DERIVED INVARIANTS OF IRREGULAR VARIETIES AND HOCHSCHILD HOMOLOGY

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Abstract. We study the behavior of cohomological support loci of the canonical bundle under derived equivalence of smooth projective varieties. This is achieved by investigating the derived invariance of a generalized version of Hochschild homology. Furthermore, using techniques coming from birational geometry, we establish the derived invariance of the Albanese dimension for varieties having non-negative Kodaira dimension. We apply our machinery to study the derived invariance of the holomorphic Euler characteristic and of certain Hodge numbers for special classes of varieties. Further applications concern the behavior of particular types of fibrations under derived equivalence.

1. Introduction

It is now well known that derived equivalent varieties share quite a few invariants. For instance: the dimension, the Kodaira dimension, the numerical dimension and the canonical ring are examples of derived invariants. In the paper [PS], by describing the behavior under derived equivalence of the Picard variety, Popa and Schnell establish the derived invariance of the number of linearly independent holomorphic one-forms. In this paper we study the behavior under derived equivalence of other fundamental objects in the geometry of *irregular* varieties, i.e. those with positive *irregularity* $q(X) := h^0(X, \Omega^1_X)$, such as the cohomological support loci and the Albanese dimension. Applications of our techniques concern the derived invariance of the holomorphic Euler characteristic of varieties with large Albanese dimension and the derived invariance of some of the Hodge numbers of fourfolds again with large Albanese dimension. A further application concerns the behavior of fibrations of derived equivalent threefolds onto irregular varieties. This work is motivated by a well-known conjecture predicting the derived invariance of all Hodge numbers and by a conjecture of Popa (see Conjectures 1.2 and 1.3 and [Po]).

The main tool we use to approach the problems described above is the comparison of the cohomology groups of twists by topologically trivial line bundles of the canonical bundles of the varieties in play. This is achieved by studying a generalized version of Hochschild homology which takes into account an important isomorphism due to Rouquier related to derived autoequivalences (see [Rou] Théorème 4.18). In this way we obtain a theoretical result of independent interest in the study of derived equivalences of smooth projective varieties, which we now present. To begin with, we recall the *Hochschild cohomology and homology* of a smooth projective variety X:

$$HH^*(X) := \bigoplus_k \mathrm{Ext}^k_{X\times X}\big(i_*\mathcal{O}_X, i_*\mathcal{O}_X\big), \quad HH_*(X) := \bigoplus_k \mathrm{Ext}^k_{X\times X}\big(i_*\mathcal{O}_X, i_*\omega_X\big)$$

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where $i : X \hookrightarrow X \times X$ is the diagonal embedding of X. The space $HH^*(X)$ has a structure of ring under composition of morphisms and $HH_*(X)$ is a graded $HH^*(X)$-module with the same operation. Results of Căldăraru and Orlov show that the Hochschild cohomology and homology are derived invariants (see [Cal] Theorem 8.1 and [Or] Theorem 2.1.8). More precisely, if $\Phi : \mathbf{D}(X) \longrightarrow \mathbf{D}(Y)$ is an equivalence of derived categories of smooth projective varieties, then it induces an isomorphism of rings $HH^*(X) \cong HH^*(Y)$ and an isomorphism of graded modules $HH_*(X) \cong HH_*(Y)$ compatible with the isomorphism $HH^*(X) \cong HH^*(Y)$. We now present the generalization of Hochschild homology mentioned above. For a triple $(\varphi, L, m) \in \mathrm{Aut}^0(X) \times \mathrm{Pic}^0(X) \times \mathbf{Z}$, we define the graded $HH^*(X)$-module

$$HH_*(X, \varphi, L, m) := \bigoplus_k \mathrm{Ext}^k_{X\times X}\big(i_*\mathcal{O}_X, (1,\varphi)_*(\omega_X^{\otimes m} \otimes L)\big)$$

with module structure given by composition of morphisms. We think of these spaces as a "twisted" version of the Hochschild homology of X. Lastly, we recall that a derived equivalence $\mathbf{D}(X) \cong \mathbf{D}(Y)$ induces an isomorphism of algebraic groups, called *Rouquier's isomorphism*

$$F : \mathrm{Aut}^0(X) \times \mathrm{Pic}^0(X) \longrightarrow \mathrm{Aut}^0(Y) \times \mathrm{Pic}^0(Y). \tag{1}$$

(An explicit description of F is given in (3) (see [Rou] Théorème 4.18, [Hu] Proposition 9.45 and [Ros] Theorem 3.1; *cf.* [PS] footnote at p. 531).) The following theorem describes the behavior of the twisted Hochschild homology under derived equivalence. Its proof follows the general strategy of the proofs of Orlov and Căldăraru, but further technicalities appear due to the possible presence of non-trivial automorphisms of X and Y; see §2 for its proof.

Theorem 1.1. *Let $\Phi : \mathbf{D}(X) \longrightarrow \mathbf{D}(Y)$ be an equivalence of derived categories of smooth projective varieties defined over an algebraically closed field and let $m \in \mathbf{Z}$. If $F(\varphi, L) = (\psi, M)$ (where F is the Rouquier isomorphism), then Φ induces an isomorphism of graded modules*

$$HH_*(X, \varphi, L, m) \cong HH_*(Y, \psi, M, m)$$

compatible with the isomorphism $HH^*(X) \cong HH^*(Y)$.

We now move our attention to the main application of Theorem 1.1, namely the behavior of *cohomological support loci* under derived equivalence. These loci are defined as

$$V^k(\omega_X) := \{L \in \mathrm{Pic}^0(X) \mid h^k(X, \omega_X \otimes L) > 0\}$$

where X is a smooth projective variety and $k \geq 0$ is an integer. From here on we work over the field of the complex numbers. The $V^k(\omega_X)$'s have been studied for instance in [GL1], [GL2], [EL], [A], [Ha], [PP2]. They are one of the most important tools in the birational study of irregular varieties; roughly speaking, they control the geometry of the Albanese map and the fibrations onto lower dimensional irregular varieties. The following conjecture, and its weaker variant, predicts the behavior of cohomological support loci under derived equivalence. As a matter of notation, we denote by $V^k(\omega_X)_0$ the union of the irreducible components of $V^k(\omega_X)$ passing through the origin.

Conjecture 1.2 ([Po] Conjecture 1.2)**.** *If X and Y are smooth projective derived equivalent varieties, then*

$$V^k(\omega_X) \cong V^k(\omega_Y) \quad \textit{for all} \quad k \geq 0.$$

Conjecture 1.3 ([Po] Variant 1.3)**.** *Under the assumptions of Conjecture 1.2, there exist isomorphisms*

$$V^k(\omega_X)_0 \cong V^k(\omega_Y)_0 \quad \textit{for all} \quad k \geq 0.$$

It is important to emphasize that for all the applications we are interested in (e.g. invariance of the Albanese dimension, invariance of the holomorphic Euler characteristic, invariance of Hodge numbers) it is in fact enough to verify Conjecture 1.3. We also remark that Conjecture 1.2 holds for varieties of general type since the cohomological support loci are birational invariants, while derived equivalent varieties of general type are birational by [Ka2] Theorem 1.4. Moreover, in [Po] §2 it has been shown that Conjecture 1.2 holds for surfaces as well.

In §3 we try to attack the above conjectures for varieties of arbitrary dimension. To begin with, we show that Theorem 1.1 implies the derived invariance of $V^0(\omega_X)$ (see Proposition 3.1). On the other hand, due to the possible presence of non-trivial automorphisms, the study of the derived invariance of the higher cohomological support loci is more involved. Nonetheless, by using a version of the Hochschild-Kostant-Rosenberg isomorphism and Brion's structural results on the actions of non-affine groups on smooth varieties, we are able to show the derived invariance of $V^1(\omega_X)_0$ (see Corollary 3.4). The next theorem summarizes the main results on the derived invariance of these loci.

Theorem 1.4. *Let X and Y be smooth projective derived equivalent varieties. Then the Rouquier isomorphism induces isomorphisms of algebraic sets*

(i). $V^0(\omega_X) \cong V^0(\omega_Y)$.

(ii). $V^0(\omega_X) \cap V^1(\omega_X) \cong V^0(\omega_Y) \cap V^1(\omega_Y)$.

(iii). $V^1(\omega_X)_0 \cong V^1(\omega_Y)_0$.

We note that (i) also holds if we consider arbitrary powers of the canonical bundle (see Proposition 3.1). We point out also that cases in which the Rouquier isomorphism induces the full isomorphism $V^1(\omega_X) \cong V^1(\omega_Y)$ occur for instance when either X is of maximal Albanese dimension (see Corollary 5.2), or when the neutral component of the automorphism group, $\mathrm{Aut}^0(X)$, is affine (see Remark 3.6); Theorem 1.4 is proved in §3.

Next we study Conjectures 1.2 and 1.3 for varieties of dimension three. In the process we recover Conjecture 1.2 in dimension two as well making the isomorphisms on cohomological support loci explicit. In the following theorem we collect all results concerning the behavior of cohomological support loci of derived equivalent threefolds. We denote by $\mathrm{alb}_X : X \longrightarrow \mathrm{Alb}(X)$ the Albanese map of X and we say that X is of *maximal Albanese dimension* if $\dim \mathrm{alb}_X(X) = \dim X$, i.e. alb_X is generically finite onto its image.

Theorem 1.5. *Let X and Y be smooth projective irregular derived equivalent threefolds. Then*

(i). *Conjecture 1.3 holds.*

(ii). *Conjecture 1.2 holds if one of the following hypotheses is satisfied*

(a) *X is of maximal Albanese dimension.*

(b) $V^k(\omega_X) = \mathrm{Pic}^0(X)$ *for some* $k \geq 0$ *(for instance, by* [PP2] *Theorem E,* $V^0(\omega_X) = \mathrm{Pic}^0(X)$ *whenever* $\mathrm{alb}_X(X)$ *is not fibered in sub-tori and* $V^0(\omega_X) \neq \emptyset$*).*

(c) $\mathrm{Aut}^0(X)$ *is affine (for instance, by a theorem of Nishi,* [Ma] *Theorem 2, this again happens when* $\mathrm{alb}_X(X)$ *is not fibered in sub-tori).*

(iii). *If* $q(X) \geq 2$*, then* $\dim V^k(\omega_X) = \dim V^k(\omega_Y)$ *for all* $k \geq 0$.

Point (iii) brings evidence to a further variant of Conjecture 1.2 predicting the invariance of the dimensions of cohomological support loci (*cf.* [Po] Variant 1.4); partial results for the case $q(X) = 1$ are described in Remark 6.10. Since the proofs of Theorems 1.4 and 1.5 extend to analogous results regarding cohomological support loci of bundles of holomorphic p-forms, when possible we will prove them in such generality. Please refer to Theorem 4.2 and §6 for the proof of Theorem 1.5.

Finally, we move our attention to applications of Theorems 1.4 and 1.5. The first regards the behavior of the Albanese dimension, $\dim \mathrm{alb}_X(X)$, under derived equivalence. According to Conjecture 1.3, the Albanese dimension is expected to be preserved under derived equivalence as it can be read off from the dimensions of the $V^k(\omega_X)_0$'s (*cf.* (5)), which is the case in dimension three thanks to Theorem 1.5. In higher dimension we establish this invariance for varieties having non-negative Kodaira dimension $\kappa(X)$, by using the derived invariance of the irregularity and an extension of a result due to Chen-Hacon-Pardini ([HP] Proposition 2.1, [CH2] Corollary 3.6) on the study of the geometry of the Albanese map via the Iitaka fibration; see §5.

Theorem 1.6. *Let* X *and* Y *be smooth projective derived equivalent varieties. If* $\dim X \leq 3$*, or if* $\dim X > 3$ *and* $\kappa(X) \geq 0$*, then*

$$\dim \mathrm{alb}_X(X) = \dim \mathrm{alb}_Y(Y).$$

The second application concerns the holomorphic Euler characteristic. This is expected to be the same for arbitrary derived equivalent smooth projective varieties since the Hodge numbers are expected to be preserved (which is known to hold in dimension up to three; *cf.* [PS] Corollary C). We deduce this for varieties of large Albanese dimension as a consequence of the previous results and generic vanishing.

Corollary 1.7. *Let* X *and* Y *be smooth projective derived equivalent varieties. If* $\dim \mathrm{alb}_X(X) = \dim X$*, or if* $\dim \mathrm{alb}_X(X) = \dim X - 1$ *and* $\kappa(X) \geq 0$*, then*

$$\chi(\omega_X) = \chi(\omega_Y).$$

An immediate consequence is the derived invariance of two of the Hodge numbers for fourfolds satisfying the hypotheses of Corollary 1.7.

Corollary 1.8. *Let* X *and* Y *be smooth projective derived equivalent fourfolds. If* $\dim \mathrm{alb}_X(X) = 4$*, or if* $\dim \mathrm{alb}_{\mathrm{X}}(X) = 3$ *and* $\kappa(X) \geq 0$*, then*

$$h^{0,2}(X) = h^{0,2}(Y) \quad \text{and} \quad h^{1,3}(X) = h^{1,3}(Y).$$

We remark that in [PS] Corollary 3.4 the authors establish the invariance of $h^{0,2}$ and $h^{1,3}$ under different hypotheses, namely when $\mathrm{Aut}^0(X)$ is not affine (we recall that $h^{0,4}$, $h^{0,3}$, $h^{0,1}$ and $h^{1,2}$ are always known to be invariant, *cf.* [PS]). Corollaries 1.7 and 1.8 are proved in §7.

We now present our last application, in a direction which is one of the main motivations for Conjectures 1.2 and 1.3 as explained in [Po]. From the classification of Fourier-Mukai equivalences for surfaces ([Ka2], [BM]), it is known that if X admits a fibration $f : X \longrightarrow C$ onto a smooth curve of genus ≥ 2, then any of its Fourier-Mukai partners admits a fibration onto the same curve. Here we use our analysis, and a theorem of Green-Lazarsfeld regarding the properties of positive-dimensional irreducible components of the cohomological support loci, to investigate the behavior of fibrations of derived equivalent threefolds onto irregular varieties. Recall that a smooth variety X is called of *Albanese general type* if alb_X is non-surjective and generically finite onto its image. The proof of the next corollary is contained in Proposition 7.3 and Remark 7.4.

Corollary 1.9. *Let X and Y be smooth projective derived equivalent threefolds. There exists a morphism $f : X \longrightarrow W$ with connected fibers onto a normal variety W of dimension ≤ 2 such that any smooth model of W is of Albanese general type if and only if Y has a fibration of the same type. Moreover, there exists a morphism $f : X \longrightarrow C$ with connected fibers onto a smooth curve of genus ≥ 2 if and only if there exists a morphism $h : Y \longrightarrow D$ with connected fibers onto a smooth curve of genus ≥ 2.*

To conclude we remark that while the approach in this paper relies in part on techniques of [PS], the key new ingredient is their interaction with the twisted Hochschild homology, introduced and studied here. We are hopeful that this general method will find further applications in the future.

2. Derived invariance of the twisted Hochschild homology

We aim to prove Theorem 1.1. Its proof is based on a technical lemma extending previous computations carried out by Căldăraru and Orlov (*cf.* [Cal] Proposition 8.1 and [Or] isomorphism (10)).

Let X and Y be smooth projective varieties defined over an algebraically closed field K, and let p and q be the projections from $X \times Y$ onto the first and second factor respectively. We denote by $\mathbf{D}(X) := \mathbf{D}^b(\mathcal{C}oh(X))$ the bounded derived category of coherent sheaves on a smooth projective variety X. When there is no possibility of ambiguity, we use the same symbol to denote a functor and its associated derived functor. An object $\mathcal{E}$ in $\mathbf{D}(X \times Y)$ defines *Fourier-Mukai functors* with *kernel* $\mathcal{E}$ as:

$$\Phi_{\mathcal{E}} : \mathbf{D}(X) \longrightarrow \mathbf{D}(Y), \quad \mathcal{F} \mapsto q_*(p^*\mathcal{F} \otimes \mathcal{E}),$$

$$\Psi_{\mathcal{E}} : \mathbf{D}(Y) \longrightarrow \mathbf{D}(X), \quad \mathcal{G} \mapsto p_*(q^*\mathcal{G} \otimes \mathcal{E}).$$

We say that X and Y are *derived equivalent* if there exists a K-linear exact equivalence of triangulated categories $\Phi : \mathbf{D}(X) \longrightarrow \mathbf{D}(Y)$. By a well-known result of Orlov, any such equivalence is of Fourier-Mukai type, i.e. there exists an object $\mathcal{E}$ in $\mathbf{D}(X \times Y)$ such that $\Phi \cong \Phi_{\mathcal{E}}$. Furthermore, the object $\mathcal{E}$ is unique up to isomorphism.

We recall that an equivalence $\Phi_{\mathcal{E}} : \mathbf{D}(X) \longrightarrow \mathbf{D}(Y)$ induces an equivalence

$$\Phi_{\mathcal{E}^* \boxtimes \mathcal{E}} : \mathbf{D}(X \times X) \longrightarrow \mathbf{D}(Y \times Y)$$

with kernel

$$\mathcal{E}^* \boxtimes \mathcal{E} \stackrel{\text{def}}{=} p_{13}^*\mathcal{E}^* \otimes p_{24}^*\mathcal{E},$$

where $\mathcal{E}^* \stackrel{\text{def}}{=} \mathbf{R}\mathcal{H}om(\mathcal{E}, \mathcal{O}_{X\times Y}) \otimes p^*\omega_X[\dim X]$ and p_{rs} is the projection from $X\times X\times Y\times Y$ onto the (r,s)-factor (*cf.* [Or] Proposition 2.1.7). Moreover, for any automorphisms $\varphi \in \mathrm{Aut}^0(X)$ and $\psi \in \mathrm{Aut}^0(Y)$ (here the supscript 0 denotes the neutral component of the corresponding group) we define the embeddings $(1,\varphi) : X \hookrightarrow X\times X$, $x \mapsto (x, \varphi(x))$ and $(1,\psi) : Y \hookrightarrow Y\times Y$, $y \mapsto (y, \psi(y))$. Finally, we denote by i and j the diagonal embeddings of X and Y respectively.

Lemma 2.1. *Let X and Y be smooth projective varieties defined over an algebraically closed field and let $\Phi_{\mathcal{E}} : \mathbf{D}(X) \longrightarrow \mathbf{D}(Y)$ be an equivalence. Denote by F the induced Rouquier isomorphism (*cf. (1)*) and let $m \in \mathbf{Z}$. If $F(\varphi, L) = (\psi, M)$, then*

$$\Phi_{\mathcal{E}^*\boxtimes\mathcal{E}}\big((1,\varphi)_*(\omega_X^{\otimes m}\otimes L)\big) \cong (1,\psi)_*(\omega_Y^{\otimes m}\otimes M).$$

Proof. We denote by t_r (*resp.* t_{rs}) the projection from $Y\times X\times Y$ onto the r-th factor (*resp.* (r,s)-th factor). Moreover, we define the morphism $\lambda : Y\times X\times Y \longrightarrow X\times X\times Y\times Y$ as $(y_1, x, y_2) \mapsto (x, \varphi(x), y_1, y_2)$, and we look at the fiber product diagram

$$\begin{array}{ccc} Y\times X\times Y & \xrightarrow{\lambda} & X\times X\times Y\times Y \\ \downarrow{\scriptstyle t_2} & & \downarrow{\scriptstyle p_{12}} \\ X & \xrightarrow{(1,\varphi)} & X\times X \end{array}$$

so that, by base change and the projection formula, we get

$$\begin{aligned} \Phi_{\mathcal{E}^*\boxtimes\mathcal{E}}\big((1,\varphi)_*(\omega_X^{\otimes m}\otimes L)\big) &\cong p_{34*}\big(p_{12}^*(1,\varphi)_*(\omega_X^{\otimes m}\otimes L)\otimes\big(\mathcal{E}^*\boxtimes\mathcal{E}\big)\big) \\ &\cong p_{34*}\big(\lambda_* t_2^*(\omega_X^{\otimes m}\otimes L)\otimes p_{13}^*\mathcal{E}^*\otimes p_{24}^*\mathcal{E}\big) \\ &\cong p_{34*}\lambda_*\big(t_2^*(\omega_X^{\otimes m}\otimes L)\otimes \lambda^* p_{13}^*\mathcal{E}^*\otimes \lambda^* p_{24}^*\mathcal{E}\big) \\ &\cong t_{13*}\big(t_2^*(\omega_X^{\otimes m}\otimes L)\otimes t_{21}^*\mathcal{E}^*\otimes t_{23}^*(\varphi\times 1)^*\mathcal{E}\big). \end{aligned} \tag{2}$$

By [Or] p. 535, the equivalence $\Phi_{\mathcal{E}}$ induces an isomorphism $\mathcal{E}\otimes p^*\omega_X \cong \mathcal{E}\otimes q^*\omega_Y$. Moreover, by [PS] Lemma 3.1, the condition $F(\varphi, L) = (\psi, M)$ is equivalent to an isomorphism of objects in $\mathbf{D}(X\times Y)$

$$(\varphi\times 1)^*\mathcal{E}\otimes p^*L \cong (1\times\psi)_*\mathcal{E}\otimes q^*M. \tag{3}$$

Therefore we get an isomorphism of objects $p^*(\omega_X^{\otimes m}\otimes L)\otimes(\varphi\times 1)^*\mathcal{E} \cong q^*(\omega_Y^{\otimes m}\otimes M)\otimes(1\times\psi)_*\mathcal{E}$, and by pulling it back via $t_{23} : Y\times X\times Y \longrightarrow X\times Y$, we finally obtain

$$t_2^*(\omega_X^{\otimes m}\otimes L)\otimes t_{23}^*(\varphi\times 1)^*\mathcal{E} \cong t_3^*(\omega_Y^{\otimes m}\otimes M)\otimes t_{23}^*(1\times\psi)_*\mathcal{E}. \tag{4}$$

At this point we rewrite the morphism $t_3 : Y\times X\times Y \longrightarrow Y$ as $t_3 = \sigma_2\circ t_{13}$ where $\sigma_2 : Y\times Y \longrightarrow Y$ is the projection onto the second factor. Moreover, we denote by $\rho : Y\times X \longrightarrow X\times Y$ the inversion morphism $(y,x)\mapsto(x,y)$. Then by (2) and (4) we obtain

$$\begin{aligned} \Phi_{\mathcal{E}^*\boxtimes\mathcal{E}}\big((1,\varphi)_*(\omega_X^{\otimes m}\otimes L)\big) &\cong t_{13*}\big(t_3^*(\omega_Y^{\otimes m}\otimes M)\otimes t_{21}^*\mathcal{E}^*\otimes t_{23}^*(1\times\psi)_*\mathcal{E}\big) \\ &\cong t_{13*}\big(t_{13}^*\sigma_2^*(\omega_Y^{\otimes m}\otimes M)\otimes t_{21}^*\mathcal{E}^*\otimes t_{23}^*(1\times\psi)_*\mathcal{E}\big) \\ &\cong \sigma_2^*(\omega_Y^{\otimes m}\otimes M)\otimes t_{13*}\big(t_{21}^*\mathcal{E}^*\otimes t_{23}^*(1\times\psi)_*\mathcal{E}\big) \\ &\cong \sigma_2^*(\omega_Y^{\otimes m}\otimes M)\otimes t_{13*}\big(t_{12}^*\rho^*\mathcal{E}^*\otimes t_{23}^*(1\times\psi)_*\mathcal{E}\big). \end{aligned}$$

Finally, by [Or] Proposition 2.1.2, we note that the object $t_{13*}\Big(t_{12}^*\rho^*\mathcal{E}^*\otimes t_{23}^*(1\times\psi)_*\mathcal{E}\Big)$ in $\mathbf{D}(Y\times Y)$ is the kernel of the composition

$$\Phi_{(1\times\psi)_*\mathcal{E}}\circ\Phi_{\rho^*\mathcal{E}^*}\cong\psi_*\circ\Phi_{\mathcal{E}}\circ\Psi_{\mathcal{E}^*}\cong\psi_*\circ\mathrm{id}_{\mathbf{D}(Y)}\cong\psi_*,$$

where we used the fact that $\Psi_{\mathcal{E}^*}$ is the right adjoint to $\Phi_{\mathcal{E}}$. On the other hand, since the kernel of the derived functor $\psi_*:\mathbf{D}(Y)\longrightarrow\mathbf{D}(Y)$ is the structure sheaf of the graph of ψ, i.e. $\mathcal{O}_{\Gamma_\psi}\cong(1,\psi)_*\mathcal{O}_Y$ (*cf.* [Hu] Example 5.4), we have an isomorphism

$$t_{13*}\big(t_{12}^*\rho^*\mathcal{E}^*\otimes t_{23}^*(1\times\psi)_*\mathcal{E}\big)\cong(1,\psi)_*\mathcal{O}_Y$$

as the kernel of an equivalence is unique up to isomorphism. To recap

$$\begin{aligned}\Phi_{\mathcal{E}^*\boxtimes\mathcal{E}}\big((1,\varphi)_*(\omega_X^{\otimes m}\otimes L)\big) &\cong \sigma_2^*(\omega_Y^{\otimes m}\otimes M)\otimes(1,\psi)_*\mathcal{O}_Y\\ &\cong (1,\psi)_*\big((1,\psi)^*\sigma_2^*(\omega_Y^{\otimes m}\otimes M)\big)\\ &\cong (1,\psi)_*\big(\psi^*(\omega_Y^{\otimes m}\otimes M)\big)\\ &\cong (1,\psi)_*(\omega_Y^{\otimes m}\otimes M).\end{aligned}$$

The last isomorphism follows as the action of $\mathrm{Aut}^0(X)$ on $\mathrm{Pic}^0(X)$ is trivial (*cf.* [PS] footnote at p. 531). □

Proof of Theorem 1.1. Let $\mathcal{E}$ be the kernel of the equivalence Φ so that $\Phi\cong\Phi_{\mathcal{E}}$. By Lemma 2.1, the equivalence $\Phi_{\mathcal{E}^*\boxtimes\mathcal{E}}$ induces isomorphisms between the graded components of $HH_*(X,\varphi,L,m)$ and $HH_*(Y,\psi,M,m)$ as follows:

$$\begin{aligned}\mathrm{Ext}^k_{X\times X}\big(i_*\mathcal{O}_X,(1,\varphi)_*(\omega_X^{\otimes m}\otimes L)\big) &\cong \mathrm{Ext}^k_{Y\times Y}\Big(\Phi_{\mathcal{E}^*\boxtimes\mathcal{E}}(i_*\mathcal{O}_X),\Phi_{\mathcal{E}^*\boxtimes\mathcal{E}}\big((1,\varphi)_*(\omega_X^{\otimes m}\otimes L)\big)\Big)\\ &\cong \mathrm{Ext}^k_{Y\times Y}\big(j_*\mathcal{O}_Y,(1,\psi)_*(\omega_Y^{\otimes m}\otimes M)\big).\end{aligned}$$

Moreover, since $\Phi_{\mathcal{E}^*\boxtimes\mathcal{E}}$ is a functor, it follows that it induces an isomorphism of graded modules. □

Theorem 1.1 will be often used in the following weaker form:

Corollary 2.2. *Let X and Y be smooth projective derived equivalent varieties defined over an algebraically closed field of characteristic zero. If $F(1,L)=(1,M)$, then for any integers m and $k\geq 0$ there exist isomorphisms*

$$\bigoplus_{q=0}^{k}H^{k-q}\big(X,\Omega_X^{\dim X-q}\otimes\omega_X^{\otimes m}\otimes L\big)\cong\bigoplus_{q=0}^{k}H^{k-q}\big(Y,\Omega_Y^{\dim Y-q}\otimes\omega_Y^{\otimes m}\otimes M\big).$$

Proof. The corollary is a consequence of Theorem 1.1 and of the general fact that the groups $\mathrm{Ext}^k_{X\times X}\big(i_*\mathcal{O}_X,i_*\mathcal{F}\big)$ decompose as $\bigoplus_{q=0}^k H^{k-q}\big(X,\Omega_X^{\dim X-q}\otimes\omega_X^{-1}\otimes\mathcal{F}\big)$ for any coherent sheaf $\mathcal{F}$ and for all $k\geq 0$ (*cf.* [Ye] Corollary 4.7 and [Sw] Corollary 2.6). □

3. Behavior of cohomological support loci under derived equivalence

In this section we study the behavior of cohomological support loci under derived equivalence. Applications of our analysis will be provided in §7. From now on we work over the field of the complex numbers.

3.1. **Cohomological support loci.** Let X be a complex smooth projective irregular variety. Given a coherent sheaf $\mathcal{F}$ on X, we define the *cohomological support loci of* $\mathcal{F}$ as

$$V^k_r(\mathcal{F}) := \{L \in \mathrm{Pic}^0(X) \mid h^k(X, \mathcal{F} \otimes L) \geq r\}$$

for all integers $k \geq 0$ and $r \geq 1$. By semicontinuity these loci are algebraic closed subsets in $\mathrm{Pic}^0(X)$. We set $V^k(\mathcal{F}) := V^k_1(\mathcal{F})$ and we denote by $V^k_r(\mathcal{F})_0$ the union of all the irreducible components of $V^k_r(\mathcal{F})$ passing through the origin of $\mathrm{Pic}^0(X)$. By following the work of Pareschi and Popa [PP2], we say that $\mathcal{F}$ is a *GV-sheaf* if

$$\mathrm{codim}_{\,\mathrm{Pic}^0(X)} V^k(\mathcal{F}) \geq k \quad \text{for all} \quad k > 0.$$

In the following we study the behavior of the loci $V^k_r(\mathcal{F})$ under equivalence of derived categories where $\mathcal{F} = \omega_X, \omega_X^{\otimes m}, \Omega^p_X \otimes \omega_X^{\otimes m}$ with $m, p \in \mathbf{Z}$ and $p \geq 0$. We recall that the cohomological support loci $V^k(\omega_X)$ associated to the canonical bundle are invariant under birational modifications for all $k \geq 0$. Furthermore, they detect the Albanese dimension of X, namely the dimension of the image of the Albanese map $\mathrm{alb}_X : X \longrightarrow \mathrm{Alb}(X)$, thanks to the following formula (*cf.* [Po] p. 7) deduced from results of [GL1] and [LP]:

$$\dim \mathrm{alb}_X(X) = \min_{k=0,\dots,\dim X}\{\dim X - k + \mathrm{codim}\, V^k(\omega_X)_0\}. \tag{5}$$

Finally, we point out that if $\dim \mathrm{alb}_X(X) = \dim X - k$, then there are inclusions

$$V^k(\omega_X) \supset V^{k+1}(\omega_X) \supset \ldots \supset V^{\dim X}(\omega_X) = \{\mathcal{O}_X\} \tag{6}$$

(*cf.* [PP2] Proposition 3.14 and [GL1] Theorem 1, or [EL] Lemma 1.8 for the case $k = 0$).

3.2. **Derived invariance of the zero-th cohomological support locus.** The following proposition proves and extends Theorem 1.4 (i).

Proposition 3.1. *Let X and Y be smooth projective varieties and let $\Phi_{\mathcal{E}} : \mathbf{D}(X) \longrightarrow \mathbf{D}(Y)$ be an equivalence. Denote by F the induced Rouquier isomorphism and let m and r be integers such that $r \geq 1$. If $L \in V^0_r(\omega_X^{\otimes m})$ and $F(1, L) = (\psi, M)$, then $\psi = 1$ and $M \in V^0_r(\omega_Y^{\otimes m})$. Moreover, F induces an isomorphism of algebraic sets*

$$V^0_r(\omega_X^{\otimes m}) \cong V^0_r(\omega_Y^{\otimes m}).$$

Proof. Let L be a line bundle in $V^0_r(\omega_X^{\otimes m})$ and suppose that $F(1, L) = (\psi, M)$ for some $\psi \in \mathrm{Aut}^0(Y)$ and $M \in \mathrm{Pic}^0(Y)$. By Theorem 1.1 and the adjunction formula we have

$$\begin{aligned} r \leq h^0(X, \omega_X^{\otimes m} \otimes L) &= \dim \mathrm{Hom}_{X\times X}\big(i_*\mathcal{O}_X, i_*(\omega_X^{\otimes m} \otimes L)\big) \\ &= \dim \mathrm{Hom}_{Y\times Y}\big(j_*\mathcal{O}_Y, (1,\psi)_*(\omega_Y^{\otimes m} \otimes M)\big) \\ &= \dim \mathrm{Hom}_Y\big((1,\psi)^* j_*\mathcal{O}_Y, \omega_Y^{\otimes m} \otimes M\big). \end{aligned}$$

Since $(1, \psi)^* j_*\mathcal{O}_Y$ is supported on the locus of fixed points of ψ (which is of codimension ≥ 1 if $\psi \neq 1$), and since there are no non-zero morphisms from a torsion sheaf to a locally free sheaf, we must have that ψ is the identity automorphism on Y and consequently that $M \in V^0_r(\omega_Y^{\otimes m})$. Therefore we have an inclusion of algebraic sets $F(1, V^0_r(\omega_X^{\otimes m})) \subset (1, V^0_r(\omega_Y^{\otimes m}))$.

In order to show the reverse inclusion, we consider the right adjoint $\Psi_{\mathcal{E}^*}$ to $\Phi_{\mathcal{E}}$ so that $\Psi_{\mathcal{E}^*} \circ \Phi_{\mathcal{E}} \cong 1_{\mathbf{D}(X)}$ and $\Phi_{\mathcal{E}} \circ \Psi_{\mathcal{E}^*} \cong 1_{\mathbf{D}(Y)}$. An easy computation shows that if F' is the Rouquier isomorphism induced by $\Psi_{\mathcal{E}^*}$, then $F' = F^{-1}$ (*cf.* [Lo] Lemma 2.1.9). Hence, by repeating the previous argument, we get an inclusion $F^{-1}(1, V^0_r(\omega_Y^{\otimes m})) \subset (1, V^0_r(\omega_X^{\otimes m}))$ inducing the wanted isomorphism. □

3.3. **Behavior of higher cohomological support loci under derived equivalence.** In this section we establish the isomorphism $V^1(\omega_X)_0 \cong V^1(\omega_Y)_0$ of Theorem 1.4. It turns out that, by using the same techniques (i.e. invariance of twisted Hochschild homology and Brion's results on actions of non-affine groups), one can show a more general result involving cohomological support loci associated to bundles of holomorphic p-forms, which we now present.

Theorem 3.2. *Let X and Y be smooth projective varieties of dimension d and let $\Phi_{\mathcal{E}} : \mathbf{D}(X) \longrightarrow \mathbf{D}(Y)$ be an equivalence. Denote by F be the induced Rouquier isomorphism and let m be an integer. If $L \in \bigcup_{p,q\geq 0} V^p\big(\Omega^q_X \otimes \omega_X^{\otimes m}\big)_0$ and $F(1, L) = (\psi, M)$, then $\psi = 1$ and $M \in \bigcup_{p,q\geq 0} V^p\big(\Omega^q_Y \otimes \omega_Y^{\otimes m}\big)_0$. Moreover, F induces isomorphisms of algebraic sets*

$$\bigcup_{q=0}^{k} V^{k-q}\big(\Omega^{d-q}_X \otimes \omega_X^{\otimes m}\big)_0 \cong \bigcup_{q=0}^{k} V^{k-q}\big(\Omega^{d-q}_Y \otimes \omega_Y^{\otimes m}\big)_0 \quad \textit{for any} \quad k \geq 0.$$

Proof. To begin with, we recall some notation and facts from [PS] Theorem A. Let $\alpha : \mathrm{Pic}^0(Y) \longrightarrow \mathrm{Aut}^0(X)$ and $\beta : \mathrm{Pic}^0(X) \longrightarrow \mathrm{Aut}^0(Y)$ be morphisms defined as

$$\alpha(M) = \mathrm{pr}_1(F^{-1}(1, M)) \quad \text{and} \quad \beta(L) = \mathrm{pr}_1(F(1, L))$$

(here pr_1 denotes the projection onto the first factor from the product $\mathrm{Aut}^0(\cdot) \times \mathrm{Pic}^0(\cdot)$). We denote by A and B the images of α and β respectively. We recall that A and B are isogenous abelian varieties.

We first consider the case when A is trivial. Then we have $F(1, \mathrm{Pic}^0(X)) = (1, \mathrm{Pic}^0(Y))$, and by Corollary 2.2 we get inclusions

$$F\big(1, \bigcup_q V^{k-q}\big(\Omega^{d-q}_X \otimes \omega_X^{\otimes m}\big)\big) \subset \big(1, \bigcup_q V^{k-q}\big(\Omega^{d-q}_Y \otimes \omega_Y^{\otimes m}\big)\big) \quad \text{for any} \quad k \geq 0.$$

In order to prove the reverse inclusions, we note that B is trivial as well and that the Rouquier isomorphism induced by the right adjoint $\Psi_{\mathcal{E}^*}$ to $\Phi_{\mathcal{E}}$ is F^{-1}. Therefore, a second application of Corollary 2.2 yields inclusions

$$F^{-1}\big(1, \bigcup_q V^{k-q}\big(\Omega^{d-q}_Y \otimes \omega_Y^{\otimes m}\big)\big) \subset \big(1, \bigcup_q V^{k-q}\big(\Omega^{d-q}_X \otimes \omega_X^{\otimes m}\big)\big) \quad \text{for any} \quad k \geq 0,$$

concluding the proof of this case.

We suppose now that both A and B are non-trivial. We first show the following

Claim 3.3. *There are inclusions $F\big(1, \bigcup_q V^{k-q}\big(\Omega^{d-q}_X \otimes \omega_X^{\otimes m}\big)_0\big) \subset \big(1, \mathrm{Pic}^0(Y)\big)$ for all integers m and $k \geq 0$.*

Proof. Brion's results on actions of non-affine algebraic groups imply that X is an étale locally trivial fibration $\xi : X \longrightarrow A/H$ where H is a finite subgroup of A (the proof of this fact is analogous to the one of [PS] Lemma 2.4; see also [Br]). Let Z be the smooth and connected fiber of ξ over the origin of A/H. Via base change we get a commutative diagram

$$\begin{array}{ccc} A\times Z & \xrightarrow{g} & X \\ \downarrow & & \downarrow \xi \\ A & \longrightarrow & A/H \end{array}$$

where $g(\varphi,z)=\varphi(z)$. Let $(z_0,y_0)\in Z\times Y$ be an arbitrary point and let

$$f=(f_1\times f_2): A\times B\longrightarrow X\times Y$$

be the orbit map $(\varphi,\psi)\mapsto(\varphi(z_0),\psi(y_0))$. In [PS] p. 533 it is shown that

$$L\in(\operatorname{Ker} f_1^*)_0 \quad\Longrightarrow\quad F(1,L)=(1,M) \quad \text{for some} \quad M\in\operatorname{Pic}^0(Y)$$

(here $(\operatorname{Ker} f_1^*)_0$ denotes the neutral component of $\operatorname{Ker} f_1^*$). So it is enough to show the inclusion

$$\bigcup_q V^{k-q}\big(\Omega_X^{d-q}\otimes\omega_X^{\otimes m}\big)_0\subset(\operatorname{Ker} f_1^*)_0 \quad \text{for any} \quad k\geq 0. \tag{7}$$

This is achieved by computing cohomology groups on $A\times Z$ via the étale morphism g and by using the fact that these computations are straightforward on A. Let p_1,p_2 be the projections from the product $A\times Z$ onto the first and second factor respectively. By denoting by $\nu: A\times\{z_0\}\hookrightarrow A\times Z$ the inclusion morphism, we have $g\circ\nu=f_1$. Moreover, via the isomorphism $\operatorname{Pic}^0(A\times Z)\cong\operatorname{Pic}^0(A)\times\operatorname{Pic}^0(Z)$, we obtain $g^*L\cong p_1^*L_1\otimes p_2^*L_2$ where $L_1\in\operatorname{Pic}^0(A)$ and $L_2\in\operatorname{Pic}^0(Z)$. Note also that $f_1^*L\cong\nu^*g^*L\cong L_1$. Finally, for all $L\in\bigcup_q V^{k-q}\big(\Omega_X^{d-q}\otimes\omega_X^{\otimes m}\big)$ there are an inclusions

$$0\neq\bigoplus_{q=0}^{k}H^{k-q}\big(X,\Omega_X^{d-q}\otimes\omega_X^{\otimes m}\otimes L\big)\subset\bigoplus_{q=0}^{k}H^{k-q}\big(A\times Z,\Omega_{A\times Z}^{d-q}\otimes\omega_{A\times Z}^{\otimes m}\otimes g^*L\big) \tag{8}$$

(*cf.* [La] Injectivity Lemma 4.1.14). Therefore, thanks to Künneth's formula, the sum on the right hand side of (8) is non-zero only if $f_1^*L\cong\mathcal{O}_A$, i.e. $L\in\operatorname{Ker} f_1^*$. This shows (7). □

By Claim 3.3 and Corollary 2.2, we obtain that for any $k\geq 0$ the Rouquier isomorphism maps

$$1\times\bigcup_q V^{k-q}\big(\Omega_X^{d-q}\otimes\omega_X^{\otimes m}\big)_0\mapsto 1\times\bigcup_q V^{k-q}\big(\Omega_Y^{d-q}\otimes\omega_Y^{\otimes m}\big)_0.$$

In complete analogy, one can also show that

$$M\in\bigcup_q V^{k-q}\big(\Omega_Y^{d-q}\otimes\omega_Y^{\otimes m}\big)_0 \quad\Longrightarrow\quad F^{-1}(1,M)=(1,L) \text{ for some } L\in\operatorname{Pic}^0(X).$$

This concludes the proof since by Corollary 2.2 F^{-1} maps

$$1\times\bigcup_q V^{k-q}\big(\Omega_Y^{d-q}\otimes\omega_Y^{\otimes m}\big)_0\mapsto 1\times\bigcup_q V^{k-q}\big(\Omega_X^{d-q}\otimes\omega_X^{\otimes m}\big)_0$$

for any $k\geq 0$. □

The following corollaries yield the proof of Theorem 1.4 (iii) and (ii).

Corollary 3.4. *Under the assumptions of Theorem 3.2, the Rouquier isomorphism F induces isomorphisms of algebraic sets*

$$V_r^1(\omega_X)_0\cong V_r^1(\omega_Y)_0 \quad \textit{for any} \quad r\geq 1.$$

Proof. Let $L \in V^1_r(\omega_X)_0$. By Theorem 3.2 we have $F(1, L) = (1, M)$ for some $M \in \mathrm{Pic}^0(Y)$, and by Corollary 2.2 we get an isomorphism

$$H^1(X, \omega_X \otimes L) \oplus H^0(X, \Omega_X^{d-1} \otimes L) \cong H^1(Y, \omega_Y \otimes M) \oplus H^0(Y, \Omega_Y^{d-1} \otimes M).$$

Moreover, by Serre duality and the Hodge linear-conjugate isomorphism, we obtain equalities

$$h^0(X, \Omega_X^{d-1} \otimes L) = h^1(X, \omega_X \otimes L) \quad \text{and} \quad h^0(Y, \Omega_Y^{d-1} \otimes M) = h^1(Y, \omega_Y \otimes M).$$

Hence $h^1(X, \omega_X \otimes L) = h^1(Y, \omega_Y \otimes M) \geq r$ and therefore F induces the wanted isomorphisms as in the proof of Theorem 3.2. □

Corollary 3.5. *Under the assumptions of Theorem 3.2, and for any integers l, m, r, s with $r, s \geq 1$, the Rouquier isomorphism F induces isomorphisms of algebraic sets*

$$V^0_r(\omega_X^{\otimes m}) \cap \Big(\bigcup_q V^{k-q}(\Omega_X^{d-q} \otimes \omega_X^{\otimes l})\Big) \cong V^0_r(\omega_Y^{\otimes m}) \cap \Big(\bigcup_q V^{k-q}(\Omega_Y^{d-q} \otimes \omega_Y^{\otimes l})\Big)$$

$$V^0_r(\omega_X^{\otimes m}) \cap V^1_s(\omega_X) \cong V^0_r(\omega_Y^{\otimes m}) \cap V^1_s(\omega_Y).$$

Proof. In Proposition 3.1 we have seen that if $L \in V^0_r(\omega_X^{\otimes m})$ then $F(1, L) = (1, M)$ for some $M \in V^0_r(\omega_Y^{\otimes m})$. We argue then as in the proofs of Theorem 3.2 and Corollary 3.4. □

Remark 3.6. It is important to note that, whenever $F(1, \mathrm{Pic}^0(X)) = (1, \mathrm{Pic}^0(Y))$, the proofs of Theorem 3.2 and Corollary 3.4 yield full isomorphisms

$$\bigcup_q V^{k-q}(\Omega_X^{d-q} \otimes \omega_X^{\otimes m}) \cong \bigcup_q V^{k-q}(\Omega_Y^{d-q} \otimes \omega_Y^{\otimes m}) \quad \text{for any} \quad k \geq 0$$

$$V^1_r(\omega_X) \cong V^1_r(\omega_Y).$$

By Theorem 3.2 this occurs either if $V^p(\Omega_X^q \otimes \omega_X^{\otimes m}) = \mathrm{Pic}^0(X)$ for some $p, q \geq 0$ and $m \in \mathbf{Z}$, or if $\mathrm{Aut}^0(X)$ is affine (since in this case the abelian variety A in the proof of Theorem 3.2 is trivial).

4. Popa's conjectures in dimension two and three

In this section we aim to prove Theorem 1.5 (i). In other words we show that Conjecture 1.3, predicting the derived invariance of cohomological support loci of type $V^k(\omega_X)_0$, holds in dimension three. The proofs of (ii) and (iii) of the same theorem are postponed in §6 since they use the derived invariance of the Albanese dimension, which will be proved in §5. Before starting with the proof of Theorem 1.5 (i), we make a couple of considerations regarding the case of surfaces.

4.1. **The case of surfaces.** In dimension two Popa proves the derived invariance of the full cohomological support loci $V^k(\omega_X)$ (*cf.* [Po] Theorem 2.1). His proof is based on an explicit computation of cohomological support loci according to the classification of surfaces up to Fourier-Mukai equivalences (*cf.* [BM]). As an application of Proposition 3.1 and Corollary 3.4, we recover this result by making the isomorphisms between cohomological support loci explicit. More precisely, if F is the Rouquier isomorphism induced by an equivalence of derived categories, then $F(1, V^k_r(\omega_X)) = (1, V^k_r(\omega_Y))$ for all integers $k \geq 0$ and $r \geq 1$. Moreover, by using the same techniques, it is possible to show that F induces further isomorphisms $V^1_r(\Omega^1_X) \cong V^1_r(\Omega^1_Y)$ for all $r \geq 1$ (*cf.* [Lo] Theorem 5.1.2 for a detailed analysis).

Example 4.1 (Elliptic surfaces)**.** Let X be an elliptic surface of Kodaira dimension one and of maximal Albanese dimension (i.e. an isotrivial elliptic surface fibered onto a curve of genus ≥ 2). By following [Be2], we recall an invariant attached to this type of surfaces. First of all we note that X admits a unique fibration $f : X \longrightarrow C$ onto a curve of genus ≥ 2 (*cf.* for instance [Po] p. 5). We then denote by G the general fiber of f, and by $\mathrm{Pic}^0(X, f)$ the kernel of the pull-back of the inclusion $u : G \hookrightarrow X$

$$0 \longrightarrow \mathrm{Pic}^0(X, f) \longrightarrow \mathrm{Pic}^0(X) \xrightarrow{u^*} \mathrm{Pic}^0(G).$$

In [Be2] (1.6) it is shown that there exists a finite group $\Gamma^0(f)$ and an isomorphism

$$\mathrm{Pic}^0(X, f) \cong f^*\mathrm{Pic}^0(C) \times \Gamma^0(f).$$

The group $\Gamma^0(f)$ is the invariant mentioned above; it is identified with the group of the connected components of $\mathrm{Pic}^0(X, f)$.

We now consider another smooth projective surface Y such that $\mathbf{D}(X) \cong \mathbf{D}(Y)$. Then, by [BM] Proposition 4.4, Y is an elliptic surface fibered onto C. Moreover, Y is of maximal Albanese dimension as well. To see this we observe that since the cohomological support loci are derived invariant in dimension two, we have $\dim \mathrm{alb}_Y(Y) = \dim \mathrm{alb}_X(X) = 2$ thanks to (5). Hence we denote by $g : Y \longrightarrow C$ the unique fibration of Y and by $\Gamma^0(g)$ its invariant. In [Ph] Theorem 5.2.7, Pham proves that the invariant $\Gamma^0(\cdot)$ attached to this kind of surfaces is a derived invariant, in other words he proves that

$$\Gamma^0(f) \cong \Gamma^0(g). \tag{9}$$

Here we note that (9) also follows from the derived invariance of the zero-th cohomological support locus. In fact, by results of Popa (*cf.* [Po] p. 5) we know that

$$V^0(\omega_X) = \mathrm{Pic}^0(X, f) \cong f^*\mathrm{Pic}^0(C) \times \Gamma^0(f)$$

and similarly for $V^0(\omega_Y)$. Therefore, Proposition 3.1 implies

$$f^*\mathrm{Pic}^0(C) \times \Gamma^0(f) \cong V^0(\omega_X) \cong V^0(\omega_Y) \cong g^*\mathrm{Pic}^0(C) \times \Gamma^0(g),$$

which in particular yields (9).

4.2. **Proof of Theorem 1.5 (i).**

Theorem 4.2. *Let X and Y be smooth projective threefolds, $\Phi_{\mathcal{E}} : \mathbf{D}(X) \longrightarrow \mathbf{D}(Y)$ be an equivalence, and let F be the induced Rouquier isomorphism. Then F induces isomorphisms of algebraic sets*

$$V^p_r(\Omega^q_X)_0 \cong V^p_r(\Omega^q_Y)_0 \quad \textit{for any} \quad p, q \geq 0 \quad \textit{and} \quad r \geq 1.$$

Proof. The isomorphisms $V^0_r(\omega_X) \cong V^0_r(\omega_Y)$ and $V^1_r(\omega_X)_0 \cong V^1_r(\omega_Y)_0$ have been proved in Proposition 3.1 and Corollary 3.4 respectively. On the other hand, the isomorphisms $V^3_r(\omega_X) \cong V^3_r(\omega_Y)$ are trivial and follow by Serre duality. We now show the isomorphisms $V^2_r(\omega_X)_0 \cong V^2_r(\omega_Y)_0$. To begin with, we note that by Claim 3.3, if $L \in V^2_r(\omega_X)_0$, then necessarily $F(1, L) = (1, M)$ for some line bundle $M \in \mathrm{Pic}^0(Y)$. Moreover, for $k = 0, 1$ we have equalities $h^k(X, \omega_X \otimes L) = h^k(Y, \omega_Y \otimes M)$ whenever $L \in V^2_r(\omega_X)_0$ and $F(1, L) = (1, M)$ (*cf.* Corollary 2.2). Finally, since the holomorphic Euler characteristic is both a derived invariant in dimension three (*cf.* [PS] Corollary C) and invariant under deformation, we have equalities $\chi(\omega_X \otimes L) = \chi(\omega_X) = \chi(\omega_Y) = \chi(\omega_Y \otimes M)$, from which we easily deduce $h^2(X, \omega_X \otimes L) = h^2(Y, \omega_Y \otimes M)$. Thus, if $L \in V^2_r(\omega_X)_0$, then $M \in V^2_r(\omega_Y)_0$ and consequently F induces inclusions $F(1, V^2_r(\omega_X)_0) \subset$

$(1, V_r^2(\omega_Y)_0)$. Since F^{-1} is the Rouquier isomorphism induced by the right adjoint $\Psi_{\mathcal{E}^*}$ to $\Phi_{\mathcal{E}}$, then we can repeat the previous argument to obtain the reverse inclusions $F^{-1}(1, V_r^2(\omega_Y)_0) \subset (1, V_r^2(\omega_X)_0)$. This in turn yields isomorphisms $V_r^0(\Omega_X^1)_0 \cong V_r^0(\Omega_Y^1)_0$ thanks to Serre duality and the Hodge linear-conjugate isomorphism.

We now prove the isomorphisms $V_r^1(\Omega_X^q)_0 \cong V_r^1(\Omega_Y^q)_0$ for $q = 1, 2$. By Claim 3.3 we have $F(1, V_r^1(\Omega_X^q)_0) \subset (1, \mathrm{Pic}^0(Y))$. By Serre duality and the Hodge linear-conjugate isomorphism we get $h^0(X, \Omega_X^1 \otimes L) = h^2(X, \omega_X \otimes L)$ and $h^0(Y, \Omega_Y^1 \otimes M) = h^2(Y, \omega_Y \otimes M)$ for all line bundles $L \in \mathrm{Pic}^0(X)$ and $M \in \mathrm{Pic}^0(Y)$. Consequently, if $L \in V^0(\Omega_X^1)_0$ and $F(1, L) = (1, M)$, then by Corollary 2.2 with $m = 0$ and $k = 2$ we have $h^1(X, \Omega_X^2 \otimes L) = h^1(Y, \Omega_Y^2 \otimes M)$. At this point, in order to prove the wanted isomorphisms, it is enough to proceed as before. In complete analogy one can also prove the isomorphisms $V_r^1(\Omega_X^1)_0 \cong V_r^1(\Omega_Y^1)_0$, this time by using Corollary 2.2 with $m = 0$ and $k = 3$. □

5. Behavior of the Albanese dimension under derived equivalence

In this section we prove Theorem 1.6. Our main tool is a generalization of a result due to Chen-Hacon-Pardini saying that if $f : X \longrightarrow Z$ is a non-singular representative of the Iitaka fibration of a smooth projective variety X of maximal Albanese dimension, then

$$q(X) - q(Z) = \dim X - \dim Z$$

(*cf.* [HP] Proposition 2.1 and [CH2] Corollary 3.6). We generalize this fact in two ways: 1) we consider all possible values of the Albanese dimension of X; and 2) we replace the Iitaka fibration with a more general class of morphisms.

Lemma 5.1. *Let X and Z be smooth projective varieties and $f : X \longrightarrow Z$ be a surjective morphism with connected fibers. If the general fiber of f is a smooth variety with surjective Albanese map, then*

$$q(X) - q(Z) = \dim \mathrm{alb}_X(X) - \dim \mathrm{alb}_Z(Z).$$

Proof. We follow [HP] Proposition 2.1 and [CH2] Corollary 3.6. Due to the functoriality of the Albanese map we get a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{alb}_X} & \mathrm{Alb}(X) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_*} \\ Z & \xrightarrow{\mathrm{alb}_Z} & \mathrm{Alb}(Z) \end{array}$$

where f_* is surjective since f is so (*cf.* [Be1] Remark V.14). Furthermore, f_* has connected fibers. To see this we denote by K the connected component of $\mathrm{Ker}\, f_*$ through the origin and set $A := \mathrm{Alb}(X)/K$. Then the natural map $\nu : A \longrightarrow \mathrm{Alb}(Y)$ is étale and f factors through the induced map $Y \times_{\mathrm{Alb}(Y)} A \longrightarrow Y$, which is étale of the same degree as ν. Since f has connected fibers, we see that ν is an isomorphism and $K = \mathrm{Ker}\, f_*$.

We now show that the image of a general fiber P of f via alb_X is a translate of $\mathrm{Ker}\, f_*$. Since alb_P is surjective, the image of P via alb_X is a translate of a sub-torus of $\mathrm{Ker}\, f_*$. Furthermore, since P moves in a continuous family, such images are all translates of a fixed sub-torus $T \subset \mathrm{Ker}\, f_*$. Our next step is to show $T = \mathrm{Ker}\, f_*$. By setting $B := \mathrm{Alb}(X)/T$, we see that the induced morphism $X \longrightarrow B$ maps a general fiber of f to a point. Therefore it induces a rational map $h : Z \dashrightarrow B$ which is a morphism since B is an abelian variety. Furthermore, $h(Z)$ generates the abelian variety B since the image of the Albanese map generates the Albanese variety. This leads to the inequality

$$\dim B \leq q(Z) = q(X) - \dim \operatorname{Ker} f_*,$$

which in turn yields $\dim T \geq \dim \operatorname{Ker} f_*$ as $\dim B = q(X) - \dim T$. For dimension reasons we get then $T = \operatorname{Ker} f_*$. In particular, this says that $\mathrm{alb}_X(X)$ is fibered in tori of dimension $q(X) - q(Z)$ over $\mathrm{alb}_Z(Z)$, and by the theorem on the dimension of the fibers of a morphism we get the stated equality. □

Proof of Theorem 1.6. We begin with the case $\dim X \leq 3$. In §4.1 and §4.2 we have seen that in dimension up to three the cohomological support loci associated to the canonical bundle around the origin are derived invariant, i.e. $V^k(\omega_X)_0 \cong V^k(\omega_Y)_0$ for all $k \geq 0$. Therefore formula (5), in combination with the fact that derived equivalent varieties have the same dimension, immediately leads to $\dim \mathrm{alb}_X(X) = \dim \mathrm{alb}_Y(Y)$.

We now assume $\dim X > 3$ and $\kappa(X) \geq 0$. If $\kappa(X) = \kappa(Y) = 0$, then the Albanese maps of X and Y are surjective by [Ka1] Theorem 1. Thus the Albanese dimensions of X and Y are $q(X)$ and $q(Y)$ respectively which are equal by work of Popa and Schnell [PS] Corollary B.

We now suppose $\kappa(X) = \kappa(Y) > 0$. Since the problem is invariant under birational modification, with a little abuse of notation, we consider non-singular representatives $f : X \longrightarrow Z$ and $g : Y \longrightarrow W$ of the Iitaka fibrations of X and Y respectively (*cf.* [Mo] (1.10)). As the canonical rings of X and Y are isomorphic (*cf.* [Or] Corollary 2.1.9), it turns out that Z and W are birational varieties (*cf.* [Mo] Proposition 1.4 or [To] p. 13). By [Ka1] Theorem 1, the morphisms f and g satisfy the hypotheses of Lemma 5.1 which yields

$$q(X) - \dim \mathrm{alb}_X(X) = q(Z) - \dim \mathrm{alb}_Z(Z) = q(W) - \dim \mathrm{alb}_W(W) = q(Y) - \dim \mathrm{alb}_Y(Y).$$

We conclude as $q(X) = q(Y)$. □

As an application of Theorem 1.6, we have the following

Corollary 5.2. *Let X and Y be smooth projective derived equivalent varieties with X of maximal Albanese dimension. If F denotes the induced Rouquier isomorphism and $F(1, L) = (\psi, M)$ with $L \in V^1_r(\omega_X)$, then $\psi = 1$ and $M \in V^1_r(\omega_Y)$. Moreover, F induces isomorphisms of algebraic sets*

$$V^1_r(\omega_X) \cong V^1_r(\omega_Y) \quad \textit{for any} \quad r \geq 1.$$

Proof. We have $\kappa(X) \geq 0$ since X is of maximal Albanese dimension. Hence Theorem 1.6 ensures that Y is of maximal Albanese dimension as well. We apply then Corollary 3.5 after having noted the inclusions $V^1_r(\omega_X) \subset V^0(\omega_X)$ and $V^1_r(\omega_Y) \subset V^0(\omega_Y)$ (*cf.* (6)). □

6. End of the proof of Theorem 1.5

6.1. **Proof of Theorem 1.5 (ii).** The following two propositions prove and extend Theorem 1.5 (ii).

Proposition 6.1. *Let X and Y be smooth projective derived equivalent threefolds and let F be the induced Rouquier isomorphism. Assume that either* $\mathrm{Aut}^0(X)$ *is affine, or that* $V^p(\Omega_X^q \otimes \omega_X^{\otimes m}) = \mathrm{Pic}^0(X)$ *for some $m, p, q \in \mathbf{Z}$ with $p, q \geq 0$. Then F induces isomorphisms of algebraic sets*

$$V_r^p(\Omega_X^q) \cong V_r^p(\Omega_Y^q) \quad \text{for all} \quad p, q \geq 0 \quad \text{and} \quad r \geq 1.$$

Proof. By Remark 3.6 we have $F(1, \mathrm{Pic}^0(X)) = (1, \mathrm{Pic}^0(Y))$. The isomorphisms $V_r^0(\omega_X) \cong V_r^0(\omega_Y)$ and $V_r^1(\omega_X) \cong V_r^1(\omega_Y)$ hold by Proposition 3.1 and Remark 3.6 respectively. The isomorphisms $V_r^2(\omega_X) \cong V_r^2(\omega_Y)$ follow since in dimension three $\chi(\omega_X) = \chi(\omega_Y)$ (*cf.* [PS] Corollary C).

We now establish the isomorphisms $V_r^1(\Omega_X^2) \cong V_r^1(\Omega_Y^2)$. Let $L \in V_r^1(\Omega_X^2)$ so that $F(1, L) = (1, M)$ for some $M \in \mathrm{Pic}^0(Y)$. By Corollary 2.2 with $m = 0$ and $k = 2$, Serre duality, and the Hodge linear-conjugate isomorphism, we get $h^1(X, \Omega_X^2 \otimes L) = h^1(Y, \Omega_Y^2 \otimes M)$. This shows that F maps $1 \times V_r^1(\Omega_X^2) \mapsto 1 \times V_r^1(\Omega_Y^2)$, inducing the wanted isomorphisms as in Proposition 3.1. Finally, the isomorphisms $V_r^1(\Omega_X^1) \cong V_r^1(\Omega_Y^1)$ are deduced in the same way by using Corollary 2.2 with $m = 0$ and $k = 3$. □

Proposition 6.2. *Let X and Y be smooth projective derived equivalent threefolds and let F be the induced Rouquier isomorphism. If X is of maximal Albanese dimension, then F induces isomorphisms of algebraic sets*

$$V_r^k(\omega_X) \cong V_r^k(\omega_Y) \quad \text{for all} \quad k \geq 0 \quad \text{and} \quad r \geq 1.$$

Proof. Proposition 3.1 and Corollary 5.2 yield the isomorphisms $V_r^k(\omega_X) \cong V_r^k(\omega_Y)$ for any $k \neq 2$, so we only focus on the remaining case. Since X is of maximal Albanese dimension, we obtain an inclusion $V_r^2(\omega_X) \subset V^0(\omega_X)$ (*cf.* (6)) leading to a further inclusion $F(1, V_r^2(\omega_X)) \subset (1, \mathrm{Pic}^0(Y))$ thanks to Proposition 3.1. Hence, by Corollary 2.2, we get that $h^k(X, \omega_X \otimes L) = h^k(Y, \omega_Y \otimes M)$ whenever $F(1, L) = (1, M)$ with $L \in V_r^2(\omega_X)$ and $k = 0, 1$. Moreover, we get $h^2(X, \omega_X \otimes L) = h^2(Y, \omega_Y \otimes M)$ since $\chi(\omega_X) = \chi(\omega_Y)$ (*cf.* [PS] Corollary C). Therefore F maps $1 \times V_r^2(\omega_X) \mapsto 1 \times V_r^2(\omega_Y)$, and by arguing as in Proposition 3.1, F^{-1} maps $1 \times V_r^2(\omega_Y) \mapsto 1 \times V_r^2(\omega_X)$ finishing the proof. □

6.2. **Proof of Theorem 1.5 (iii).** We show now the proof of Theorem 1.5 (iii). Before jumping into technicalities, we first present the plan of its proof.

Thanks to Propositions 6.1 and 6.2 we can assume that X is a threefold with $\dim \mathrm{alb}_X(X) \leq 2$, $V^0(\omega_X) \subsetneq \mathrm{Pic}^0(X)$ and with non-affine automorphism group $\mathrm{Aut}^0(X)$. In particular, we can suppose that X is not of general type and that $\chi(\omega_X) = 0$ (*cf.* [PS] Corollary 2.6). Thanks to Proposition 3.1, Theorem 1.6, and [PS] Theorem A (1), the Fourier-Mukai partner Y of X satisfies the same hypotheses as X. Hence, Theorem 1.5 (iii) follows as soon as we classify $\dim V^i(\omega_X)$ in terms of derived invariants. This classification is carried out in the following Propositions 6.5 - 6.9 where $\dim V^1(\omega_X)$ and $\dim V^2(\omega_X)$ are computed in terms of $\kappa(X)$, $q(X)$, $\dim \mathrm{alb}_X(X)$ and $\dim V^0(\omega_X)$.

The main tools we use towards the proofs of Propositions 6.5 - 6.9 are generic vanishing theorems (*cf.* [GL1] Theorem 1 and [PP2] Theorem 5.8), Kollár's result on higher direct images of the canonical bundle (*cf.* [Ko2] Theorem 3.1, [Ko1] Theorem 2.1 and [Ko1] Proposition 7.6), and the classification of smooth projective surfaces (*cf.* for instance [Be1]). The following two lemmas will be useful to our analysis.

Lemma 6.3. *Let X and Y be smooth projective varieties and $f : X \longrightarrow Y$ be a surjective morphism with connected fibers. If h denotes the dimension of the general fiber of f, then*

$$f^*V^k(\omega_Y) \subset V^{k+h}(\omega_X) \quad \textit{for any} \quad k = 0, \ldots, \dim Y.$$

Proof. By [Ko1] Theorem 2.1 and Proposition 7.6 we have $R^h f_*\omega_X \cong \omega_Y$ and $R^k f_*\omega_X = 0$ for $k > h$. Moreover, by [Ko2] Theorem 3.1 we obtain decompositions

$$H^{k+h}(X, \omega_X \otimes f^*L) \cong H^k(Y, \omega_Y \otimes L) \oplus \bigoplus_{l \neq k} H^l(Y, R^{h+k-l} f_*\omega_X \otimes L)$$

for any $L \in \mathrm{Pic}^0(Y)$. At this point it is enough to note that the pull-back homomorphism $f^* : \mathrm{Pic}^0(Y) \longrightarrow \mathrm{Pic}^0(X)$ is injective as the fibers of f are connected. □

Lemma 6.4. *Let X be a smooth projective variety with $\kappa(X) = -\infty$. Then $V^0(\omega_X^{\otimes m}) = \emptyset$ for any $m > 0$.*

Proof. Suppose that $L \in V^0(\omega_X^{\otimes m})$ for some $m > 0$. By [CH2] Theorem 3.2 we can assume that L is a line bundle of finite order, say of oder e. If $\mathcal{O}_X \longrightarrow \omega_X^{\otimes m} \otimes L$ is a non-zero section of $\omega_X^{\otimes m} \otimes L$, then it induces a non-zero section $\mathcal{O}_X \longrightarrow \omega_X^{\otimes me}$; this yields a contradiction as $\kappa(\omega_X) = -\infty$. □

Proposition 6.5. *Let X be a smooth projective threefold such that $\kappa(X) = 2$, $\dim \mathrm{alb}_X(X) = 2$, $\chi(\omega_X) = 0$ and $V^0(\omega_X) \subsetneq \mathrm{Pic}^0(X)$. If $q(X) = 2$, then we have: i) $\dim V^2(\omega_X) = 0$; ii) $\dim V^1(\omega_X) = 1$ if and only if $\dim V^0(\omega_X) = 1$; and iii) $\dim V^1(\omega_X) = 0$ if and only if $\dim V^0(\omega_X) \leq 0$. If $q(X) > 2$, then we have $\dim V^1(\omega_X) = \dim V^2(\omega_X) = q(X) - 1$.*

Proof. Since the problem is invariant under birational modification, with a little abuse of notation we consider a non-singular representative $f : X \longrightarrow S$ of the Iitaka fibration of X (*cf.* [Mo] (1.10)), so that X and S are smooth varieties and f is an algebraic fiber space. We divide the proof into three cases according to the values of the Albanese dimension of S.

Case I: $\dim \mathrm{alb}_S(S) = 2$. By the classification theory of smooth projective surfaces, S is either a surface of general type, or birational to an abelian surface, or birational to an elliptic surface fibered onto a curve of genus ≥ 2. Moreover, by Lemma 5.1, we have $q(X) = q(S)$.

If S is of general type, then by Castelnuovo's Theorem (*cf.* [Be1] Theorem X.4) we have $\chi(\omega_S) > 0$ and hence $V^0(\omega_S) = \mathrm{Pic}^0(S)$. Therefore, by Lemma 6.3, we get $V^1(\omega_X) = \mathrm{Pic}^0(X)$, and consequently $V^0(\omega_X) = \mathrm{Pic}^0(X)$ since $\chi(\omega_X) = 0$ and $V^2(\omega_X) \subsetneq \mathrm{Pic}^0(X)$ (*cf.* (5)). This contradicts our hypotheses and hence this case does not occur.

If S is birational to an abelian surface, then we have $q(X) = q(S) = 2$ and $f^*\mathrm{Pic}^0(S) = \mathrm{Pic}^0(X)$. By using [Ko2] Theorem 3.1, we obtain decompositions

$$H^2(X, \omega_X \otimes f^*L) \cong H^2(S, f_*\omega_X \otimes L) \oplus H^1(S, R^1 f_*\omega_X \otimes L)$$

for any $L \in \mathrm{Pic}^0(S)$. Moreover, we note that $R^1 f_*\omega_X \cong \omega_S$ and $R^2 f_*\omega_X = 0$ (*cf.* [Ko1] Proposition 7.6 and Theorem 2.1). Therefore, since by [PP2] Theorem 5.8 $f_*\omega_X$ is a GV-sheaf on S (i.e. $\mathrm{codim}_{\mathrm{Pic}^0(S)} V^k(f_*\omega_X) \geq k$ for $k > 0$), we get $\dim V^2(\omega_X) = 0$. At this point the statements ii) and iii) of the proposition follow as $\chi(\omega_X) = 0$ and $\dim V^1(\omega_X) \geq 0$ (note that $\mathcal{O}_X \in V^1(\omega_X)$ since $q(X) = 2$).

If S is birational to an elliptic surface $h : S \longrightarrow C$ fibered onto a curve C of genus $g(C) = q(S) - 1 = q(X) - 1 \geq 2$, then X is fibered onto C as well. Therefore we have $V^0(\omega_C) = \mathrm{Pic}^0(C)$, and consequently $V^2(\omega_X)$ is of codimension one in $\mathrm{Pic}^0(X)$ by Lemma 6.3 and (5). Since $\chi(\omega_X) = 0$, $V^1(\omega_X)$ is of codimension one as well.

Case II: $\dim \mathrm{alb}_S(S) = 1$. We have $q(X) = q(S) + 1$ by Lemma 5.1. Moreover, alb_S has connected fibers, and by [Be1] Proposition V.15 $\mathrm{alb}_S(S)$ is a smooth curve of genus $q(S)$. We distinguish two subcases: $q(S) = 1$ and $q(S) \geq 2$.

If $q(S) = 1$, then $q(X) = 2$ and alb_X is surjective. Let $X \xrightarrow{b} Z \longrightarrow \mathrm{Alb(X)}$ be the Stein factorization of alb_X, and let $b' : X' \longrightarrow Z'$ be a non-singular representative of b. We note that Z' is a smooth surface with $q(Z') = 2$ and of maximal Albanese dimension. Therefore either Z' is of general type, or it is birational to an abelian surface. However, we have just seen that Z' cannot possibly be of general type, therefore Z' is birational to an abelian surface and the same calculations of the previous case apply.

If $q(S) \geq 2$, then the Albanese map of S induces a fibration of S onto a smooth curve C of genus $g(C) = q(S)$. Therefore X is fibered onto C as well and we conclude as in the previous case.

Case III: $\dim \mathrm{alb}_S(S) = 0$. As we have seen in the proof of Lemma 5.1, the image of a general fiber of f is mapped via alb_X onto a fiber of the induced morphism $f_* : \mathrm{Alb}(X) \longrightarrow \mathrm{Alb}(S)$. On the other hand, if $\dim \mathrm{alb}_S(S) = 0$, then $\mathrm{Alb}(S)$ is trivial. This yields a contradiction and therefore this case does not occur. □

Proposition 6.6. *Let X be a smooth projective threefold such that $\kappa(X) = 2$, $\dim \mathrm{alb}_X(X) = 1$, $\chi(\omega_X) = 0$ and $V^0(\omega_X) \subsetneq \mathrm{Pic}^0(X)$. If $q(X) = 1$, then we have $\dim V^1(\omega_X) \leq 0$ and $\dim V^2(\omega_X) = 0$. On the other hand, if $q(X) > 1$, then we have $V^1(\omega_X) = V^2(\omega_X) = \mathrm{Pic}^0(X)$.*

Proof. As in the previous proof we denote by $f : X \longrightarrow S$ a non-singular representative of the Iitaka fibration of X. We distinguish two cases: $\dim \mathrm{alb}_S(S) = 0$ and $\dim \mathrm{alb}_S(S) = 1$.

If $\dim \mathrm{alb}_S(S) = 0$, then we have $q(S) = 0$ and therefore $q(X) = 1$ by Lemma 5.1. Moreover, by [Ue] Lemma 2.11, alb_X is surjective and has connected fibers. We set $E := \mathrm{Alb}(X)$, $a := \mathrm{alb}_X$ and we note that by [Ko1] Proposition 7.6 there is an isomorphism $R^2 a_* \omega_X \cong \mathcal{O}_E$. Finally, by [Ko2] Theorem 3.1, we get isomorphisms

$$H^2(X, \omega_X \otimes a^* L) \cong H^1(E, R^1 a_* \omega_X \otimes L) \oplus H^0(E, L)$$

for any $L \in \mathrm{Pic}^0(E) \cong \mathrm{Pic}^0(X)$. By [Ha] Corollary 4.2, $R^1 a_* \omega_X$ is a GV-sheaf on E. Hence $\dim V^2(\omega_X) = 0$, and consequently $V^1(\omega_X)$ is either empty or zero-dimensional as $V^0(\omega_X) \subsetneq \mathrm{Pic}^0(X)$ and $\chi(\omega_X) = 0$.

We now suppose $\dim \mathrm{alb}_S(S) = 1$. In this case alb_S has connected fibers and its image is a smooth curve B of genus $g(B) = q(S) > 1$. Moreover, we have $q(X) = q(S)$ by Lemma 5.1. We distinguish two subcases: $q(S) = 1$ and $q(S) > 1$. If $q(S) = 1$, then the image of alb_X is an elliptic curve and the same argument of the previous case applies. If $q(S) = g(B) > 1$, then we get $V^0(\omega_B) = \mathrm{Pic}^0(B)$ and $\mathrm{Pic}^0(X) \cong \mathrm{Pic}^0(S) \cong \mathrm{Pic}^0(B)$. Hence, by Lemma 6.3, there are inclusions

$$\mathrm{alb}_S^*\mathrm{Pic}^0(B) = \mathrm{alb}_S^* V^0(\omega_B) \subset V^1(\omega_S) \subset \mathrm{Pic}^0(S)$$

leading to $V^1(\omega_S) = \mathrm{Pic}^0(S)$. Moreover, a second application of Lemma 6.3 gives

$$f^* V^1(\omega_S) \subset V^2(\omega_X) \subset \mathrm{Pic}^0(X)$$

showing that $V^2(\omega_X) = \mathrm{Pic}^0(X)$. Finally, we also have $V^1(\omega_X) = \mathrm{Pic}^0(X)$ as $\chi(\omega_X) = 0$. □

Proposition 6.7. *Let X be a smooth projective threefold such that $\kappa(X) = 1$, $\chi(\omega_X) = 0$ and $V^0(\omega_X) \subsetneq \mathrm{Pic}^0(X)$.*

(i). *Assume $\dim \mathrm{alb}_X(X) = 2$. If $q(X) = 2$, then we have: i) $\dim V^2(\omega_X) = 0$; ii) $\dim V^1(\omega_X) = 1$ if and only if $\dim V^0(\omega_X) = 1$; and iii) $\dim V^1(\omega_X) = 0$ if and only if $\dim V^0(\omega_X) \le 0$. If $q(X) \ge 3$, then we have $\dim V^1(\omega_X) = \dim V^2(\omega_X) = q(X) - 1$.*

(ii). *Assume $\dim \mathrm{alb}_X(X) = 1$. If $q(X) = 1$, then we have $\dim V^1(\omega_X) \le 0$ and $\dim V^2(\omega_X) = 0$. If $q(X) \ge 2$, then we obtain $V^1(\omega_X) = V^2(\omega_X) = \mathrm{Pic}^0(X)$.*

Proof. We start with the case $\dim \mathrm{alb}_X(X) = 2$. Let $f : X \longrightarrow C$ be a non-singular representative of the Iitaka fibration of X where C is a smooth curve.

If $g(C) \ge 2$, then by Lemma 5.1 we have $q(X) = g(C) + 1 \ge 3$, and by Lemma 6.3 we obtain a series of inclusions $f^*\mathrm{Pic}^0(C) = f^* V^0(\omega_C) \subset V^2(\omega_X) \subset \mathrm{Pic}^0(X)$. We conclude that

$$\dim V^2(\omega_X) = q(X) - 1$$

since $V^2(\omega_X) \subsetneq \mathrm{Pic}^0(X)$ by (5). Therefore we see that $V^1(\omega_X) \subsetneq \mathrm{Pic}^0(X)$ as $\chi(\omega_X) = 0$ and $V^0(\omega_X) \subsetneq \mathrm{Pic}^0(X)$. Finally, thanks to the inclusion $V^1(\omega_X) \supset V^2(\omega_X)$ of (6) we obtain $\dim V^1(\omega_X) = q(X) - 1$.

If $g(C) \le 1$, then $q(X) = 2$ and $a := \mathrm{alb}_X$ is surjective. Let $b : X' \longrightarrow Z'$ be a non-singular representative of the Stein factorization of a. Then, as we have seen in the proof of Proposition 6.5, Z' is birational to an abelian surface, and therefore $\dim V^2(\omega_X) = 0$. Since $\mathcal{O}_X \in V^1(\omega_X)$, we obtain the statements ii) and iii) of the proposition.

We now study the case $\dim \mathrm{alb}_X(X) = 1$. If $g(C) \ge 2$, then it turns out that $q(X) = g(C)$ and $f^*\mathrm{Pic}^0(C) = \mathrm{Pic}^0(X)$. Therefore, by Lemma 6.3, we get $V^2(\omega_X) = \mathrm{Pic}^0(X)$, and hence we have $V^1(\omega_X) = \mathrm{Pic}^0(X)$. On the other hand, if $g(C) \le 1$, then $q(X) = 1$ and $\mathrm{alb}_X : X \longrightarrow \mathrm{Alb}(X)$ is an algebraic fiber space onto an elliptic curve. We conclude then as in the proof of Proposition 6.6. □

Proposition 6.8. *Let X be a smooth projective threefold such that $\kappa(X) = 0$ and $\chi(\omega_X) = 0$. If $\dim \mathrm{alb}_X(X) = 2$, then we have $\dim V^1(\omega_X) = \dim V^2(\omega_X) = 0$. On the other hand, if $\dim \mathrm{alb}_X(X) = 1$, then we have $\dim V^1(\omega_X) \le 0$ and $\dim V^2(\omega_X) = 0$.*

Proof. We recall that, by [CH1] Lemma 3.1, $V^0(\omega_X)$ consists of at most one point. We start with the case $\dim \mathrm{alb}_X(X) = 2$. By [Ka1] Theorem 1, alb_X is surjective and has connected fibers. Therefore we have $q(X) = h^2(X, \omega_X) = 2$ and hence $\mathcal{O}_X \in V^1(\omega_X)$ since $\chi(\omega_X) = 0$. We set $a := \mathrm{alb}_X$ and we note that, by [Ha] Corollary 4.2, $a_*\omega_X$ is a GV-sheaf, i.e.

$$\mathrm{codim}\, V^1(a_*\omega_X) \ge 1 \quad \text{and} \quad \mathrm{codim}\, V^2(a_*\omega_X) \ge 2.$$

By using that $R^1a_*\omega_X \cong \mathcal{O}_{\mathrm{Alb}(X)}$ and $R^2a_*\omega_X = 0$ (*cf.* [Ko1] Proposition 7.6 and Theorem 2.1), and by using [Ko2] Theorem 3.1, we get isomorphisms

$$H^1(X, \omega_X \otimes a^*L) \cong H^1(\mathrm{Alb}(X), a_*\omega_X \otimes L) \oplus H^0(\mathrm{Alb}(X), L)$$

for any $L \in \mathrm{Pic}^0(\mathrm{Alb}(X)) \cong \mathrm{Pic}^0(X)$. Therefore we have

$$\operatorname{codim} V^1(\omega_X) \geq 1 \quad \text{and} \quad \operatorname{codim} V^2(\omega_X) \geq 2,$$

and consequently the hypothesis $\chi(\omega_X) = 0$ implies $\dim V^1(\omega_X) = 0$.

If $\dim \mathrm{alb}_X(X) = 1$, then as in the previous case we have $\dim V^2(\omega_X) = 0$. Therefore $V^1(\omega_X)$ is either empty or of dimension zero since $\chi(\omega_X) = 0$. □

Proposition 6.9. *Let X be a smooth projective threefold such that $\kappa(X) = -\infty$ and $\chi(\omega_X) = 0$.*

(i). *Suppose* $\dim \mathrm{alb}_X(X) = 2$. *If $q(X) = 2$, then we have $V^1(\omega_X) = V^2(\omega_X) = \{\mathcal{O}_X\}$. If $q(X) > 2$, then we obtain* $\dim V^1(\omega_X) = \dim V^2(\omega_X) = q(X) - 1$.

(ii). *Suppose* $\dim \mathrm{alb}_X(X) = 1$. *If $q(X) = 1$, then we have* $\dim V^1(\omega_X) \leq 0$ *and* $\dim V^2(\omega_X) = 0$. *If $q(X) > 1$, then we obtain $V^1(\omega_X) = V^2(\omega_X) = \mathrm{Pic}^0(X)$.*

Proof. We start with the case $\dim \mathrm{alb}_X(X) = 2$. Let $a : X \longrightarrow S \subset \mathrm{Alb}(X)$ be the Albanese map of X, $b : X \longrightarrow S'$ be the Stein factorization of a, and let $c : X' \longrightarrow S''$ be a non-singular representative of b. We can easily check that $q(X') = q(S'')$, $\dim \mathrm{alb}_S(S) = 2$, and hence that $\kappa(S'') \geq 0$. Furthermore, we have $c_*\omega_{X'} = 0$. To see this, we point out that by [PP2] Theorem 5.8 $c_*\omega_{X'}$ is a GV-sheaf on S'', and moreover that by Lemma 6.4 $V^0(c_*\omega_{X'}) = V^0(\omega_{X'}) = V^0(\omega_X) = \emptyset$. This immediately implies $c_*\omega_{X'} = 0$ as a GV-sheaf $\mathcal{F}$ is non-zero if and only if $V^0(\mathcal{F}) \neq \emptyset$. We distinguish now three cases according to the values of $\kappa(S'')$.

If $\kappa(S'') = 0$, then S'' is birational to an abelian surface. This forces $q(X) = q(X') = q(S'') = 2$ and $c^*\mathrm{Pic}^0(S'') = \mathrm{Pic}^0(X')$. By [Ko2] Theorem 3.1, [Ko1] Theorem 2.1, and [Ko1] Proposition 7.6, we obtain isomorphisms

$$H^2(X', \omega_{X'} \otimes c^*L) \cong H^1(S'', \omega_{S''} \otimes L), \quad H^1(X', \omega_{X'} \otimes c^*L) \cong H^0(S'', \omega_{S''} \otimes L)$$

for any $L \in \mathrm{Pic}^0(S'')$. Therefore we have $V^2(\omega_X) \cong V^2(\omega_{X'}) = c^*V^1(\omega_{S''}) = \{\mathcal{O}_{X'}\}$ and $V^1(\omega_X) \cong V^1(\omega_{X'}) = c^*V^0(\omega_{S''}) = \{\mathcal{O}_{X'}\}$.

If $\kappa(S'') = 1$, then S'' is birational to an elliptic surface of maximal Albanese dimension fibered onto a curve of genus $g(C) \geq 2$. Thus X is fibered onto C as well and $q(X') = q(S'') = g(C) + 1$. By Lemma 6.3 and (5), we deduce $\dim V^2(\omega_{X'}) = g(C) = q(X') - 1$, and therefore we get $\dim V^1(\omega_{X'}) = q(X') - 1$ as $\chi(\omega_{X'}) = 0$ and $V^0(\omega_{X'}) = \emptyset$.

If $\kappa(S'') = 2$, then by Castelnuovo's Theorem we have $\chi(\omega_{S''}) > 0$, which immediately yields $V^0(\omega_{S''}) = \mathrm{Pic}^0(S'')$. By using Lemma 6.3, we see that $\dim V^0(\omega_{X'}) > 0$. This contradicts Lemma 6.4 and hence this case does not occur.

We now suppose $\dim \mathrm{alb}_X(X) = 1$. Let $a : X \longrightarrow C \subset \mathrm{Alb}(X)$ be the Albanese map of X where $C := \operatorname{Im} a$. Then a has connected fibers and $q(X) = g(C)$ by [Ue] Lemma 2.11. As in the previous case, we note that $a_*\omega_X = 0$. Moreover, by [Ko2] Theorem 3.1 and [Ko1]

Proposition 7.6, we obtain isomorphisms

$$H^1(X,\omega_X\otimes a^*L)\cong H^0(C,R^1a_*\omega_X\otimes L)$$
$$H^2(X,\omega_X\otimes a^*L)\cong H^1(C,R^1a_*\omega_X\otimes L)\oplus H^0(C,\omega_C\otimes L)$$

for any $L\in \mathrm{Pic}^0(C)$. At this point we distinguish two cases: $g(C)=1$ and $g(C)>1$. If $g(C)=q(X)>1$, then we have $V^0(\omega_C)=\mathrm{Pic}^0(C)$, and by Lemma 6.3 we get $V^2(\omega_X)=V^1(\omega_X)=\mathrm{Pic}^0(X)$. On the other hand, if $g(C)=q(X)=1$, then by [Ha] Corollary 4.2 $R^1a_*\omega_X$ is a GV-sheaf on $C=\mathrm{Alb}(X)$. Hence we obtain $\dim V^2(\omega_X)=0$, and consequently we see that $\dim V^1(\omega_X)\leq 0$ since $\chi(\omega_X)=0$ and $V^0(\omega_X)=\emptyset$. □

Remark 6.10. In the case $q(X)=1$, the previous propositions yield the following statement: for each k, $\dim V^k(\omega_X)=1$ if and only if $\dim V^k(\omega_Y)=1$. In general, we have not been able to show that if a locus $V^k(\omega_X)$ is empty (*resp.* of dimension zero) then the corresponding locus $V^k(\omega_Y)$ is empty (*resp.* of dimension zero). This ambiguity is mainly caused by the possible presence of non-trivial automorphisms.

An application of a sheafified version of the derivative complex (*cf.* [EL] Theorem 3 and [LP]) can be shown to yield Conjecture 1.2 for threefolds having $q(X)=2$ (*cf.* [Lo] Proposition 5.2.15).

7. Applications

In this final section we prove Corollaries 1.7, 1.8 and 1.9. Moreover, we present a further result regarding the invariance of the Euler characteristic of powers of the canonical bundle for derived equivalent smooth minimal varieties of maximal Albanese dimension.

7.1. **Holomorphic Euler characteristic and Hodge numbers.**

Proof of Corollary 1.7. Let $d:=\dim X=\dim Y$. We begin with the case $\dim \mathrm{alb}_X(X)=d$. By Theorem 1.6 Y is of maximal Albanese dimension, and by (5) we get inequalities

$$\operatorname{codim} V^1(\omega_X)\geq 1 \quad \text{and} \quad \operatorname{codim} V^1(\omega_Y)\geq 1.$$

We distinguish two cases: $V^0(\omega_X)\subsetneq \mathrm{Pic}^0(X)$ and $V^0(\omega_X)=\mathrm{Pic}^0(X)$. If $V^0(\omega_X)\subsetneq\mathrm{Pic}^0(X)$, then we also have $V^0(\omega_Y)\subsetneq \mathrm{Pic}^0(Y)$ by Proposition 3.1. Moreover, there are inclusions $\mathrm{Pic}^0(X)\supsetneq V^0(\omega_X)\supset V^1(\omega_X)\supset\cdots\supset V^d(\omega_X)=\{\mathcal{O}_X\}$, and similarly for the loci $V^k(\omega_Y)$ (*cf.* (6)). Therefore if $L\notin V^0(\omega_X)$ and $M\notin V^0(\omega_Y)$, then $h^k(X,\omega_X\otimes L)=h^k(Y,\omega_Y\otimes M)=0$ for all $k\geq 0$. Since the holomorphic Euler characteristic is invariant under deformation, we finally obtain

$$\chi(\omega_X)=\chi(\omega_X\otimes L)=0=\chi(\omega_Y\otimes M)=\chi(\omega_Y).$$

On the other hand, if $V^0(\omega_X)=\mathrm{Pic}^0(X)$, then by Proposition 3.1 we have $F(1,\mathrm{Pic}^0(X))=(1,\mathrm{Pic}^0(Y))$, and thus

$$\exists\, L_0\in V^0(\omega_X)\backslash\big(\cup_{k=1}^d V^k(\omega_X)\big) \text{ such that } F(1,L_0)=(1,M_0) \text{ with } M_0\in V^0(\omega_Y)\backslash\big(\cup_{k=1}^d V^k(\omega_Y)\big).$$

Hence, by using Corollary 2.2 with $m=k=0$, we have

$$\chi(\omega_X)=\chi(\omega_X\otimes L_0)=h^0(X,\omega_X\otimes L_0)=h^0(Y,\omega_Y\otimes M_0)=\chi(\omega_Y\otimes M_0)=\chi(\omega_Y).$$

We suppose now $\dim \mathrm{alb}_X(X) = d-1$ and $\kappa(X) \geq 0$. By Theorem 1.6 we have $\dim \mathrm{alb}_Y(Y) = d-1$, and therefore there are inclusions $V^1(\omega_X) \supset V^2(\omega_X) \supset \ldots \supset V^d(\omega_X)$ and $V^1(\omega_Y) \supset V^2(\omega_Y) \supset \ldots \supset V^d(\omega_Y)$. We distinguish four cases.

The first case is when $V^0(\omega_X) = V^1(\omega_X) = \mathrm{Pic}^0(X)$. By Proposition 3.1 and Corollary 3.4, it turns out that $V^0(\omega_Y) = V^1(\omega_Y) = \mathrm{Pic}^0(Y)$ as well. We claim that

$$\exists\, \mathcal{O}_X \neq L_1 \in V^0(\omega_X)\backslash V^2(\omega_X) \;\text{ such that }\; F(1, L_1) = (1, M_1) \;\text{ with }\; \mathcal{O}_Y \neq M_1 \in V^0(\omega_Y)\backslash V^2(\omega_Y).$$

In fact, by Remark 3.6 the Rouquier isomorphism maps $F(1, \mathrm{Pic}^0(X)) = (1, \mathrm{Pic}^0(Y))$ and therefore it is enough to choose the image under F^{-1} of a generic element $(1, M)$ with $M \notin V^2(\omega_Y)$. By using Corollary 2.2 twice, first with $k = 0$ and then with $k = 1$, we obtain

$$\chi(\omega_X) = \chi(\omega_X \otimes L_1) = h^0(X, \omega_X \otimes L_1) - h^1(X, \omega_X \otimes L_1) =$$
$$h^0(Y, \omega_Y \otimes M_1) - h^1(Y, \omega_Y \otimes M_1) = \chi(\omega_Y \otimes M_1) = \chi(\omega_Y).$$

The second case is when $V^0(\omega_X) = \mathrm{Pic}^0(X)$ and $V^1(\omega_X) \subsetneq \mathrm{Pic}^0(X)$. By Proposition 3.1 and Corollary 3.4, we have $V^0(\omega_Y) = \mathrm{Pic}^0(Y)$ and $V^1(\omega_Y) \subsetneq \mathrm{Pic}^0(Y)$. As before, $F(1, \mathrm{Pic}^0(X)) = (1, \mathrm{Pic}^0(Y))$, and hence we can pick an element

$$\mathcal{O}_X \neq L_2 \in V^0(\omega_X)\backslash V^1(\omega_X) \;\text{ such that }\; F(1, L_2) = (1, M_2) \;\text{ with }\; \mathcal{O}_Y \neq M_2 \in V^0(\omega_Y)\backslash V^1(\omega_Y).$$

Hence we obtain equalities $\chi(\omega_X) = \chi(\omega_X \otimes L_2) = h^0(X, \omega_X \otimes M_2) = h^0(Y, \omega_Y \otimes M_2) = \chi(\omega_Y \otimes M_2) = \chi(\omega_Y)$.

The third case is when $V^0(\omega_X) \subsetneq \mathrm{Pic}^0(X)$ and $V^1(\omega_X) = \mathrm{Pic}^0(X)$. By using Proposition 3.1 and Corollary 3.4, it is easy to see that $V^0(\omega_Y) \subsetneq \mathrm{Pic}^0(Y)$ and $V^1(\omega_Y) = \mathrm{Pic}^0(Y)$. Moreover, Remark 3.6 yields $F(1, \mathrm{Pic}^0(X)) = (1, \mathrm{Pic}^0(Y))$. Therefore, similarly to the previous cases, there exists a pair $(L_3, M_3) \neq (\mathcal{O}_X, \mathcal{O}_Y)$ such that

$$F(1, L_3) = (1, M_3) \;\text{ with }\; L_3 \notin V^0(\omega_X) \cup V^2(\omega_X) \;\text{ and }\; M_3 \notin V^0(\omega_Y) \cup V^2(\omega_Y)$$

and, by Corollary 2.2, we have $\chi(\omega_X) = \chi(\omega_X \otimes L_3) = -h^1(X, \omega_X \otimes L_3) = -h^1(Y, \omega_Y \otimes M_3) = \chi(\omega_Y \otimes M_3) = \chi(\omega_Y)$.

The last case is when both $V^0(\omega_X)$ and $V^1(\omega_X)$ are proper subvarieties of $\mathrm{Pic}^0(X)$. Then $V^0(\omega_Y)$ and $V^1(\omega_Y)$ are proper subvarieties as well and hence $\chi(\omega_X) = \chi(\omega_Y) = 0$.

□

Proof of Corollary 1.8. By the derived invariance of Hochschild homology $HH_0(X) \cong HH_0(Y)$ and $HH_1(X) \cong HH^1(Y)$, we have $h^0(X, \omega_X) = h^0(Y, \omega_Y)$ and $h^1(X, \omega_X) = h^1(Y, \omega_Y)$. Therefore Corollary 1.7 implies $h^{0,2}(X) = h^{0,2}(Y)$ since $h^3(X, \omega_X) = q(X) = q(Y) = h^3(Y, \omega_Y)$ and $h^4(X, \omega_X) = 1 = h^4(Y \omega_Y)$.

For the second equality we apply Corollary 2.2 with $(L, M) = (\mathcal{O}_X, \mathcal{O}_Y)$ and $k = 2$, so that $h^2(X, \omega_X) + h^1(X, \Omega^3_X) + h^0(X, \Omega^2_X) = h^2(Y, \omega_Y) + h^1(Y, \Omega^3_Y) + h^0(Y, \Omega^2_Y)$. Therefore we obtain $h^{1,3}(X) = h^{1,3}(Y)$ since Serre duality and the Hodge linear-conjugate isomorphism yield equalities $h^2(X, \omega_X) = h^0(X, \Omega^2_X)$ and $h^2(Y, \omega_Y) = h^0(Y, \Omega^2_Y)$. □

By using a result in [PP2], we can also derive a consequence about pluricanonical bundles.

Corollary 7.1. *Let X and Y be smooth projective derived equivalent varieties with X of maximal Albanese dimension and minimal. Then*

$$\chi(\omega_X^{\otimes m}) = \chi(\omega_Y^{\otimes m}) \quad \textit{for all} \quad m \geq 2.$$

Proof. By [PP2] Corollary 5.5, $\omega_X^{\otimes m}$ and $\omega_Y^{\otimes m}$ are GV-sheaves on X and Y respectively for any $m \geq 2$.[1] In particular, this implies that $\operatorname{codim} V^1(\omega_X^{\otimes m}) \geq 1$ and $\operatorname{codim} V^1(\omega_Y^{\otimes m}) \geq 1$. At this point we argue as in the first part of the proof of Corollary 1.7, after having noted the inclusions $V^0(\omega_X^{\otimes m}) \supset V^1(\omega_X^{\otimes m})$ and $V^0(\omega_Y^{\otimes m}) \supset V^1(\omega_Y^{\otimes m})$ (*cf.* [PP2] Proposition 3.14). □

7.2. **Fibrations.** In this subsection we study the behavior of particular types of fibrations under derived equivalence. We begin by recalling some terminology from [Cat] and [LP].

A smooth projective variety X is of *Albanese general type* if it is of maximal Albanese dimension and has non-surjective Albanese map. An *irregular fibration* (*resp. higher irrational pencil*) is a surjective morphism with connected fibers $f : X \longrightarrow Z$ onto a normal variety Z with $0 < \dim Z < \dim X$ and such that any smooth model of Z is of maximal Albanese dimension (*resp.* Albanese general type).

In [Po] Corollary 3.4 Popa observes that a consequence of Conjecture 1.3 is that if X admits a fibration onto a variety having non-surjective Albanese map, then any Fourier-Mukai partner of X admits an irregular fibration. With Theorem 1.4 at hand, we can verify this statement under an additional hypothesis on X.

Proposition 7.2. *Let X and Y be smooth projective derived equivalent varieties with* $\dim \mathrm{alb_X}(X) \geq \dim X - 1$*. If X admits a surjective morphism $f : X \longrightarrow Z$ with connected fibers onto a normal variety Z having non-surjective Albanese map and such that* $\dim X > \dim Z$*, then Y admits an irregular fibration.*

Proof. Let $Z \xrightarrow{f'} Z' \longrightarrow \mathrm{alb}_Z(Z)$ be the Stein factorization of alb_Z. By taking a non-singular representative of f' we can assume Z' smooth. We can easily check that Z' is of maximal Albanese dimension (so that $\mathcal{O}_{Z'} \in V^0(\omega_{Z'})$), and that $\mathrm{alb}_{Z'}$ is not surjective. Hence, by [EL] Proposition 2.2, there exists a positive-dimensional irreducible component V of $V^0(\omega_{Z'})$ passing through the origin. Moreover, by Lemma 6.3, we have $(f \circ f')^* V \subset V^k(\omega_X)_0$ where $k = \dim X - \dim Z'$, and by (5) we get $(f \circ f')^* V \subset V^k(\omega_X)_0 \subset V^1(\omega_X)_0$. Finally, by Theorem 1.4 (iii) there exists a positive-dimensional irreducible component $V' \subset V^1(\omega_Y)_0$ We conclude then by applying [GL2] Theorem 0.1. □

We point out that, thanks to Theorem 1.5, we can remove the hypothesis "$\dim \mathrm{alb}_X(X) \geq \dim X - 1$" from the above proposition in the case of threefolds. The following proposition, together with the subsequent remark, provides the proof of Corollary 1.9.

Proposition 7.3. *Let X and Y be smooth projective derived equivalent threefolds. Fix k to be either 1 or 2. Then X admits a higher irrational pencil $f : X \longrightarrow Z$ with $0 < \dim Z \leq k$ if and only if Y admits a higher irrational pencil $g : Y \longrightarrow W$ with $0 < \dim W \leq k$.*

[1] The minimality condition is necessary; see [PP2] Example 5.6.

Proof. We start with the case $k = 1$, and therefore we consider a higher irrational pencil $f : X \longrightarrow Z$ onto a smooth curve Z of genus $g(Z) \geq 2$. By Lemma 6.3, we have $f^*V^0(\omega_Z) = f^*\mathrm{Pic}^0(Z) \subset V^2(\omega_X)_0$, and by Theorem 1.5 (i) there exists a component $T \subset V^2(\omega_Y)_0$ such that

$$\dim T \geq q(Z) \geq 2. \tag{10}$$

Moreover, by [GL2] Theorem 0.1 or by [Be2] Corollaire 2.3, there exists an irrational fibration $g : Y \longrightarrow W$ onto a smooth curve W such that $T \subset g^*\mathrm{Pic}^0(W) + \gamma$ for some $\gamma \in \mathrm{Pic}^0(Y)$. Therefore we obtain the inequality

$$q(W) = g(W) \geq \dim T \geq 2 \tag{11}$$

ensuring that g is a higher irrational pencil.

We suppose now $k = 2$, and we consider a higher irrational pencil $f : X \longrightarrow Z$ onto a surface. It is a general fact that, by possibly replacing Z with a lower dimensional variety, one can furthermore assume $\chi(\omega_{Z'}) > 0$ for any smooth model Z' of Z (see [PP1] p. 271). If $\dim Z = 1$, then we apply the argument of the previous case. On the other hand, if $\dim Z = 2$ then by Lemma 6.3 we get

$$f^*V^0(\omega_Z) = f^*\mathrm{Pic}^0(Z) \subset V^1(\omega_X)_0.$$

Moreover, by Theorem 1.5, there exists a component $T \subset V^1(\omega_Y)_0$ such that $\dim T \geq q(Z') \geq 3$, and by [GL2] Theorem 0.1 there exists an irregular fibration $g : Y \longrightarrow W$ such that $T \subset g^*\mathrm{Pic}^0(W) + \gamma$ for some $\gamma \in \mathrm{Pic}^0(Y)$. Therefore $q(W) \geq \dim T \geq 3$ and g is a higher irrational pencil. □

Remark 7.4. We can slightly improve the statement of Proposition 7.3 in the case of fibrations onto curves. In fact, by going back to the proof of Proposition 7.3 in the case $k = 1$, we see that from the inequalities (10) and (11) we obtain the inequality $q(W) \geq q(Z)$. Then the following holds. Fix an integer $g \geq 2$. The variety X admits a higher irrational pencil $f : X \longrightarrow C$ onto a curve of genus $g(C) \geq g$ if and only if Y admits a higher irrational pencil $h : Y \longrightarrow D$ onto a curve of genus $g(D) \geq g$.

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