I. The exponential Equation or Natural Growth Equation

We are going to consider an initial value problem that serves as a mathematical model for a wide range of natural phenomena. This model is called exponential initial value problem or natural growth initial value problem. It is defined as follows:

\[
\begin{align*}
\frac{dx}{dt} &= kx \\
x(0) &= x_0
\end{align*}
\]

where \(k\) is a constant (it can be either positive or negative) and \(x = x(t)\) is a function of \(t\). In practice, the function \(x(t)\) represents a certain quantity evolving during the time \(t\), and \(x_0\) represents that quantity at time \(t = 0\). For instance, you can think that \(x(t)\) is the number of a certain population at time \(t\), or the number of atoms of a certain radioactive isotope contained in a given sample. In particular, we can suppose that both \(x(t)\) and \(x_0\) are positive since they represent physical quantities that cannot be negative. The constant \(k\) depends on the given problem and usually we need to determine it.

1.1. Solving the exponential initial value problem. We aim to solve the initial value problem \([1]\) with positive initial condition \(x_0 > 0\). We will see that its solution is \(x(t) = x_0 e^{kt}\).

The differential equation in \([1]\) is a separable differential equation. Hence we solve it by separating the variables. As we would like to divide by \(x\) the both sides of the equation, but we can not divide by a quantity that is zero, we first need to discuss the case \(x(t) = 0\). However we can easily check that the function \(x(t) = 0\) is a solution of the equation \(\frac{dx}{dt} = kx\) (but not of the initial value problem as in \([1]\) since we are assuming that \(x(0) = x_0 > 0\)). Now we suppose that \(x \neq 0\) and hence divide the equation \(\frac{dx}{dt} = kx\) by \(x\). Hence we have

\[
\frac{1}{x} \frac{dx}{dt} = k.
\]

Now we multiply both sides of the equation by \(dt\) to have:

\[
\frac{1}{x} dx = k dt.
\]

We now integrate the above equation to find:

\[
\int \frac{1}{x} dx = \int k dt + C
\]

where \(C\) is an arbitrary constant. By solving the integrals we have that

\[
\ln |x| = kt + C,
\]
and by applying the exponential function to both sides, we have

$$ |x| = e^{kt+C} = e^{kt}e^C = Ae^{kt} \text{ where we set } A = e^C. $$

Now we impose the initial condition $x(0) = x_0$. To this end it is enough to plug $t = 0$ and $x = x_0$ in the above equation. We have then (note that since $x_0$ is positive, we have $|x_0| = x_0$):

$$ x_0 = |x_0| = Ae^{k(0)} = A. $$

Hence the solution to the IVP (1) is

$$ x(t) = x_0e^{kt}. $$

2. Population growth

Let $P(t)$ be a function that denotes the number of individuals of a certain population having constant birth rate $\beta$ and constant death rate $\delta$. Moreover let $P_0$ be the number of individuals at time $t = 0$. We will find a differential equation that predicts how the population $P(t)$ will vary during the time.

For the population $P$ to have constant birth and death rates, it means that on an small interval of time $\Delta t$ there approximately $\beta P \Delta t$ births and $\delta P \Delta t$ deaths occurring in the interval of time $\Delta t$. Hence the variation of the population $\Delta P$ is given by:

$$ \Delta P = \beta P \Delta t - \delta P \Delta t = (\beta - \delta)P \Delta t. $$

Hence, by using the definition of the derivative as a limit, we have

$$ \frac{dP}{dt} = \lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} = \lim_{\Delta t \to 0} (\beta - \delta)P = (\beta - \delta)P = kP $$

where we set $k := \beta - \delta$.

Therefore the model that describes the evolution of a population $P$ with constant birth and death rates and with $P_0$ individuals at time $t = 0$ is

$$ \begin{cases} \frac{dP}{dt} = kP \\ P(0) = P_0 \end{cases} $$

where $k$ is a constant. This is an exponential initial value problem as in §1, and hence its solution is given by the function:

$$ P(t) = P_0e^{kt}. $$

2.1. Exercise. World’s total population was 6 billion in 1999 and was then increasing at a rate of 212 thousand persons per day throughout the year 1999. (a) What will be the world population in 2050? (b) How long will it take the world population to increase tenfold?

We start by solving question (a). The model that represents our problem is

$$ \begin{cases} \frac{dP}{dt} = kP \\ P(0) = P_0 \end{cases} $$
where $t$ is the time measured in years, the time $t = 0$ corresponds to the year 1999, $P(t)$ denotes the number of persons at the year $t$, and $P_0 = 6$ billion is the population at time $t = 0$ (i.e. in the year 1999). First of all we need to determine the constant $k$.

The information that the world population is increasing by 212 thousand, or equivalently by $0.000212$ billion, persons per day for the entire 1999, it means that at time $t = 0$ (or in 1999) the population increased by $(0.000212)(365.25) = 0.07743$ billion persons. This value is exactly the derivative of $P$ at time $t = 0$, hence

$$\left(\frac{dP}{dt}\right)_{t=0} = 0.07443.$$ 

We can then find the constant $k$ by plugging $t = 0$ into the equation $\frac{dP}{dt} = kP$, which becomes

$$\left(\frac{dP}{dt}\right)_{t=0} = k P(0)$$

and hence

$$0.07443 = k \cdot 6.$$ 

Solving for $k$ we find

$$k = \frac{0.07443}{6} = 0.0129.$$ 

The solution of the initial value problem (3) is

$$P(t) = P_0 e^{kt} = 6 e^{0.0129t}.$$ 

To find the population in the year 2050, it is then enough to plug $t = 51$ in the previous equation:

$$P(51) = \text{population in 2050} = 6 e^{0.0129(51)} = 11.58 \text{ billion}.$$ 

Now we solve question (b). To see at what time $t$ the population increases tenfold, it is enough to solve for $t$ the equation

$$P(t) = 10 P_0.$$ 

Since $P(t) = 6 e^{0.0129t}$ and $P_0 = 6$, we need to solve for $t$ the following equation

$$6 e^{0.0129t} = 60.$$ 

This gives $t = 178$ years that corresponds to the year 2177.

3. Radioactive Decay

Consider a sample of material that contains $N(t)$ atoms of a certain radioactive isotope at time $t$. It has been observed that a constant fraction of those atoms will spontaneously decay. Hence the sample behaves like a population with constant birth rate $\beta = 0$ and constant death rate $\delta$, which we call in this situation $k$.

The model that describes $N(t)$ is given then by

$$\frac{dN}{dt} = (\beta - \delta)N = -kN$$

where $k = \delta > 0$ is a positive constant.
The associated initial value problem is

\[
\begin{aligned}
\frac{dN}{dt} &= -kN \\
N(0) &= N_0
\end{aligned}
\]

where \( N_0 \) denotes the number of atoms at time \( t = 0 \). Hence \( N_0 \) is positive. To solve the previous IVP, we use the techniques of §1, hence we have:

\[ N(t) = N_0 e^{-kt}. \]

3.1. **Exercise.** A specimen of charcoal turns out to contain 63% as much \(^{14}\text{C}\) as a sample of present-day charcoal of equal mass. What is the age of the sample? Assume that the half-life of \(^{14}\text{C}\) is 5700 years.

The model that describes our problem is

\[
\begin{aligned}
\frac{dN}{dt} &= -kN \\
N(0) &= N_0
\end{aligned}
\]

where \( k > 0 \) is a positive constant. The solution of this problem is

\[ N(t) = N_0 e^{-kt}. \]

First of all we determine the constant \( k \). The information on the half-life, it simply means that

\[ N(5700) = \frac{1}{2} N_0, \]

i.e. after 5700 years the amount of \(^{14}\text{C}\) in the sample has halved. We then have

\[ N_0 e^{-k(5700)} = \frac{1}{2} N_0 \]

and we solve this equation for \( k \). First of all we divide both sides by \( N_0 \), and then we apply the \( \ln \) function to both sides. Hence one finds that

\[ k = -\ln \left( \frac{1}{2} \right) \frac{1}{5700} = 0.0001216. \]

Now we answer to the question of the problem. Notice that it is just asking at what time \( t \) we have that

\[ N(t) = 63\% \text{ of } N_0 = (0.63) N_0. \]

Since \( N(t) = N_0 e^{-0.0001216t} \), we need to solve for \( t \) the equation

\[ N_0 e^{-0.0001216t} = (0.63) N_0. \]

This gives \( t = 3800 \) years.