MATH 211: Introduction to Linear Algebra - Fall 2017

Homework 6 – Due date 10/20/17

Read: Section 3.2, Section 3.3 and Theorem 2.1.3 in section 2.1. Keywords: Linear independence of vectors, non-trivial relations, subspaces of $\mathbf{R}^n$, bases of a subspace, spans kernels and images are subspaces, linear transformations preserve the sum of vectors, and the scalar multiplication of a scalar and a vector.

Problem 1: Consider the following vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ in $\mathbf{R}^2$.

i) Write $\vec{v}_3$ as linear combination of $\vec{v}_1$ and $\vec{v}_2$.
ii) Write a non-trivial relation between $\vec{v}_1, \vec{v}_2, \vec{v}_3$.
iii) Are $\vec{v}_1, \vec{v}_2, \vec{v}_3$ linearly independent? Do they span $\mathbf{R}^2$? Do they form a basis of $\mathbf{R}^2$?

Problem 2: Repeat the first 2 points of Problem 1 with the following vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$. Then answer to the following questions: Are $\vec{v}_1, \vec{v}_2, \vec{v}_3$ linearly independent? Do they span $\mathbf{R}^3$? Do they form a basis of $\mathbf{R}^3$?

Problem 3: Prove (or explains with your own words) why we can find at most n linearly independent vectors in $\mathbf{R}^n$. (*Hint:* The answer to this problem is very similar to that of Problem 5 (iv) in Homework 4.)

Problem 4: Make a sketch of the following subset $W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \text{ s.t. } xy \leq 0 \right\}$ consisting of the union of the second and fourth quadrants, including the axes. Show that W is not closed under addition, by writing down two vectors that belong to W, but such that their sum is not in W.

Problem 5: Repeat Problem 4 with the following subset $W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \text{ s.t. } x^2 - y^2 = 0 \right\}$. (*Hint*: Note that $x^2 - y^2 = (x+y)(x-y) = 0$, so that W is simply the union of the two lines $y = x$ and $y = -x$. Make a sketch of W!)

Problem 6: Prove that the image of a linear transformation $T\colon \mathbf{R}^m \to \mathbf{R}^n$ is a linear subspace of $\mathbf{R}^n$. You can read the solution from your textbook (Edition 5: Theorem 3.1.4 at page 114; Edition 4: Theorem 3.1.4 at page 105).

Problem 7: Find a basis for the following subset $U = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{ in } \mathbf{R}^4 \text{ s.t. } x_2 + x_3 + x_4 = 0 \text{ and } x_3 - x_4 = 0 \right\}$ of $\mathbf{R}^4$. Write then U as kernel of some matrix.

Problem 8: Consider the following subset

$$U = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \text{ s.t. } \begin{cases} y_1 = x_1 + 2x_2 + 3x_3 \\ y_2 = 4x_1 + 5x_2 + 6x_3 \\ y_3 = 7x_1 + 8x_2 + 9x_3 \end{cases} \text{ where } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ is in } \mathbf{R}^3 \right\}.$$

Convince yourself that U is nothing else than the image of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. In this way we have proved that U is a subspace of $\mathbf{R}^3$ as the image of any linear transformation is a subspace. Find a basis of U.

Problem 9: Find a basis of the subspace in $\mathbf{R}^4$ defined by $2x_1 - x_2 + 2x_3 + 4x_4 = 0$.

Problem 10: Consider the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. Is $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ in the image of A? Is $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ in the kernel of A? Perform your calculations and justify your answers.