MATH 211: Introduction to Linear Algebra - Fall 2017

Answer Keys to Practice Midterm 2

Problem 1. (1). A basis of U_2 is $\mathcal{B} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$. The dimension of U_2 is 3. An isomorphism between U_2 and $\mathbf{R}^3$ is given by $L\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. The inverse is given by $L^{-1}\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} \right) = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$.

(2). A basis of W is $\mathcal{B} = \left(\begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$. The dimension of W is 2.

Problem 2. A basis of the subspace of even polynomials of degree at most 3 is $\mathcal{B} = (1, x^2)$. The dimension is 2.

Problem 3. (1). A basis of the image of T is $\mathcal{B} = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix} \right)$. The dimension of the image is 4. The dimension of the kernel is $\dim \operatorname{Ker}(T) = \dim \mathbf{R}^{2\times 2} - \dim \operatorname{Im}(T) = 4 - 4 = 0$. Therefore the empty set is a basis of the kernel. The linear function T is an isomorphism.

(2). A basis of the image of T is $\mathcal{B} = (1, 2x - 2, 3x^2 - 6x)$. Its dimension is 3. A basis of the kernel of T is $\mathcal{C} = (1)$. Its dimension is 1. The linear function T is not an isomorphism.

(3). A basis of the image is $\mathcal{B} = (-2, -2x, 2 - 2x^2)$. The dimension of the image is 3. Therefore the dimension of the kernel is $\dim \operatorname{Ker}(T) = 3 - 3 = 0$. It follows that $\operatorname{Ker}(T) = \{0\}$. The linear function T is an isomorphism.

(4). A basis of the image of T is $\mathcal{B} = \left(\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right)$. Its dimension is 2. A basis of the kernel of T is $\mathcal{C} = (x^2 - x)$. Its dimension is 1. The linear function T is not an isomorphism.

Problem 4. A basis of U is $\left(\begin{pmatrix} 1/2 \\ 5/4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right)$. A basis of $U^\perp$ is $\left(\begin{pmatrix} 1 \\ 0 \\ -1/2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -5/4 \\ 1 \end{pmatrix} \right)$. Both U and $U^\perp$ have dimension 2.

Problem 5. One possible basis of $\mathbf{R}^3$ is $\mathcal{B} = \left(\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \right)$. The diagonal matrix B that represents T with respect to $\mathcal{B}$ is $B = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$. The linear transformation T is an isomorphism because B is invertible (its rank is 3).

Problem 6. A basis of the span is $\mathcal{B} = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right)$. Its dimension is 3. The vector $\begin{pmatrix} 7 \\ 3 \\ 3 \\ 3 \end{pmatrix}$ can be written as a linear combination of the vectors of the basis $\mathcal{B}$ as follows

$$\begin{pmatrix} 7 \\ 3 \\ 3 \\ 3 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

Therefore its $\mathcal{B}$-coordinates are $\left[\begin{pmatrix} 7 \\ 3 \\ 3 \\ 3 \end{pmatrix} \right]_{\mathcal{B}} = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$.

Problem 7. The given vectors form a basis of $\mathbf{R}^3$ if $k \neq 1$ and $k \neq -1$.

Problem 8. F, F, T, T.