MATH 211: Introduction to Linear Algebra - Fall 2017

Notes about isomorphisms

Isomorphisms

Definition of isomorphism. A *real* vector space V is a vector space in which the scalars are real numbers. In our course and in the rest of these notes we only consider real vector spaces.

A function $T\colon V \to W$ between two vector spaces V and W is called ***linear*** if it preserves the sum and the scalar multiplication. In formulas this amounts to say that

$$T(x+y) = T(x) + T(y) \quad \text{for all } x, y \text{ in } V$$

and

$$T(cx) = cT(x) \quad \text{for all } x \text{ in } V \text{ and all scalars } c \text{ in } \mathbf{R}.$$

We say that a function $T\colon V \to W$ is an ***isomorphism*** if T is linear and T is invertible. We recall that T is invertible if there exists a linear function $T^{-1}\colon W \to V$ such that

$$T^{-1}\big(T(x)\big) = x \quad \text{for all } x \text{ in } V$$
$$T\big(T^{-1}(y)\big) = y \quad \text{for all } y \text{ in } W.$$

In many cases it is a difficult task to find the inverse T^{-1} of a linear function T. However in order to check that T is an isomorphism, we will usually calculate its kernel and image, as stated in the following theorem.

Theorem 1. *Suppose that $T\colon V \to W$ is a linear function between two vector spaces. Then T is an isomorphism if and only if $\mathrm{Im}(T) = W$ and $\mathrm{Ker}(T) = \{0\}$.*

The above theorem is very general, and it works even if either V or W has infinite dimension.

Can you think about a vector space of infinite dimension? If not, here is an example. The set of all infinite sequences of real numbers

$$(x_0, x_1, x_2, \ldots) \text{ such that } x_i \text{ are in } \mathbf{R} \text{ for all } i = 0, 1, 2, \ldots$$

is a vector space of infinite dimension. We denote it by $\mathbf{R}^\infty$.

Another example of a vector space of infinite dimension is the set $F(\mathbf{R}, \mathbf{R})$ of all functions $f\colon \mathbf{R} \to \mathbf{R}$ from $\mathbf{R}$ to $\mathbf{R}$. The vector spaces P_n of polynomials of degree at most n are finite-dimensional subspaces of $F(\mathbf{R}, \mathbf{R})$. Try to prove that e^x and e^{2x} are linearly independent elements of $F(\mathbf{R}, \mathbf{R})$, but that $\cos^2(x), \sin^2(x), 1$ are linearly dependent in $F(\mathbf{R}, \mathbf{R})$.

Criteria for finite dimensional vector spaces. We have easier criteria when the vector spaces V and W have both finite dimensions. (In our course we will only work with finite-dimensional vector spaces.) The big advantage of working with finite-dimensional spaces is that we have the Rank-Nullity Theorem, which states:

$$\dim \mathrm{Ker}(T) + \dim \mathrm{Im}(T) = \dim V$$

for any linear function $T\colon V \to W$.

In this way we obtain two criteria for a linear function between finite-dimensional vector spaces to be an isomorphism.

Theorem 2 (Criterion 1)**.** *Suppose that $T\colon V \to W$ is a linear function between two vector spaces of finite dimension. Then T is an isomorphism if and only if $\dim \operatorname{Ker}(T) = 0$ and $\dim V = \dim W$.*

Theorem 3 (Criterion 2)**.** *Suppose that $T\colon V \to W$ is a linear function between two vector spaces of finite dimension. Then T is an isomorphism if and only if $\dim \operatorname{Im}(T) = \dim W$ and $\dim V = \dim W$.*

Exercise. Prove the above criteria.

In particular we note that if $T\colon V \to W$ is a linear function between two finite-dimensional vector spaces having different dimensions, then T is never an isomorphism. An important theorem in linear algebra states that if V is a real vector spaces of finite dimension n, then we can always find an isomorphism between V and $\mathbf{R}^n$. Therefore V is isomorphic to $\mathbf{R}^n$, and any two real vector spaces of dimension n are isomorphic (because they are both isomorphic to $\mathbf{R}^n$).

Problem. I am going to solve Problem 3 part (3) of the Practice Midterm 2. The problem is the following. Let $T\colon P_2 \to P_2$ be the linear function defined as $T\big(f(x)\big) = f''(x) - 2f(x)$. Find bases of the image and kernel and their dimensions. Say whether T is an isomorphism.

Solution. A polynomial $f(x)$ in P_2 looks like $f(x) = a + bx + cx^2$. The derivatives are $f'(x) = b + 2cx$ and $f''(x) = 2c$. Therefore T can be described as

$$T\big(f(x)\big) = 2c - 2(a + bx + cx^2) = a(-2) + b(-2x) + c(2 - 2x^2).$$

Writing $T(f(x))$ in this way it gives immediately a basis of the image of T. In fact a basis of the image is

$$\mathcal{B} = \Big(-2, -2x, 2 - 2x^2 \Big).$$

The dimension of the image is 3. Thanks to Theorem 3 we deduce that T is an isomorphism because $\dim \operatorname{Im}(T) = \dim P_2 = 3$. It follows that the kernel has dimension 0 by the Rank-Nullity Theorem. The only subspace that has dimension 0 is the zero subspace. Hence $\operatorname{Ker}(T) = \{0\}$ and the zero polynomial is its basis.

Problem. Consider the linear function $T\colon P_2 \to P_2$ defined as $T(f(x)) = f'(x)$. Say whether T is an isomorphism.

Solution. A polynomial $f(x)$ in P_2 looks like $f(x) = a + bx + cx^2$. Its derivative is $f'(x) = b + 2cx$. Therefore we can write $T(f(x))$ as

$$T(f(x)) = b + 2cx = b(1) + c(2x).$$

Hence a basis of the image is $\mathcal{B} = \big(1, 2x\big)$. Its dimension is 2. Thanks to Theorem 3 we already see that T is not an isomorphism because $\dim \operatorname{Im}(T) \neq \dim P_2$. The dimension of the kernel is 1, by the Rank-Nullity Theorem. To find a basis of the kernel we solve $T(f(x)) = 0$. Hence we want

$$b + 2cx = 0.$$

As a polynomial is equal to the zero polynomial if and only if all of its coefficients are zero, we obtain

$$b = 0, \quad 2c = 0.$$

Thus a polynomial $f(x) = a + bx + cx^2$ is in the kernel of T if and only if $b = c = 0$. It follows that all polynomials $f(x) = a = a(1)$ are in the kernel, and therefore $\mathcal{B} = \big(1\big)$ is a basis of $\operatorname{Ker}(T)$. There is no surprise here, because we already know from calculus that the first derivative of any constant is zero.

Further reading: Complex Numbers

I am going to show that the set of complex numbers has a vector space structure (over $\mathbf{R}$). What I am going to say here is not relevant to the course, but if you are interested you should continue reading.

Consider the set $\mathbf{C}$ of complex numbers defined as:

$$\mathbf{C} \ = \ \{x + yi \textit{ such that } x \textit{ and } y \textit{ are in } \mathbf{R} \textit{ and } i^2 = -1\}.$$

Note that i is not a real number! In fact the square root of a negative number does not exist. You should think at i as a symbol living outside $\mathbf{R}$, but satisfying some relations with real numbers. Note that any real number x is in particular a complex number because $x = x + 0i$.

The reason why one works with $\mathbf{C}$ rather than $\mathbf{R}$ is that the polynomial $x^2 + 1$ has no real roots, but the complex number i is a root of $x^2 + 1$ (the other root is $-i$). A deep result in algebra is that every non-constant one-single variable polynomial with real coefficients admits one complex root (this theorem is called Fundamental Theorem of Algebra, its proof is very difficult!).

Since we can write any complex number $x + yi$ as

$$x + yi \ = \ x \cdot 1 + y \cdot i,$$

where the elements 1 and i are in $\mathbf{C}$, and x and y are scalars in $\mathbf{R}$, we deduce that $\mathbf{C}$ is a real vector space whose basis is $\mathcal{B} = \big(1, i\big)$. The dimension of $\mathbf{C}$ is 2 and $\mathbf{C}$ is isomorphic to $\mathbf{R}^2$ as a vector space.

Exercise. Find an isomorphism between $\mathbf{C}$ and $\mathbf{R}^2$.

Note that $\mathbf{C}$ has a richer structure than $\mathbf{R}^2$. In fact we can multiply two complex numbers to obtain a third complex number, but we cannot multiply two vectors in $\mathbf{R}^2$ to obtain a third vector of $\mathbf{R}^2$. Certainly we can perform the dot product of two vectors of $\mathbf{R}^2$, but the result will be no longer a vector, rather a scalar.