MATH 211: Introduction to Linear Algebra - Fall 2017

Solution Problem 21 in Section 3.4

Point (a). We first solve point (a) and find the matrix B via the formula $B = S^{-1}AS$. The matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ is the matrix that represents a linear transformation T with respect to the standard basis. In order words $T(\vec{x}) = A\vec{x}$. The matrix B is the matrix that represents T with respect to the new basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2)$. In other words $[T(\vec{x})]_{\mathcal{B}} = B[\vec{x}]_{\mathcal{B}}$. In this problem $\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. The matrix S is the matrix of change of basis from the basis $\mathcal{B}$ to the standard basis, its columns are the vectors forming the basis $\mathcal{B}$. Hence

$$S = \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}.$$

The inverse of S is

$$S^{-1} = \frac{1}{7}\begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}.$$

Now we can find B according to the formula

$$B = S^{-1}AS = \frac{1}{7}\begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}.$$

Solve now the triple product of matrices. As matrix multiplication is associative, it is the same thing if you first multiply the first matrix with the second matrix, and then with the third matrix, or if you first multiply the second matrix with the third matrix, and then with the first matrix. You should find $B = \begin{pmatrix} 7 & 0 \\ 0 & 0 \end{pmatrix}$.

Point (c). Now we solve point (c). We want to construct B column by column. In order to do this we recall how B is constructed: the columns of B are the $\mathcal{B}$-coordinates of the images (via A) of the vectors forming the basis $\mathcal{B}$. In other words

$$B = \begin{pmatrix} | & | \\ [T\begin{pmatrix} 1 \\ 3 \end{pmatrix}]_{\mathcal{B}} & [T\begin{pmatrix} -2 \\ 1 \end{pmatrix}]_{\mathcal{B}} \\ | & | \end{pmatrix}.$$

Let's compute the first column of B. The vector $T\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is computed as

$$T\begin{pmatrix} 1 \\ 3 \end{pmatrix} = A\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 21 \end{pmatrix}.$$

Now we look for the coordinates of $\begin{pmatrix} 7 \\ 21 \end{pmatrix}$ with respect to the basis $\mathcal{B}$. In order to do this we want to write $\begin{pmatrix} 7 \\ 21 \end{pmatrix}$ as linear combination of $\vec{v}_1, \vec{v}_2$:

$$\begin{pmatrix} 7 \\ 21 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2\begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

One minute of thinking (or by solving the system for c_1 and c_2) gives that $c_1 = 7$ and $c_2 = 0$. Therefore

$$[\begin{pmatrix} 7 \\ 21 \end{pmatrix}]_{\mathcal{B}} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}.$$

Now we compute the second column of B. We first compute

$$T\begin{pmatrix} 1 \\ 3 \end{pmatrix} = A\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Afterwards we write $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as linear combination of the elements of the basis $\mathcal{B}$, but this is very easy:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 0 \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Therefore

$$[\begin{pmatrix} 0 \\ 0 \end{pmatrix}]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and the matrix B is

$$B = \begin{pmatrix} 7 & 0 \\ 0 & 0 \end{pmatrix}.$$

Notice that B looks much simpler than A, so that working in a opportune basis different from the standard one may simplify the matrix that represents a transformation, and hence calculations.