MATH 211: Introduction to Linear Algebra - Fall 2017

Homework 10 – Due date November 29th

Read. "Formulas about base change" in BB. "Exercises with solutions about diagonalization" in BB. Sections 4.3, 7.2, 7.3 and Example 2 in 7.4 of the Textbook.

Problem 1. For each matrix find its eigenvalues together with their algebraic multiplicities, find bases of the eigenspaces associated to each eigenvalue, find the geometric multiplicity of each eigenvalue. Then say whether the matrix is diagonalizable and specify which criterion you use to answer to this question. If a matrix A is diagonalizable, then write a diagonal matrix B and an invertible matrix S such that $B = S^{-1}AS$ (do not perform this product, unless you want to check that calculations are correct).

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & -1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

(*Hint:* For the first two matrices you can check that your work is correct by looking at the file "Exercises with solutions about diagonalization" in BB. For the third matrix, recall that the determinant of an upper-triangular matrix is the product of the entries along the main diagonal.)

Problem 2. Consider the linear function $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ defined as $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x - 4y \\ 2x - 1y \end{pmatrix}$. Write the matrix A that represents the linear function T with respect to the standard basis of $\mathbf{R}^2$. Find a basis $\mathcal{B}$ of $\mathbf{R}^2$ such that the matrix B that represents T with respect to $\mathcal{B}$ is diagonal. In other words you want to diagonalize A.

Problem 3. Say whether $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is diagonalizable. If not say why.

Problem 4. Write the matrix that represents the rotation in $\mathbf{R}^2$ by 90 degrees counterclockwise with respect to the standard basis. Find its eigenvalues. Do you notice anything strange? Could you provide a geometric explanation?

Problem 5. (i). Make a picture of a linear function that is an isomorphism. **(ii)**. Make a picture of a linear function that is not an isomorphism. **(iii)**. Is is true that the determinant of a matrix is equal to the determinant of its rref? If yes prove it, if not give an example (*Hint:* The answer is negative!).

Problem 6. For Problems 6 and 7 you may want to read the notes "Formulas about base change" I posted in BB.

Check that $\mathcal{B} = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)$ is a basis of $\mathbf{R}^2$. Write the matrix of change of basis $S_{\mathcal{B}\to\mathcal{E}}$ from the basis $\mathcal{B}$ to the standard basis $\mathcal{E}$ of $\mathbf{R}^2$. Calculate the standard coordinates of a vector knowing that its $\mathcal{B}$-coordinates are $\begin{pmatrix} -4/5 \\ 3/5 \end{pmatrix}$ (you should use the matrix $S_{\mathcal{B}\to\mathcal{E}}$). Compute the $\mathcal{B}$-coordinates of $\begin{pmatrix} -12 \\ 7 \end{pmatrix}$ (you should use the matrix $S^{-1}_{\mathcal{B}\to\mathcal{E}}$).

Now consider the linear function $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ defined as $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x - y \\ -x + 2y \end{pmatrix}$. Write the matrix A that represents T with respect to the standard basis $\mathcal{E}$ of $\mathbf{R}^2$. Write the matrix B that represents T with respect to the basis $\mathcal{B}$. What relation should A and B satisfy? (Do not perform any product.) Find the $\mathcal{B}$-coordinates of $\begin{pmatrix} 33 \\ 101 \end{pmatrix}$. Find the $\mathcal{B}$-coordinates of $T\begin{pmatrix} 33 \\ 101 \end{pmatrix}$. Find the standard coordinates of $T\begin{pmatrix} 33 \\ 101 \end{pmatrix}$. Is T an isomorphism?

Problem 7. In class we showed that $\mathcal{B} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$ is a basis of $\mathbf{R}^{2\times 2}$, the set of 2×2 matrices with real coefficients. Hence the dimension of $\mathbf{R}^{2\times 2}$ is 4. Show that the following set of matrices $\mathcal{C} = \left(\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \right)$ form a basis of $\mathbf{R}^{2\times 2}$ (you should first find the $\mathcal{B}$-coordinates of the matrices of the set $\mathcal{C}$). Write the matrix of change of basis $S_{\mathcal{C}\to\mathcal{B}}$. Find the matrix whose $\mathcal{C}$-coordinates are $\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$.

Now consider the linear function $T\colon \mathbf{R}^{2\times 2} \to \mathbf{R}^{2\times 2}$ defined as $T(M) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} M$. Write the matrix $C_{\mathcal{B}}$ that represents T with respect to the basis $\mathcal{B}$. Write the matrix $C_{\mathcal{C}}$ that represents T with respect to the basis $\mathcal{C}$. What is the relation between $C_{\mathcal{C}}$ and $C_{\mathcal{B}}$? Find bases for the kernel and image of T (you can work out these bases either using $C_{\mathcal{B}}$ or $C_{\mathcal{C}}$). Is T an isomorphism?

Find the $\mathcal{C}$-coordinates of $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$. Find the $\mathcal{C}$-coordinates of $T\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$. Find the $\mathcal{B}$-coordinates of $T\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$. Find $T\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$.

Problem 8 (Optional). Recall that the formula of the orthogonal projection in $\mathbf{R}^3$ onto a plane V of equation $ax + by + cz = 0$ is

$$\text{proj}_V(\vec{x}) = \vec{x} - \left(\frac{\vec{x}\cdot\vec{r}}{\vec{r}\cdot\vec{r}}\right)\vec{r}, \qquad \text{where} \qquad \vec{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

(as usual the dot denotes the dot product). Write the matrix that represents with respect to the standard basis of $\mathbf{R}^3$ the transformation $T(\vec{x}) = -3\,\text{proj}_V(\vec{x})$, where V is the plane of equation $x + y + z = 0$ (you should find $A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$). Find a basis $\mathcal{B}$ of $\mathbf{R}^3$ so that the matrix B that represents T with respect to $\mathcal{B}$ is diagonal. Is $\mathcal{B}$ formed by 2 vectors spanning the plane V and one other perpendicular to V? Find bases for the kernel and image of T.