GV-SUBSCHEMES AND THEIR EMBEDDINGS IN PRINCIPALLY POLARIZED ABELIAN VARIETIES

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Abstract. We prove that the embedding of a GV-subscheme in a principally polarized abelian variety does not factor through any nontrivial isogeny. As an application, we present a new proof of a theorem of Clemens–Griffiths identifying the intermediate Jacobian of a smooth cubic threefold in $\mathbf{P}^4$ to the Albanese variety of its Fano surface of lines.

1. Introduction

Given a principally polarized abelian variety (A,Θ) with $\dim A = g$, we say that a reduced subscheme $X \subseteq A$ of dimension d has *minimal cohomology class* if $[X] = \frac{[\Theta]^{g-d}}{(g-d)!}$ in $H^{2g-2d}(A,\mathbf{Z})$. Beside the theta divisor of A itself, at the moment there are only two known families of subvarieties having minimal cohomology class. These are the Abel–Jacobi images of a symmetric product of a curve into its Jacobian and the Fano surfaces of lines embedded in intermediate Jacobians attached to smooth cubic threefolds in $\mathbf{P}^4$ (see [BL] and [CG]). Furthermore, Debarre in [D] conjectures that, extending in higher dimension questions of Beauville and Ran, these are the only possible examples when A is indecomposable.

The Matsusaka–Ran criterion yields this conjecture in the case of curves. Moreover, as shown in [D, Theorem 5.1], the only reduced subschemes of a Jacobian having minimal cohomology class are the aforementioned Brill–Noether loci. Finally, in [H2], Höring proves that if (A,Θ) is a general intermediate Jacobian of a smooth cubic threefold in $\mathbf{P}^4$, then the Fano surface of lines is the only subscheme having minimal cohomology class, with the exception, of course, of the theta divisor.

In [PP1] the authors propose to attack Debarre's conjecture by studying a cohomological condition on the twisted ideal sheaf $\mathscr{I}_X(\Theta)$ of a reduced closed subscheme X in a principally polarized abelian variety (A,Θ), by means of Fourier–Mukai transforms. More precisely we say that a subscheme X embedded in (A,Θ) is a *GV-subscheme* if

$$\operatorname{codim}_{\widehat{A}} \left\{ \alpha \in \widehat{A} \,|\, h^i(A,\mathscr{I}_X(\Theta)) > 0 \right\} \geq i \quad \text{for all} \quad i > 0$$

(*cf.* Definition 2.3). One of the main results of [PP1] is that geometrically nondegenerate[1] GV-subschemes have minimal cohomology class. Furthermore, Pareschi–Popa also prove

[1] A d-dimensional subvariety X of an abelian variety A is geometrically nondegenerate if the kernel of the restriction map $H^0(A,\Omega^d_A) \to H^0(X_{reg},\Omega^d_{X_{reg}})$ contains no nonzero decomposable d-forms.

that the only geometrically nondegenerate GV-subschemes in A of dimensions 1 and $g-2$ are the Abel–Jacobi images of a curve in its Jacobian and the Brill–Noether loci W_{g-2} respectively.

In this notes we derive some further geometric properties of GV-subschemes. Our main result is the following:

Theorem 1.1. *Let X be a reduced closed GV-subscheme of a principally polarized abelian variety (A, Θ) and let $\iota : X \hookrightarrow A$ be the closed immersion.*

(1) *If X generates A as a group, then the immersion $\iota : X \hookrightarrow A$ does not factor through any nontrivial isogeny.*
(2) *If X is of positive dimension and generates a proper abelian subvariety J of A, then A is a product of principally polarized abelian varieties having J as one of the factors. Furthermore, X is a GV-subscheme in J.*

In order to prove Theorem 1.1 we will use the theory of GV and M*-regular sheaves* on abelian varieties as described by Pareschi–Popa in [PP2, PP3] (*cf.* Definition 3.1). We will proceed as follows. First of all we show that in case the embedding ι factors through a nontrivial isogeny $\varphi : B \to A$, then $\mathscr{I}_{X/B} \otimes \varphi^*\mathscr{O}_A(\Theta)$, the ideal sheaf of X in B twisted by the pull-back of Θ, is a GV-sheaf on B. By using a characterization of GV-sheaves this means that the Fourier-Mukai transform of the dual of $\mathscr{I}_{X/B} \otimes \varphi^*\mathscr{O}_A(\Theta)$ can be written as a tensor product of an ideal sheaf of a subscheme of codimension at least two and a line bundle M on $\widehat{B}$. Finally, the last step is to deduce positivity properties on M. In particular, by using our generation hypothesis on X, we show that the line bundle M is ample on $\widehat{B}$ and moreover that the pull-back φ^*M is a principal polarization. This immediately yields that $\deg \varphi = h^0(B, \varphi^*M) = 1$.

In the last section we give an application of Theorem 1.1 by providing a new proof of a theorem of Clemens–Griffiths stating that the intermediate Jacobian of a smooth cubic threefold in $\mathbf{P}^4$ is isomorphic to the Albanese variety of its Fano surface of lines ([CG, Theorems 11.19]). Besides Theorem 1.1 this proof also relies on a theorem of Höring showing that the Fano surface of lines of a smooth cubic threefold in $\mathbf{P}^4$ is a GV-subscheme ([H1, Theorem 1.2]).

Notation. Throughout this paper we work over an algebraically closed field K of characteristic zero unless otherwise specified. Given a smooth variety Z over K, we will denote by $\mathbf{D}(Z)$ the derived category of bounded complexes of quasi-coherent sheaves on Z having coherent cohomology. If A is an abelian variety, we denote by 0_A the neutral element of A, by $(-1)_A$ the multiplication by -1, and by $\widehat{A}$ the dual abelian variety. Moreover, if $\Theta \subseteq A$ is a principal polarization, we denote by $\widehat{\Theta} \subset \widehat{A}$ the dual polarization.

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2. Background Material

2.1. **Fourier–Mukai Transforms.**

One of the main tools we will use in this paper is the Fourier–Mukai transform between the derived category of an abelian variety A of dimension g and the derived category of its dual $\widehat{A}$. This is defined as follows. Let $\mathscr{P}$ be a normalized Poincaré line bundle on $A \times \widehat{A}$, then we define the functor

$$\mathbf{R}S_A : \mathbf{D}(A) \longrightarrow \mathbf{D}(\widehat{A})$$

by setting

$$\mathbf{R}S_A(-) = \mathbf{R}p_{\widehat{A}\,*}(p_A^*(-) \otimes \mathscr{P})$$

where p_A and $p_{\widehat{A}}$ are the first and second projections from the product $A \times \widehat{A}$, respectively. Similarly we can also consider the Fourier–Mukai transform in the other direction, again induced by the Poincaré line bundle $\mathscr{P}$

$$\mathbf{R}\widehat{S}_A : \mathbf{D}(\widehat{A}) \longrightarrow \mathbf{D}(A)$$

by setting

$$\mathbf{R}\widehat{S}_A(-) = \mathbf{R}p_{A\,*}(p_{\widehat{A}}^*(-) \otimes \mathscr{P}).$$

Mukai's inversion theorem [M1, Theorem 2.2] tells us that these functors are equivalences of triangulated categories. More precisely the following formulas hold:

$$\mathbf{R}S_A \circ \mathbf{R}\widehat{S}_A \simeq (-1_{\widehat{A}})^* \circ [-g], \quad \mathbf{R}\widehat{S}_A \circ \mathbf{R}S_A \simeq (-1_A)^* \circ [-g]; \tag{2.1}$$

where $[-]$ stands for, as usual, the shift functor in a triangulated category.

For an arbitrary abelian variety A we denote by

$$\mathbf{R}\Delta_A : \mathbf{D}(A) \to \mathbf{D}(A), \quad \mathscr{F} \mapsto \mathbf{R}\mathcal{H}om_A(\mathscr{F}, \mathscr{O}_A)$$

the derived dual functor. The commutativity between the derived dual and the Fourier–Mukai transforms $\mathbf{R}S_A$ and $\mathbf{R}\widehat{S}_A$ is described by the following formulas, again due to Mukai ([M1, (3.8) and Theorem 3.13]):

$$\mathbf{R}\Delta_{\widehat{A}} \circ \mathbf{R}S_A \simeq ((-1_{\widehat{A}})^* \circ \mathbf{R}S_A \circ \mathbf{R}\Delta_A)[g], \quad \mathbf{R}\Delta_A \circ \mathbf{R}\widehat{S}_A \simeq ((-1_A)^* \circ \mathbf{R}\widehat{S}_A \circ \mathbf{R}\Delta_{\widehat{A}})[g]. \tag{2.2}$$

2.2. ***GV*-subschemes.** In this subsection we recall a few facts regarding *GV-sheaves* and introduce the main characters of this work, i.e. *GV-subschemes*.

Let A be an abelian variety of dimension g and let $\mathscr{F}$ be a coherent sheaf on A. For any $i \geq 0$ we define the *i-th cohomological support locus of $\mathscr{F}$* as

$$V_A^i(\mathscr{F}) := \{\alpha \in \widehat{A} \,|\, h^i(A, \mathscr{F} \otimes \alpha) > 0\}.$$

Definition 2.1. Given an integer $k \geq 0$, we say that $\mathscr{F}$ is a GV_k-sheaf if one of the following equivalent conditions holds:[2]

(1) $\operatorname{codim}_{\widehat{A}} V_A^i(\mathscr{F}) \geq i + k$ for every $i > 0$

(2) $\operatorname{codim}_{\widehat{A}} \operatorname{Supp} R^i S_A(\mathscr{F}) \geq i + k$ for every $i > 0$.

Usually GV_0-sheaves are simply called GV, while GV_1-sheaves are said *M-regular sheaves*. Also note that M-regular sheaves are GV. We highlight the following result giving equivalent conditions for a sheaf to be either GV or M-regular (we refer to [PP2, PP3] for further generalities regarding GV_k-sheaves).

Theorem 2.2 ([PP3, Theorem 2.3 and Proposition 2.8])**.** *Let A be an abelian variety of dimension g over an algebraically closed field and let $\mathscr{F}$ be a coherent sheaf on A.*

(1) *The sheaf $\mathscr{F}$ is GV if and only if the cohomology sheaves $R^i S_A \mathbf{R}\Delta_A(\mathscr{F})$ vanish for every $i \neq g$. Furthermore, if this is the case, then we have $\operatorname{rk} R^g S_A \mathbf{R}\Delta_A(\mathscr{F}) = \chi(A, \mathscr{F})$.*

(2) *If $\mathscr{F}$ is a GV-sheaf, then it is M-regular if and only if $R^g S_A \mathbf{R}\Delta_A(\mathscr{F})$ is a torsion free sheaf.*

Definition 2.3. We say that a reduced closed subscheme X of a principally polarized abelian variety (A, Θ) defined by an ideal sheaf $\mathscr{I}_X \subset \mathscr{O}_A$ is a *GV-subscheme* if $\mathscr{I}_X(\Theta) := \mathscr{I}_X \otimes \mathscr{O}_A(\Theta)$ is a GV-sheaf.

A useful criterion to detect GV-subschemes is the following:

Proposition 2.4 ([PP1, Lemma 3.3])**.** *Let $\iota : X \hookrightarrow A$ be a reduced closed subscheme of a principally polarized abelian variety (A, Θ) of dimension g. Then X is a GV-subscheme if and only if $\iota_* \mathscr{O}_X(\Theta)$ is an M-regular sheaf with $\chi(A, \iota_* \mathscr{O}_X(\Theta)) = 1$. In particular, if X is a GV-subscheme, then there is an isomorphism of complexes*

$$\mathbf{R} S_A \mathbf{R}\Delta_A(\iota_* \mathscr{O}_X(\Theta)) \simeq \mathscr{I}_Z \otimes L[-g], \tag{2.3}$$

where $\mathscr{I}_Z \subset \mathscr{O}_{\widehat{A}}$ is an ideal sheaf whose zero locus does not contain any divisorial components and L is the reflexive hull of $R^g S_A \mathbf{R}\Delta_A(\iota_* \mathscr{O}_X(\Theta))$.

[2] A proof of their equivalence is in [PP2, Lemma 3.6].

Remark 2.1. With notation as in the previous proposition, we observe that from Proposition 2.4 and [M1, Theorem 3.13 (5)] there is an inclusion of sheaves $\mathscr{I}_Z \otimes L \hookrightarrow \mathscr{O}_{\widehat{A}}(\widehat{\Theta})$ obtained by applying the functor $\mathbf{R}S_A \mathbf{R}\Delta_A$ to the exact sequence

$$0 \longrightarrow \mathscr{I}_X(\Theta) \longrightarrow \mathscr{O}_A(\Theta) \longrightarrow \iota_* \mathscr{O}_X(\Theta) \longrightarrow 0.$$

Moreover, by taking $\mathscr{H}om$'s, we get a further inclusion

$$\mathscr{O}_{\widehat{A}}(-\widehat{\Theta}) \hookrightarrow L^{-1}$$

so that line bundle $\mathscr{O}_{\widehat{A}}(\widehat{\Theta}) \otimes L^{-1}$ has one nonzero global section which can be written as $\mathscr{O}_{\widehat{A}}(E)$ for some effective divisor E on $\widehat{A}$. Suppose now that L itself has one nonzero global section so that $L \simeq \mathscr{O}_{\widehat{A}}(E')$ for some effective divisor E', then only two possibilities occur:

(1) either L is ample, and therefore $E = 0$ and $E' \simeq \Theta$;

(2) or L is not ample. In this case we have then that the divisor $\widehat{\Theta} = E + E'$ is reducible as $\widehat{\Theta}$ is a principal polarization, and hence, by the Decomposition Theorem in [BL, Theorem 4.3.1], A is a product of nontrivial principally polarized abelian varieties.

3. Proof of the main theorem

The results stated in the Introduction are consequences of the following technical statement:

Proposition 3.1. *Let $\iota : X \hookrightarrow A$ be a GV-subscheme of a principally polarized abelian variety (A, Θ) and suppose that the inclusion $\iota : X \hookrightarrow A$ factors through a nontrivial isogeny φ as in the diagram below*

$$\begin{tikzcd} X \arrow[rr, "a"] \arrow[dr, "i"'] & & B \arrow[dl, "\varphi"] \\ & A. & \end{tikzcd}$$

Then there is an isomorphism of complexes of sheaves

$$\mathbf{R}S_B \mathbf{R}\Delta_B(a_* \mathscr{O}_X \otimes \varphi^* \mathscr{O}_A(\Theta)) \simeq \mathscr{I}_W \otimes M[-g]$$

where $\mathscr{I}_W$ is an ideal sheaf on $\widehat{B}$ whose zero locus does not contain any divisorial components, M is a line bundle on $\widehat{B}$ such that $\widehat{\varphi}^* M \simeq L$, and L is the line bundle appearing in (2.3).

Proof. In [M1, (3.4) p. 159] Mukai studied the behavior of the Fourier–Mukai transform with respect to an arbitrary isogeny. Using his result, together to Proposition 2.4, we get the following chain of isomorphisms in the derived category $\mathbf{D}(\widehat{A})$:

$$\begin{aligned}\mathscr{I}_Z \otimes L[-g] &\simeq \mathbf{R}S_A \mathbf{R}\Delta_A(\iota_* \mathscr{O}_X(\Theta)) \\ &\simeq \mathbf{R}S_A \mathbf{R}\Delta_A(\varphi_* a_* \mathscr{O}_X(\Theta)) \\ &\simeq \mathbf{R}S_A \mathbf{R}\Delta_A(\varphi_*(a_* \mathscr{O}_X \otimes \varphi^* \mathscr{O}_A(\Theta))) \\ &\simeq \widehat{\varphi}^* \mathbf{R}S_B \mathbf{R}\Delta_B(a_* \mathscr{O}_X \otimes \varphi^* \mathscr{O}_A(\Theta)).\end{aligned}$$

Therefore, by using the fact that φ is a flat morphism, we have that

$$\widehat{\varphi}^*(R^i S_B \mathbf{R}\Delta_B(a_* \mathscr{O}_X \otimes \varphi^* \mathscr{O}_A(\Theta)) = 0 \quad \text{for} \quad i < g.$$

From this it follows easily that

$$R^i S_B \mathbf{R}\Delta_B(a_* \mathscr{O}_X \otimes \varphi^* \mathscr{O}_A(\Theta)) = 0 \quad \text{for } i < g,$$

and therefore

$$\mathbf{R}S_B \mathbf{R}\Delta_B(a_* \mathscr{O}_X \otimes \varphi^* \mathscr{O}_A(\Theta)) \simeq \mathscr{F}[-g]$$

for some coherent sheaf $\mathscr{F}$ on $\widehat{B}$.

By shifting complexes to the left, we get an isomorphism of sheaves

$$\mathscr{I}_Z \otimes L \simeq \widehat{\varphi}^* \mathscr{F}$$

from which we can deduce that $\mathscr{F}$ has generic rank equals to 1. Furthermore, we get an inclusion of sheaves

$$\mathscr{F} \hookrightarrow \widehat{\varphi}_*(\widehat{\varphi}^* \mathscr{F}) \simeq \widehat{\varphi}_*(\mathscr{I}_Z \otimes L),$$

from which we notice that since the latter is torsion free, then so is $\mathscr{F}$. Therefore we can write

$$\mathscr{F} \simeq \mathscr{I}_W \otimes M$$

where M is the reflexive hull of $\mathscr{F}$ and $\mathscr{I}_W$ is an ideal sheaf whose zero locus does not contain any divisorial components. Finally, by applying $\mathscr{H}om_{\mathscr{O}_{\widehat{A}}}(-, \mathscr{O}_{\widehat{A}})$ to the isomorphism $\widehat{\varphi}^*(\mathscr{I}_W \otimes M) \simeq \mathscr{I}_Z \otimes L$, we get $(\widehat{\varphi}^* M)^{-1} \simeq L^{-1}$. As a consequence, we have $\widehat{\varphi}^* M \simeq L$ and the statement is proved. □

We now divide the proof of Theorem 1.1 into two cases: when X generates A, and when it does not.

3.1. **Case 1: When X generates A (proof of Theorem 1.1 (1)).**

Suppose that the hypotheses of Theorem 1.1 (1) hold, i.e. $\iota : X \hookrightarrow (A, \Theta)$ is an embedding of a reduced GV-subscheme in a g-dimensional principally polarized abelian variety such that X generates A as a group. We are going to show that the inclusion $\iota : X \hookrightarrow A$ does not factor through any nontrivial isogeny. By arguing by contradiction, if it did, then we would be in the same situation of diagram (3.4) under the extra information that $a(X)$ spans B. Then, by using Proposition 3.1 we can conclude that

$$\mathbf{R}S_B \mathbf{R}\Delta_B(a_* \mathscr{O}_X \otimes \varphi^* \mathscr{O}_A(\Theta)) \simeq \mathscr{I}_W \otimes M[-g]. \tag{3.1}$$

The key point of our argument is the following general:

Lemma 3.2. *Let $\mathscr{F}$ be an M-regular sheaf on an abelian variety A of dimension g whose support spans A. If $\chi(A,\mathscr{F})=1$, then the reflexive hull of $R^gS_A\mathbf{R}\Delta_A(\mathscr{F})$ is an ample line bundle on $\widehat{A}$.*

Proof. Denote by L the reflexive hull of $R^gS_A\mathbf{R}\Delta_A(\mathscr{F})$. By Theorem 2.2 it is a line bundle on $\widehat{A}$ such that

$$\mathbf{R}S_A\mathbf{R}\Delta_A(\mathscr{F})\simeq\mathscr{I}\otimes L[-g]$$

for some ideal sheaf $\mathscr{I}$ on $\widehat{A}$. Moreover, by applying Mukai's inversion theorem we get an isomorphism

$$\mathbf{R}\widehat{S}_A\mathbf{R}\Delta_{\widehat{A}}(\mathscr{I}\otimes L)\simeq\mathscr{F}[-g]. \tag{3.2}$$

Let $\mathscr{P}$ be a Poincaré line bundle on $\widehat{A}\times A$ and denote by $\mathscr{P}_x$ the restriction $\mathscr{P}_{|\widehat{A}\times\{x\}}$. If by contradiction L were not ample, then there would exist a proper abelian subvariety $G\subsetneq A$ such that $H^0(\widehat{A},L\otimes\mathscr{P}_x)=0$ for any $x\notin G$. Therefore

$$\begin{aligned}0 &= H^0(\widehat{A},\mathscr{I}\otimes L\otimes\mathscr{P}_x)\\ &\simeq \mathrm{Hom}_{\mathbf{D}(\widehat{A})}(\mathscr{O}_{\widehat{A}},\mathscr{I}\otimes L\otimes\mathscr{P}_x)\\ &\simeq \mathrm{Hom}_{\mathbf{D}(\widehat{A})}(\mathbf{R}\Delta_{\widehat{A}}(\mathscr{I}\otimes L),\mathscr{P}_x),\end{aligned}$$

and by applying the Fourier–Mukai transform $\mathbf{R}\widehat{S}_A$, together to the formulas (2.1) and (3.2), we would get

$$\begin{aligned}0 &= \mathrm{Hom}_{\mathbf{D}(A)}(\mathbf{R}\widehat{S}_A\circ\mathbf{R}\Delta_{\widehat{A}}(\mathscr{I}\otimes L),\mathbf{R}\widehat{S}_A(\mathscr{P}_x))\\ &\simeq \mathrm{Hom}_{\mathbf{D}(A)}(\mathscr{F}[-g],\mathscr{O}_x[-g])\\ &\simeq \mathrm{Hom}_{\mathbf{D}(A)}(\mathscr{F},\mathscr{O}_x)\end{aligned}$$

where $\mathscr{O}_x$ denotes the skyscraper sheaf at x. This says that the point x does not belong to the support of $\mathscr{F}$. Thus we conclude that $\mathrm{Supp}(\mathscr{F})\subset G$ which contradicts the hypothesis that $\mathrm{Supp}(\mathscr{F})$ generates A. □

Corollary 3.3. *The line bundle M in (3.1) is ample on $\widehat{B}$.*

Proof. From Proposition 3.1 and Theorem 2.2 we deduce that $a_*\mathscr{O}_X\otimes\varphi^*\mathscr{O}_A(\Theta)$ is M-regular with Euler characteristic equals to 1. The hypotheses of our setting grant that $\mathrm{Supp}\left(a_*\mathscr{O}_X\otimes\varphi^*\mathscr{O}_A(\Theta)\right)=a(X)$ spans B. The corollary then follows by the lemma above. □

As a consequence of the previous discussion, the line bundle $L\simeq\widehat{\varphi}^*M$ is ample too, and therefore, by Remark 2.1, it is a principal polarization. We deduce immediately that $\widehat{\varphi}$ is an isomorphism, and thus so is φ.

3.2. Case 2: When X does not generate A (proof of Theorem 1.1 (2)).

Suppose now that X generates a proper abelian subvariety $J = \langle X \rangle \subsetneq A$. Then by Poincaré's reducibility theorem there exists an abelian subvariety $B \subset A$ such that the restriction $\psi : J \times B \to A$ of the multiplication map from A to $J \times B$ is an isogeny such that

$$\psi^* \mathscr{O}_A(\Theta) \simeq \mathscr{O}_A(\Theta)_{|J} \boxtimes \mathscr{O}_A(\Theta)_{|B}. \tag{3.3}$$

From the very definition of ψ there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\iota'"] \arrow[d, "\iota"] & J \arrow[d, "(\mathrm{id}_J, 0_B)"] \\ A & J \times B \arrow[l, "\psi"] \end{tikzcd} \tag{3.4}$$

where ι and ι' are the inclusion maps of X in A and X in J respectively, and $(\mathrm{id}_J, 0_B)$ is the morphism taking p to $(p, 0_B)$. Finally, we denote by $a : X \to J \times B$ the composition $(\mathrm{id}_J, 0_B) \circ \iota'$.

Observe that by Proposition 2.4 we have an isomorphism

$$\mathbf{R}S_A \mathbf{R}\Delta_A(\iota_* \mathscr{O}_X(\Theta)) \simeq \mathscr{I}_Z \otimes L[-\dim A]$$

where $\mathscr{I}_Z$ is a sheaf of ideals such that Z does not contain any divisorial component and L is a line bundle on $\widehat{A}$. Moreover, by Proposition 3.1, there is an isomorphism

$$\mathbf{R}S_{J\times B} \mathbf{R}\Delta_{J \times B}(a_* \mathscr{O}_X \otimes \psi^* \mathscr{O}_A(\Theta)) \simeq \mathscr{I}_W \otimes M[-g] \tag{3.5}$$

where $\mathscr{I}_W$ is an ideal sheaf on $\widehat{J} \times \widehat{B}$ whose zero locus does not contain any divisorial component, and M is a line bundle on $\widehat{J} \times \widehat{B}$ such that $\widehat{\psi}^* M \simeq L$. Finally we notice that the left hand side of the above isomorphism can be written as follows:

$$\begin{aligned} &\mathbf{R}S_{J\times B} \mathbf{R}\Delta_{J \times B}(a_* \mathscr{O}_X \otimes \psi^* \mathscr{O}_A(\Theta)) \\ &\simeq \mathbf{R}S_{J\times B} \mathbf{R}\Delta_{J \times B}(a_* \mathscr{O}_X \otimes (\mathscr{O}_A(\Theta)_{|J} \boxtimes \mathscr{O}_A(\Theta)_{|B})) \\ &\simeq \mathbf{R}S_{J\times B} \mathbf{R}\Delta_{J \times B}(\iota'_*(\mathscr{O}_X \otimes \mathscr{O}_A(\Theta)_{|J}) \boxtimes (\mathscr{O}_{0_B} \otimes \mathscr{O}_A(\Theta)_{|B})) \\ &\simeq \mathbf{R}S_J \mathbf{R}\Delta_J(\iota'_* \mathscr{O}_X \otimes \mathscr{O}_A(\Theta)_{|J}) \boxtimes \mathbf{R}S_B(\mathbf{R}\Delta_B(\mathscr{O}_{0_B})) \\ &\simeq \mathbf{R}S_J \mathbf{R}\Delta_J(\iota'_* \mathscr{O}_X \otimes \mathscr{O}_A(\Theta)_{|J}) \boxtimes \mathscr{O}_{\widehat{B}}[-\dim \widehat{B}]. \end{aligned}$$

where $\mathscr{O}_{0_B}$ denotes the skyscraper sheaf at 0_B. We deduce that

$$R^i S_J \mathbf{R}\Delta_J(\iota'_* \mathscr{O}_X \otimes \mathscr{O}_A(\Theta)_{|J}) = 0 \quad \text{for every} \quad i < \dim J,$$

and that $R^{\dim J} S_J \mathbf{R}\Delta_J(\iota'_* \mathscr{O}_X \otimes \mathscr{O}_A(\Theta)_{|J})$ is a torsion free sheaf of rank 1. Therefore we can write

$$\mathbf{R}S_J \mathbf{R}\Delta_J(\iota'_* \mathscr{O}_X \otimes \mathscr{O}_A(\Theta)_{|J}) \simeq \mathscr{I}_{W'} \otimes M'[-\dim J] \tag{3.6}$$

where $\mathscr{I}_{W'}$ is an ideal sheaf such that its zero locus does not contain any divisorial components and M' is a line bundle on $\widehat{J}$. Then, by combining (3.6) with (3.5) we obtain a series of isomorphisms

$$\mathscr{I}_W \otimes M[-g] \simeq \big(\mathscr{I}_{W'} \otimes M'[-\dim J]\big) \boxtimes \mathscr{O}_{\widehat{B}}[-\dim \widehat{B}] \simeq \big(\mathscr{I}_{W'} \otimes M'\big) \boxtimes \mathscr{O}_{\widehat{B}}[-g]$$

from which we see that $M \simeq M' \boxtimes \mathscr{O}_{\widehat{B}}$ since they are reflexive hulls of isomorphic sheaves.

Finally, by using Lemma 3.2, and by reasoning as in Corollary 3.3, we get the following

Corollary 3.4. *The line bundle M' in (3.6) is ample on $\widehat{J}$.*

Therefore we can conclude that

$$L \simeq \widehat{\psi}^* M \simeq \widehat{\psi}^*(M' \boxtimes \mathscr{O}_{\widehat{B}})$$

is effective but not ample. Furthermore, from Remark 2.1 $\widehat{A}$ can be written as a non-trivial product $\widehat{A} \simeq \widehat{A}_1 \times \widehat{A}_2$ and, up to switching factors, there is an isomorphism $L \simeq p_1^*(\mathscr{O}_{\widehat{A}_1}(\Theta_1))$ where Θ_1 is a principal polarization on $\widehat{A}_1$. However more is true. In fact, by Künneth formula, we obtain inequalities

$$0 < h^0(\widehat{J}, M') = h^0(\widehat{J}, M') \cdot h^0(\widehat{B}, \mathscr{O}_{\widehat{B}}) \leq h^0(\widehat{A}, L) = 1$$

which imply that M' is a principal polarization on $\widehat{J}$. Furthermore, since L is obviously trivial on $\widehat{\psi}^{-1}(\{0_{\widehat{J}}\} \times \widehat{B})$, we have the following commutative diagram:

$$\begin{tikzcd}
\widehat{A} \simeq \widehat{A}_1 \times \widehat{A}_2 \arrow[r, "\widehat{\psi}"] \arrow[d, "p_1"'] & \widehat{J} \times \widehat{B} \arrow[d, "q_J"] \\
\widehat{A}_1 \arrow[r, "\pi"'] & \widehat{J} \simeq \widehat{A}_1 / p_1\big(\widehat{\psi}^{-1}\big(\{0_{\widehat{J}}\} \times \widehat{B}\big)\big)
\end{tikzcd}$$

Finally, by projection formula we have the following chain of equalities of algebraic sets

$$A_1 \times \{0_{A_2}\} = V^0_{\widehat{A}_1 \times \widehat{A}_2}(L) = \psi(V^0_{\widehat{J} \times \widehat{B}}(M' \boxtimes \mathscr{O}_{\widehat{B}})) = \psi(J \times \{0_B\})$$

from which we conclude that π is an isogeny. We deduce that A_1 and J have the same dimension and further that $h^0(A_1, \pi^* M') = 1$ since $L \simeq p_1^* \pi^* M'$. Therefore we conclude that π is an isomorphism and that $\widehat{J}$ is a direct factor of $\widehat{A}$ such that $\mathscr{O}_{\widehat{A}}(\Theta)_{|\widehat{J}}$ is a principal polarization. This immediately says that J is a direct factor of A and that $\mathscr{O}_A(\Theta)_{|J}$ is a principal polarization on J.

In order to complete the proof of Theorem 1.1 we need to show that $\mathscr{I}_{X/J} \otimes \mathscr{O}_A(\Theta)_{|J}$ is a GV-sheaf. To this end, consider the short exact sequence

$$0 \longrightarrow \mathscr{I}_{X/J} \otimes \mathscr{O}_A(\Theta)_{|J} \longrightarrow \mathscr{O}_A(\Theta)_{|J} \longrightarrow \iota'_* \mathscr{O}_X \otimes \mathscr{O}_A(\Theta)_{|J} \longrightarrow 0$$

to which we apply the functor $\mathbf{R}S_J \mathbf{R}\Delta_J$. By taking cohomology we see that

$$R^i S_J \mathbf{R}\Delta_J(\mathscr{I}_{X/J} \otimes \mathscr{O}_A(\Theta)_{|J}) \simeq R^{i+1} S_J \mathbf{R}\Delta_J(\mathscr{O}_X \otimes \mathscr{O}_A(\Theta)_{|J}) = 0 \quad \text{for} \quad i < g-1.$$

In addition, we have that

$$R^{g-1} S_J \mathbf{R}\Delta_J(\mathscr{I}_{X/J} \otimes \mathscr{O}_A(\Theta)_{|J}) = 0$$

since it is the kernel of a nonzero morphism between torsion free sheaves of equal rank. We conclude by invoking Theorem 2.2

4. An application

We apply Theorem 1.1 to provide a new proof of a theorem of Clemens–Griffiths [CG, Theorem 11.19 and (0.8)].

Theorem 4.1 ([CG, Theorem 11.19])**.** *Let $Y \subset \mathbf{P}^4$ be a smooth cubic threefold and let S be the Fano scheme parametrizing lines of $\mathbf{P}^4$ contained in Y. Moreover, denote by $\mathrm{Alb}(S)$ the Albanese variety of S and by $J(Y)$ the intermediate Jacobian of Y. Then $\mathrm{Alb}(S)$ and $J(Y)$ are isomorphic.*

Proof. They are classical results that the irregularity $q(S) = \dim H^1(S, \mathscr{O}_S)$ of S equals 5, and that the map $\pi : \mathrm{Alb}(S) \to J(Y)$ induced by the universal property of the Albanese variety is an isogeny. For instance, this was observed by Fano and Gherardelli over the complex numbers, and by Altman–Kleiman and Murre over any algebraically closed field of characteristic different from 2 [G, AK, M2]. In particular, we notice that S generates $J(Y)$ as a group. Moreover, Höring in [H1] proves that S is a GV-subscheme in $J(Y)$ by only exploiting the structure of a Prym variety on the intermediate Jacobian $J(Y)$. Thus all the hypotheses of Theorem 1.1 (1) are satisfied and so π is an isomorphism. □

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