $u_{X,2,q}$ and $u_{Y,2,q}$ the assignments defined in Theorem 16 for the varieties X and Y, respectively. Due to the symmetric nature of the previous argument, we conclude that the composition $\nu_q \stackrel{\text{def}}{=} u_{Y,2,q}^{-1}\, F\, u_{X,2,q} \colon F_X^{2,q} \to F_Y^{2,q}$ defines a bijection such that if $(v\colon X \to S) \in F_X^{2,q}$ and $\nu_q(v) = (w\colon Y \to T)$, then

$$F\big(\mathrm{id}_X, \pi_{X*}\widetilde{v}^*\operatorname{Pic}^0(\widetilde{S})\big) \;=\; \big(\mathrm{id}_Y, \pi_{Y*}\widetilde{w}^*\operatorname{Pic}^0(\widetilde{T})\big) \tag{8}$$

where π_X and π_Y are the birational morphisms defining the non-singular representatives $\widetilde{v}$ and $\widetilde{w}$, respectively:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\pi_X} & X \\ \downarrow{\scriptstyle \widetilde{v}} & & \downarrow{\scriptstyle v} \\ \widetilde{S} & \xrightarrow{\rho_S} & S \end{array} \qquad\qquad \begin{array}{ccc} \widetilde{Y} & \xrightarrow{\pi_Y} & Y \\ \downarrow{\scriptstyle \widetilde{w}} & & \downarrow{\scriptstyle w} \\ \widetilde{T} & \xrightarrow{\rho_T} & T. \end{array}$$

In order to finish the proof we will prove that $S \simeq T$. We denote by $\mathcal{E}$ the kernel of the D-equivalence $\mathbf{D}(X) \simeq \mathbf{D}(Y)$ and by p_1, p_2 the projections from $X \times Y$ onto the X and Y, respectively. Moreover we denote by π the product $\pi_X \times \pi_Y$, and by $\xi\colon \operatorname{Pic}^0(\widetilde{S}) \to \operatorname{Pic}^0(\widetilde{T})$ the isomorphism induced by F. By Proposition 8 there is an induced isomorphism $\operatorname{Pic}^0(S) \simeq \operatorname{Pic}^0(T)$, which we continue to denote abusively by ξ. Finally we denote by $\widetilde{p}_1$ and $\widetilde{p}_2$ the projections from $\widetilde{X}\times\widetilde{Y}$ onto the first and second factor, respectively. By the equality (8), the action of the Rouquier isomorphism described in (3) gives rise to a collection of isomorphisms of complexes:

$$\pi_*\,\widetilde{p}_1^*\,\widetilde{v}^*\alpha \otimes \mathcal{E} \;\simeq\; \pi_*\,\widetilde{p}_2^*\,\widetilde{w}^*\xi(\alpha) \otimes \mathcal{E} \quad \text{for all} \quad \alpha \in \operatorname{Pic}^0(\widetilde{S}). \tag{9}$$

By Proposition 8 the isomorphisms (9) are equivalent to the following:

$$\pi_*\,\widetilde{p}_1^*\,\widetilde{v}^*\rho_S^*\gamma \otimes \mathcal{E} \;\simeq\; \pi_*\,\widetilde{p}_2^*\,\widetilde{w}^*\rho_T^*\xi(\gamma) \otimes \mathcal{E} \quad \text{for all} \quad \gamma \in \operatorname{Pic}^0(S).$$

Denote now by t_1 and t_2 the projections from $S \times T$ onto S and T, respectively. By tensoring the previous isomorphisms by a sufficiently ample line bundle L on $X \times Y$, and by pushing them forward to $S \times T$, we obtain new isomorphisms:

$$t_1^*\gamma \otimes \mathcal{E}' \;\simeq\; t_2^*\xi(\gamma) \otimes \mathcal{E}' \quad \text{for all} \quad \gamma \in \operatorname{Pic}^0(S) \tag{10}$$

where

$$\mathcal{E}' \stackrel{\text{def}}{=} \mathbf{R}(v \times w)_*\big(\mathcal{E} \otimes L\big).$$

Recall that $a\colon S \to \operatorname{Alb}(\widetilde{S})$ and $b\colon T \to \operatorname{Alb}(\widetilde{T})$ are finite morphisms to abelian varieties, previously defined as part of data in the χ-positive 2-irrational pencils v and w. Moreover set

$$\mathcal{E}'' \stackrel{\text{def}}{=} \mathbf{R}(a \times b)_*\mathcal{E}' = (a \times b)_*\mathcal{E}'.$$

Finally, denote by q_1 and q_2 the two projections from $\operatorname{Alb}(\widetilde{S}) \times \operatorname{Alb}(\widetilde{T})$ onto the first and second factor, respectively. By pushing the isomorphisms (10) forward to $\operatorname{Alb}(\widetilde{S})\times\operatorname{Alb}(\widetilde{T})$, we obtain new isomorphisms:

$$(a \times b)_*\big(t_1^*\gamma \otimes \mathcal{E}'\big) \;\simeq\; (a \times b)_*\big(t_2^*\xi(\gamma) \otimes \mathcal{E}'\big) \quad \text{for all} \quad \gamma \in \operatorname{Pic}^0(S).$$

By noting that every $\gamma \in \operatorname{Pic}^0(S)$ can be written as $\gamma = a^*\beta$ for a unique $\beta \in \operatorname{Pic}^0(\operatorname{Alb}(\widetilde{S}))$, projection formula yields that

$$q_1^*\beta \otimes \mathcal{E}'' \simeq q_2^*\xi(\beta) \otimes \mathcal{E}'' \qquad \text{for all} \quad \beta \in \operatorname{Pic}^0(\operatorname{Alb}(\widetilde{S})). \tag{11}$$

Denote now by $\mathcal{E}''_s \overset{\text{def}}{=} \mathcal{E}''_{|q_1^{-1}(s)}$ the derived restriction of $\mathcal{E}''$ to the fiber $q_1^{-1}(s)$. By restricting the isomorphisms (11) to each fiber of q_1, we have isomorphisms

$$\mathcal{E}''_s \otimes \beta \simeq \mathcal{E}''_s \qquad \text{for all} \quad \beta \in \operatorname{Pic}^0(\operatorname{Alb}(\widetilde{S})) \quad \text{and} \quad s \in \operatorname{Alb}(\widetilde{S})$$

(with an abuse of notation we still denote by ξ the isomorphism $\operatorname{Pic}^0(\operatorname{Alb}(\widetilde{S})) \simeq \operatorname{Pic}^0(\operatorname{Alb}(\widetilde{T}))$. As $\mathcal{E}''_s$ has only a finite number of non-zero cohomology sheaves, [Muk81, Lemma 3.3] implies that $\operatorname{Supp}\big(\mathcal{E}''_s\big)$ is 0-dimensional for all s. Similarly, also the other derived restrictions $\mathcal{E}''_t \overset{\text{def}}{=} \mathcal{E}''_{|q_2^{-1}(t)}$ have 0-dimensional support for all $t \in \operatorname{Alb}(\widetilde{T})$. We conclude that $\dim \operatorname{Supp}\big(\mathcal{E}'\big) \leq 2$ as $(a \times b)$ is a finite morphism, and that the restrictions $q_1\colon \operatorname{Supp}\big(\mathcal{E}''\big) \to \operatorname{Alb}(\widetilde{S})$ and $q_2\colon \operatorname{Supp}\big(\mathcal{E}''\big) \to \operatorname{Alb}(\widetilde{T})$ are finite morphisms. We also conclude that the morphisms $t_1\colon \operatorname{Supp}(\mathcal{E}') \to S$ and $t_2\colon \operatorname{Supp}(\mathcal{E}') \to T$ are finite morphisms as showed by the diagrams:

$$\begin{array}{ccc} \operatorname{Supp}\big(\mathcal{E}'\big) & \xrightarrow{t_1} & S \\ \Big\downarrow{\scriptstyle a\times b} & & \Big\downarrow{\scriptstyle a} \\ \operatorname{Supp}\big(\mathcal{E}''\big) & \xrightarrow{q_1} & \operatorname{Alb}(\widetilde{S}) \end{array} \qquad\qquad \begin{array}{ccc} \operatorname{Supp}\big(\mathcal{E}'\big) & \xrightarrow{t_2} & T \\ \Big\downarrow{\scriptstyle a\times b} & & \Big\downarrow{\scriptstyle b} \\ \operatorname{Supp}\big(\mathcal{E}''\big) & \xrightarrow{q_2} & \operatorname{Alb}(\widetilde{T}). \end{array}$$

To conclude, we consider the following commutative diagrams:

$$\begin{array}{ccc} \operatorname{Supp}\big(\mathcal{E} \otimes L\big) & \xrightarrow{p_1} & S \\ \Big\downarrow{\scriptstyle v\times w} & & \Big\downarrow{\scriptstyle v} \\ \operatorname{Supp}\big(\mathcal{E}'\big) & \xrightarrow{t_1} & S \end{array} \qquad\qquad \begin{array}{ccc} \operatorname{Supp}\big(\mathcal{E} \otimes L\big) & \xrightarrow{p_2} & T \\ \Big\downarrow{\scriptstyle v\times w} & & \Big\downarrow{\scriptstyle w} \\ \operatorname{Supp}\big(\mathcal{E}'\big) & \xrightarrow{t_2} & T, \end{array}$$

in which the restriction morphism $v \times w\colon \operatorname{Supp}\big(\mathcal{E} \otimes L\big) \to \operatorname{Supp}\big(\mathcal{E}'\big)$ is well-defined thanks to an argument similar to that of Claim 18. We note that as p_1 and p_2 are surjective morphisms with connected fibers, also t_1 and t_2 are surjective with connected fibers. From this we deduce that $\operatorname{Supp}\big(\mathcal{E}'\big)$ is irreducible and that $S \simeq \operatorname{Supp}\big(\mathcal{E}'\big) \simeq T$.

Finally we prove the last statement of the theorem. Assume that $\delta\colon X \to S$ is a surjective morphism onto the basis of some χ-positive 2-irrational pencil, and let $X \overset{\delta'}{\to} S' \overset{\delta''}{\to} S$ be the Stein factorization. Then S' is a normal variety and admits a finite morphism to an abelian variety. Let $\widetilde{\delta''}\colon \widetilde{S'} \to \widetilde{S}$ be a non-singular representative of δ''. We claim that in order to finish the proof, it is enough to prove that $\chi(\omega_{\widetilde{S'}}) > 0$. In fact, granting the claim for this time being, the fibration δ' would give rise to a class in $F_X^{2,q'}$ for some $q' \geq q$, and hence it would induce the desired surjection as $Y \to T \overset{\sim}{\to} S' \overset{\delta''}{\to} S$, in which the fibration $Y \to T$ is defined as $\nu_{q'}(\delta')$. The claim is a consequence of the following lemma.

Lemma 20. *Let $f\colon X \to Y$ be a surjective morphism between two smooth projective varieties of the same dimension. If Y is of maximal Albanese dimension, then we have $\chi(\omega_X) \geq \chi(\omega_Y)$.*

Proof. There exists a decomposition $f_*\omega_X \simeq \omega_Y \oplus \mathcal{E}$ where $\mathcal{E}$ is a sheaf on Y. Moreover $f_*\omega_X$ is a GV-sheaf by [PP11, Theorem 5.8] so that $\mathcal{E}$ is itself GV. Hence $\chi(\mathcal{E}) \geq 0$ and the degeneration of the Leray spectral sequence yields

$$\chi(\omega_X) = \chi(f_*\omega_X) = \chi(\omega_Y) + \chi(\mathcal{E}) \geq \chi(\omega_Y).$$

□

5. Applications of Theorem 1

The fundamental group $\pi_1(X)$ of a smooth projective complex variety is not in general a derived invariant, as shown for instance by Schnell in [Sch12]. Nevertheless, as an application of Theorem 1, we deduce the derived invariance of one of its properties.

Corollary 21. *Let $g \geq 2$ be an integer. If X and Y are smooth projective D-equivalent varieties, then $\pi_1(X)$ admits a surjective homomorphism onto the fundamental group of a Riemann surface of genus g if and only if $\pi_1(Y)$ does. Hence $\pi_1(X)$ admits a surjective homomorphism onto a non-abelian free group if and only if $\pi_1(Y)$ does.*

Proof. In the Appendix of [Cat91] Beauville proves that for any integer $g \geq 2$ there exists a surjective homomorphism $\pi_1(X) \to \Gamma_g$ onto the fundamental group of a Riemann surface of genus g if and only if X admits an irrational pencil of genus greater or equal to g. Hence the first statement of the corollary is an application of Theorem 1. The second statement is a consequence of Castelnuovo–De Franchis' Theorem as explained in [ABC$^+$96, §2] or [KK10, Theorem 1.1]. □

Remark 22. In [Cat91, Theorem 1.10] (*cf.* also [KK10, Theorem 1.1]) the author relates the existence of irrational pencils of genus $g \geq 2$ to the existence of maximal isotropic subspaces in $H^1(X, \mathbf{C})$ (we recall that a subspaces $U \subset H^1(X, \mathbf{C})$ is said *isotropic* if the map $U \wedge U \to H^2(X, \mathbf{C})$ is the trivial map). Hence, as an application of Theorem 1, we deduce that X admits a maximal isotropic subspace $U \subset H^1(X, \mathbf{C})$ of dimension $g \geq 2$ if and only if Y does. More refined statements along these lines can be formulated by means of [Cat91, Theorem 2.25].

A further application of Theorem 1 involves the cup product map

$$\varphi_X\colon H^1(X, \mathbf{C}) \wedge H^1(X, \mathbf{C}) \to H^2(X, \mathbf{C})$$

of a smooth projective complex variety. In view of Castelnuovo–De Franchis' theorem, Theorem 1 implies that if X and Y are D-equivalent, then φ_X is injective on pure forms of type $\omega_1 \wedge \omega_2$ if and only if φ_Y has the same property. In addition we have the following result concerning the injectivity of the whole cup product map.

Corollary 23. *Suppose that X and Y are smooth projective D-equivalent varieties such that $p_g(X) = h^0(X, \omega_X) = 1$. Then φ_X is injective if and only if φ_Y is injective.*

Proof. In [CJT17, Theorem 1.3] the authors prove that if $p_g(X) = 1$, then φ_X is injective if and only if X does not admit any irrational pencil of genus 2. As the geometric genus of a variety is a derived invariant, the corollary is again a simple application of Theorem 1. □

6. Further results and Proof of Corollary 5

In this section we are going to prove that the property for a variety to admit a χ-positive irrational pencil over some positive-dimensional basis is invariant under derived equivalence. This is achieved by exploiting the derived invariance of the Hodge number $h^{1,n}(X)$ obtained as an application of the derived invariance of Hochschild homology (*cf.* [Huy06, Remark 5.40]). At the end of this section we will prove Corollary 5.

We continue to denote by F the Rouquier isomorphism induced by an equivalence $\mathbf{D}(X) \simeq \mathbf{D}(Y)$ of triangulated categories. We suppose that $f\colon X \to V$ is a fibration in $F_X^{n-1,q}$ (with $q = q(V) \geq n-1$), and denote by Z the irreducible component of $V^1(\omega_X)$ induced by f (*i.e.* $Z \stackrel{\text{def}}{=} \widetilde{f}^* \operatorname{Pic}^0(\widetilde{V})$ where $\widetilde{f}\colon \widetilde{X} \to \widetilde{V}$ is a non-singular representative of f). By Theorem 6 the locus $Z' \stackrel{\text{def}}{=} F(\operatorname{id}_X, Z) \subset \operatorname{Pic}^0(Y)$ is an irreducible component of $V^1(\omega_Y)_0$, and hence by Lemma 13 it induces a fibration $h\colon Y \to W$ onto a normal projective variety of dimension $0 < \dim W \leq n-1$. Moreover, if $\widetilde{h}\colon \widetilde{Y} \to \widetilde{W}$ is a non-singular representative of h, then $\widetilde{W}$ is of maximal Albanese dimension, $\chi(R^1\widetilde{h}_*\omega_{\widetilde{Y}}) > 0$, and $Z' = \widetilde{h}^* \operatorname{Pic}^0(\widetilde{W})$. We note that if $\dim W = \dim V$, then h is a fibration in $F_Y^{n-1,q}$ as $R^1\widetilde{h}_*\omega_{\widetilde{Y}} \simeq \omega_{\widetilde{W}}$ (*cf.* [Kol86, Proposition 7.6]).

Proposition 24. *If $\dim W < \dim V$, then Y admits a fibration $p\colon Y \to U$ in $F_Y^{k,q'}$ for some integers $0 < k < n-1$ and $2 \leq q' \leq q$. Moreover there exists a dominant rational map $\gamma\colon W \dashrightarrow U$ such that $p = \gamma\, h$.*

Proof. As $q(\widetilde{W}) = q \geq n-1 > \dim \widetilde{W}$, we have $\dim V^0(\omega_{\widetilde{W}})_0 > 0$ by [EL97, Proposition 2.2]. If $V^0(\omega_{\widetilde{W}}) = \operatorname{Pic}^0(\widetilde{W})$, then h is a fibration in $F_Y^{\dim W,q}$. On the other hand, if $V^0(\omega_{\widetilde{W}}) \neq \operatorname{Pic}^0(\widetilde{W})$ and $Z'' \subset V^0(\omega_{\widetilde{W}})_0$ is a positive-dimensional irreducible component of codimension $0 < j \stackrel{\text{def}}{=} \operatorname{codim} Z'' < q(\widetilde{W})$, then Z'' is also an irreducible component of $V^j(\omega_{\widetilde{W}})_0$ (*cf.* [Par17, Corollary 3.3]). Therefore there exists a further fibration $h'\colon \widetilde{W} \to W'$ such that $0 < \dim W' \leq \dim W - j < n-1-j$. Moreover, if $\widetilde{h}'\colon \widetilde{\widetilde{W}} \to \widetilde{W}'$ a non-singular representative of h', then $\widetilde{\widetilde{W}}$ is of maximal Albanese dimension, $\chi(R^j\widetilde{h}'_*\omega_{\widetilde{\widetilde{W}}}) > 0$, $Z'' = \widetilde{h}'^* \operatorname{Pic}^0(\widetilde{W}')$, and $\dim V^0(\omega_{\widetilde{W}'})_0 > 0$. In this way we obtain a fibration $l\colon \widetilde{Y} \to W'$

with the same properties of h, but such that $\dim W' < \dim W$. By proceeding in this way, we will eventually arrive to a fibration $p\colon Y^* \to U$ in $F^{k,q'}_{Y^*}$, where Y^* is a smooth projective variety birational to Y, $0 < k < n-1$, and $\dim U \leq q' \leq q$. However, if U is a curve, then we must have $q' \geq 2$, as by [Bea92, Corollaire 2.3] the $(n-1)$-th non-vanishing locus attached to the canonical bundle of a variety does not contain any component of dimension one passing through the origin. As U admits a finite morphism to $\mathrm{Alb}(\widetilde{U})$, where $\widetilde{U}$ is a resolution of singularities of U, that factors through $alb_{\widetilde{U}}$, the rational dominant map $Y \dashrightarrow U$ extends to a morphism p' such that the composition $Y^* \to Y \xrightarrow{p'} U$ equals p. The last statement is a consequence of the construction of the fibrations. $\square$

Remark 25. Proposition 24 generalizes to the case of fibrations in $F^{p,q}_X$ once one assumes the derived invariance of the Hodge number $h^{0,p}(X)$ in dimension n.

In the case of fourfolds we can be more precise about the type of fibrations that Y may posses. In this case the Rouquier isomorphism induces isomorphisms of algebraic sets $V^i(\omega_X)_0 \simeq V^i(\omega_Y)_0$ for each $i \geq 0$ (*cf.* Theorem 6 and [Abu17, Theorem 1.0.3]). We continue to denote by $f\colon X \to V$ a fibration in $F^{3,q}_X$, by $\widetilde{f}\colon \widetilde{X} \to \widetilde{V}$ a non-singular representative of f, any by $h\colon Y \to W$ the fibration induced by the irreducible component $Z' \stackrel{\text{def}}{=} F\big(\mathrm{id_X}, Z \stackrel{\text{def}}{=} \widetilde{f}^* \operatorname{Pic}^0(\widetilde{V})\big)$ in $V^1(\omega_Y)_0$. In particular by the proof of Theorem 16 we have an equality $Z' = \widetilde{h}^* \operatorname{Pic}^0(\widetilde{W})$ where $\widetilde{h}\colon \widetilde{Y} \to \widetilde{W}$ is a non-singular representative of h.

Corollary 26. *If* $\dim W < 3$, *then* $\widetilde{W}$ *is either birational to an isotrivial elliptic surface fibered over a curve of genus* $q(\widetilde{W}) - 1$, *or isomorphic to a smooth projective curve.*

Proof. Suppose that $\dim W = 2$. We are going to prove that $\chi(\omega_{\widetilde{W}}) = 0$, so that $\widetilde{W}$ is birational to an isotrivial elliptic surface of Kodaira dimension one thanks to the classification theory of complex algebraic surfaces (recall that $\widetilde{W}$ is of maximal Albanese dimension with $q(\widetilde{W}) \geq 3$). If by contradiction $\chi(\omega_{\widetilde{W}}) > 0$ so that $V^0(\omega_{\widetilde{W}}) = \operatorname{Pic}^0(\widetilde{W})$, then the irreducible component $Z' = \widetilde{h}^* \operatorname{Pic}^0(\widetilde{W})$ would lie in $V^2(\omega_Y)_0$. Hence by Theorem 6 the component $Z = F^{-1}(\mathrm{id}_Y, Z')$ would be an irreducible component of $V^2(\omega_X)_0$, which is impossible as Z induces a fibration onto V which is a variety of dimension 3. The case $\dim W = 1$ is obvious. $\square$

Proof of Corollary 5. We use the notation of the paragraph before Corollary 26. If X does not carry any irrational pencil of genus $g \geq 2$, then so does Y by Theorem 1. This exclude the case $\dim W = 1$. Suppose now that $\dim W = 2$. By Corollary 26 the variety $\widetilde{W}$ admits an elliptic fibration onto a smooth curve B of genus $g \geq 2$. Hence there exists a rational map $Y \dashrightarrow B$ which extends to a morphism as the Albanese morphism of B is injective (*cf.* also the proof of Proposition 27). Therefore Y would admit an irrational pencil of genus $g \geq 2$ which is impossible. We conclude that $\dim W = 3$, and hence that the assignment $\eta_q\colon F^{3,q}_X \to F^{3,q}_Y$ defined by $u^{-1}_{Y,3,q}\, F\, u_{X,3,q}$ yields a bijection of sets. The fact that η_q preserves the bases of the fibrations up to isomorphism follows as in the proof of Theorem 3. $\square$

7. Higher irrational pencils and K-equivalence

A conjecture of Kawamata predicts that K-equivalent varieties are D-equivalent (*cf.* [Kaw02, Conjecture 1.2] and the first paragraph of the Introduction for the definition of K-equivalence). On the other hand, K-equivalent varieties are trivially birational. In this section we reveal the behavior of the sets $F_X^{k,q}$ under K- and birational equivalence of varieties.

To begin with, we define a *k-irrational pencil* $f\colon X \to V$ as a χ-positive k-irrational pencil without the condition on the positivity of $\chi(\omega_{\widetilde{V}})$. Notice that our definition of a k-irrational pencil differs from that given in [Cat91]. For every pair of integers $q \geq k \geq 1$ we denote by $P_X^{k,q}$ the set of ismorphism classes of k-irrational pencils $f\colon X \to V$ such that $\dim V = k$ and $q(V) = q$. Note that $P_X^{k,q} \supset F_X^{k,q}$.

Proposition 27. *Let X and Y be smooth projective birational varieties. Then for every pair of integers $q \geq k \geq 1$ there exists a bijection of sets $\tau_{k,q}\colon P_X^{k,q} \to P_Y^{k,q}$ preserving the bases of the fibrations up to isomorphism.*

Proof. Let $\varphi\colon Y \dashrightarrow X$ be a birational map, and let $f\colon X \to V$ be a k-irrational pencil in $P_X^{k,q}$. Consider the composition $h = f \circ \varphi\colon Y \dashrightarrow V$. As V admits a finite morphism to $\mathrm{Alb}(\widetilde{V})$, and the composition $Y \dashrightarrow V \to \mathrm{Alb}(\widetilde{V})$ extends to a morphism, we get that h is itself a morphism. The assignment $\tau_{k,q}(f) = h$ defines the wanted bijection. □

Remark 28. Given the conclusions of Theorem 1, Theorem 3, and Proposition 27, we speculate that if X and Y are D-equivalent varieties, then for every pair of integers $q \geq k \geq 3$ there are bijections of sets $F_X^{k,q} \to F_Y^{k,q}$ preserving the bases of the fibrations.

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