Hints for Homework 8

Problem. Find a basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ of $\mathbf{R}^3$ such that the matrix B representing the reflection in $\mathbf{R}^3$ about the plane P defined by $x + y + z = 0$ is diagonal. Then find the $\mathcal{B}$-coordinates of the image under the reflection about P of the vector $\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$.

Solution. We are looking for a smart choice of a basis of $\mathbf{R}^3$, keeping in mind that we are dealing with a plane. I claim that a suitable basis would be that consisting of two vectors $\vec{v}_1$ and $\vec{v}_2$ that span the plane P, and a third vector $\vec{v}_3$ that is perpendicular to the plane. In the last part of this note we will check that the matrix B is diagonal.

We now find $\vec{v}_1$ and $\vec{v}_2$. These vectors are two vectors that span the plane. To find them, we write the solutions of $x+y+z = 0$ in parametric form. As $x = -y-z$, we have that $y = t$ and $z = s$ are free variables. The solutions are

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore we take

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Now we look for a vector $\vec{v}_3$ perpendicular to the plane. Recall that two vectors are perpendicular if their dot product is zero. Moreover, a vector is perpendicular to the plane if it is perpendicular to the vectors spanning the plane. Hence we need to find a vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 0.$$

Solving the dot products we end up with two equations:

$$-x + y = 0 \quad \text{and} \quad -x + z = 0.$$

One solution of this system is $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Now we check that the matrix B that represents the reflection T about the plane P with respect to the basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2 \vec{v}_3)$ is diagonal. Denote by T the reflection about the plane $x + y + z = 0$. The matrix B is given by the formula:

$$B = \begin{pmatrix} | & | & | \\ [T(\vec{v}_1)]_{\mathcal{B}} & [T(\vec{v}_2)]_{\mathcal{B}} & [T(\vec{v}_3)]_{\mathcal{B}} \\ | & | & | \end{pmatrix}.$$

Now compute the first column of B. As $\vec{v}_1$ is a vector lying in the plane P, when we reflect it about the same plane we will end up with the same vector. Therefore $T(\vec{v}_1) = \vec{v}_1$. The $\mathcal{B}$-coordinates of $\vec{v}_1$ are

$$[\vec{v}_1]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

because $\vec{v}_1 = 1\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3$. Similarly $T(\vec{v}_2) = \vec{v}_2$ and

$$[\vec{v}_2]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Finally, as $\vec{v}_3$ is perpendicular to the plane, its reflection is equal to $-\vec{v}_3$, namely

$$T(\vec{v}_3) = -\vec{v}_3.$$

The $\mathcal{B}$-coordinates of $-\vec{v}_3$ are

$$[\vec{v}_3]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Finally the matrix B is

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Now we solve the last question. Recall that the meaning of B is that it transforms the $\mathcal{B}$-coordinates of a vector, into the $\mathcal{B}$-coordinates of the image under T of that vector. In other words, we have the relation $[T(\vec{x})]_{\mathcal{B}} = B[\vec{x}]_{\mathcal{B}}$ for all vectors $\vec{x}$ in $\mathbf{R}^3$. Therefore the question simply asks to find $[T\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}]_{\mathcal{B}}$, which is then equal to $B[\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}]_{\mathcal{B}}$. The $\mathcal{B}$-coordinates of $\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ are $[\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}]_{\mathcal{B}} = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$ because $\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} = 2\vec{v}_1 - 3\vec{v}_2 + 2\vec{v}_3$. We finally find that

$$[T\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}]_{\mathcal{B}} = B[\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix}.$$

Problem. Consider a basis $\mathcal{B}$ of $\mathbf{R}^3$ consisting of vectors $\mathcal{B} = \left(\vec{u}, \vec{v}, \vec{w}\right)$. Say whether the following vectors are linearly independent or not:

$$\vec{u} - \vec{v} + \vec{w}, \quad \vec{v} - \vec{u} - 3\vec{w}, \quad \vec{u} - \vec{v} - \vec{w}.$$

Then say what is the dimension of their span, and describe it geometrically.

Solution. We first find the $\mathcal{B}$-coordinates of the vectors $\vec{u} - \vec{v} + \vec{w}$, $\vec{v} - \vec{u} - 3\vec{w}$, and $\vec{u} - \vec{v} - \vec{w}$. As

$$\vec{u} - \vec{v} + \vec{w} \ = \ 1\vec{u} - 1\vec{v} + 1\vec{w},$$

we have that

$$[\vec{u} - \vec{v} + \vec{w}]_{\mathcal{B}} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Now we move to the second vector. We want to write it as a linear combination of the elements of the basis $\mathcal{B}$. As

$$\vec{v} - \vec{u} - 3\vec{w} \ = \ -1\vec{u} + 1\vec{v} - 3\vec{w},$$

we have that

$$[\vec{v} - \vec{u} - 3\vec{w}]_{\mathcal{B}} = \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix}.$$

Finally we study the third vector. Proceeding as before we find

$$[\vec{u} - \vec{v} - \vec{w}]_{\mathcal{B}} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

Now we put the three $\mathcal{B}$-coordinates vectors in a matrix

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -3 & -1 \end{pmatrix}.$$

The rref of this matrix is

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore only the first two vectors $\vec{u} - \vec{v} + \vec{w}, \quad \vec{v} - \vec{u} - 3\vec{w}$ are linearly independent. They span a subspace of dimension two in $\mathbf{R}^3$. This subspace looks like a plane of $\mathbf{R}^3$ passing through the origin.

Problem. Show that the subset W consisting of all 2×2 matrices A such that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A = A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is a subspace of $\mathbf{R}^{2\times 2}$. Find a basis and the dimension of W.

Solution. First of all we write W as a subset:

$$W = \{A \text{ in } \mathbf{R}^{2\times 2} \text{ such that } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A = A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}.$$

In order to show that W is a subspace of $\mathbf{R}^{2\times 2}$, we need to show that: 1) W contains the zero matrix, 2) W is closed under addition, and 3) W is closed under scalar multiplication.

We start with 1). The zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ belongs to W because the following equation

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is a true statement, as both sides are equal to the zero matrix.

No we prove that W is closed under addition. In other words we need to prove that if A_1 and A_2 are two elements of W, then also $A_1 + A_2$ is an element of W. To say that A_1 belongs to W it means that:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_1 = A_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

Similarly, to say that A_2 is an element of W it means that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_2 = A_2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

We would like to prove that $A_1 + A_2$ is also an element of W. In other words we need to prove the equality

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (A_1 + A_2) = (A_1 + A_2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

In order to do so, we start with the left hand side of equation (3), and we try to rewrite it until we get the right hand side of equation (3):

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (A_1 + A_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_1 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_2 = A_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + A_2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (A_1 + A_2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that we have used the distributive property of matrix multiplication, and both equations (1) and (2).

Now we prove that W is closed under scalar multiplication. In other words we want to prove that if A is in W, then also cA is in W for any scalars c. To say that A is in W it means that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A = A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

Recall that we want to prove that cA is also in W, this simply means that we need to show that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (cA) \;=\; (cA) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

We start with left hand side of equation (5), and through a series of equalities, we will get to its right hand side:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (cA) \;=\; c\Big(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A \Big) = c\Big(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A \Big) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (cA).$$

Note that we have used the associativity property of matrix multiplication, and equation (4).

Now we look for a basis of W. First of all we need to understand how we can write a typical element of W in terms of some arbitrary constants. In our case, an element of W is a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We can rewrite this equation by performing the two matrix multiplications:

$$\begin{pmatrix} c & d \\ a & b \end{pmatrix} \;=\; \begin{pmatrix} a & -b \\ c & -d \end{pmatrix}.$$

We conclude that $c = a$ and $d = -b$. Plugging these relations into $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have that a typical element of W is of the form

$$A = \begin{pmatrix} a & b \\ a & -b \end{pmatrix}$$

which can be rewritten in terms of the constants a and b as

$$A = \begin{pmatrix} a & b \\ a & -b \end{pmatrix} \;=\; a \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \;+\; b \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}.$$

Therefore the matrices $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$ span W, and since they are also linearly independent, they actually form a basis of W. The dimension of W is 2.

Problem. Show that the subset W consisting of polynomials $p(x)$ of degree at most 3 such that $2p'(1) - 3p(1) = 0$ is a subspace of P_3. Find then a basis and the dimension of W.

Solution. We can write W as

$$W \;=\; \{\; p(x) = a + bx + cx^2 + dx^3 \text{ such that } a, b, c, d \text{ are scalars and } 2p'(1) - 3p(1) = 0 \;\}.$$

Obviously the zero polynomial belongs to W because the derivative of 0 is 0. To prove that W is closed under addition, we take two polynomials $p(x)$ and $q(x)$ in W and prove that $p(x) + q(x)$ still belongs to W. This amounts to prove that $2\big(p+q\big)'(1) - 3\big(p+q\big)(1) = 0$, but this is true because

$$\begin{aligned} 2\big(p+q\big)'(1) - 3\big(p+q\big)(1) \;&=\; 2\big(p'(1) + q'(1)\big) - 3\big(p(1) + q(1)\big) = \\ &=\; \big(2p'(1) - 3p(1)\big) \;+\; \big(2q'(1) - 3q(1)\big) = 0 + 0 = 0. \end{aligned}$$

Notice that the quantities $2p'(1) - 3p(1)$ and $2q'(1) - 3q(1)$ are both equal to zero because p and q are two polynomials in W.

Finally, in order to prove that W is closed under scalar multiplication, we take a polynomial $p(x)$ in W and prove that $cp(x)$ is in W for any scalar c. As

$$2(cp)'(1) - 3(cp)(1) \;=\; 2cp'(1) - 3cp(1) = c\big(2p'(1) - 3p(1)\big) = c \cdot 0 = 0,$$

we have that $cp(x)$ is in W.

Now we look for a basis of W. To this end we write a typical element of W in terms of arbitrary constants. A typical element of W is a polynomial $p(x) = a + bx + cx^2 + dx^3$ such that $2p'(1) - 3p(1) = 0$. We write this condition explicitly. The derivative of $p(x)$ is $p'(x) = b + 2cx + 3dx^2$. Moreover we note that $p(1) = a + b + c + d$ and $p'(1) = b + 2c + 3d$. To say that $2p'(1) - 3p(1) = 0$ it means that

$$2\big(b + 2c + 3d\big) - 3\big(a + b + c + d\big) = 0.$$

But this is equivalent to

$$-3a - b + c + 3d = 0.$$

We can solve for c, for instance, so that $c = 3a + b - 3d$. Then a typical element of W looks like

$$p(x) = a + bx + (3a + b - 3d)x^2 + dx^3 = a(1 + 3x^2) \ + \ b(x + x^2) \ + \ d(-3x^2 + x^3).$$

Therefore a basis of W is formed by the polynomials $\big(1 + 3x^2, x + x^2, -3x^2 + x^3\big)$, and the dimension of W is 3. In practice what this problem is saying is that the the polynomials of degree at most three satisfying $2p'(1) - 3p(1) = 0$ must be linear combinations of $1 + 3x^2, x + x^2$ and $-3x^2 + x^3$.

Problem. Consider the function $T\colon \mathbf{R}^{2\times 2} \to \mathbf{R}^{2\times 2}$ defined by $T(A) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} A$. Show that T is linear. Then find bases of the kernel and image of T. Say whether T is an isomorphism.

Solution. In order to prove that T is linear, we need to prove that it preserves the sum and the scalar multiplication. In other words we need to prove that

$$T(A + B) \ = \ T(A) + T(B) \quad \text{for all } A \text{ and } B \text{ in } \mathbf{R}^{2\times 2}$$

and

$$T(cA) \ = \ cT(A) \quad \text{for all } A \text{ in } \mathbf{R}^{2\times 2} \text{ and scalars } c.$$

We start by proving the first condition:

$$T(A + B) \ = \ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}(A + B) \ = \ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} A \ + \ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} B \ = \ T(A) \ + \ T(B).$$

Notice that we have only used the distributive property of matrix multiplication. Now we prove the other condition:

$$T(cA) \ = \ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}(cA) \ = \ c\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} A \ = \ cT(A).$$

In order to find the kernel of T, we solve the equation $T(A) = 0$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix in $\mathbf{R}^{2\times 2}$. Then

$$T(A) \ = \ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} A \ = \ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ = \ \begin{pmatrix} a + 2c & b + 2d \\ 3a + 4c & 3b + 4d \end{pmatrix} \ = \ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

yields equations:

$$a + 2c = 0, \qquad b + 2d = 0, \qquad 3a + 4c = 0, \qquad 3b + 4d = 0.$$

But these equations admit the only solution $a = 0$, $b = 0$, $c = 0$, $d = 0$. Therefore $Ker(T) = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\}$. The basis of $Ker(T)$ is formed by the zero matrix, and its dimension is zero.

Now we find a basis of the image of T. For this we want solve $B = T(A)$. This is equivalent to

$$B \ = \ T(A) \ = \ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} A \ = \ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ = \ \begin{pmatrix} a + 2c & b + 2d \\ 3a + 4c & 3b + 4d \end{pmatrix}.$$

Now we want to write the previous matrix in terms of the free variables a, b, c, d:

$$\begin{pmatrix} a + 2c & b + 2d \\ 3a + 4c & 3b + 4d \end{pmatrix} \ = \ a\begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} \ + \ b\begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} \ + \ c\begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix} \ + \ d\begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix}.$$

Therefore a basis of the image is formed by the matrices $\begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix}$ and its dimension is 4. Since also $\mathbf{R}^{2\times 2}$ is of dimension 4, we have that $Im(T) = \mathbf{R}^{2\times 2}$.

We can say that T is an isomorphism because $Ker(T) = \{0\}$ and $Im(T) = \mathbf{R}^{2\times 2}$.