Why are diagonal matrices important?

Introduction. Two frequent questions I get asked are: *Why do we need to perform a change of basis in order to get a diagonal matrix? Why are diagonal matrices "simpler" than other matrices?* I will try to convince you that diagonal matrices are important by describing a problem in Calculus. The content of these pages goes a little beyond the scope of our course. However if you feel somewhat comfortable with first-order differential equations, you may find the following example interesting.

A simple reason. First of all, a diagonal matrix is a matrix whose entries are zero outside the main diagonal (of course zero entries along the main diagonal are allowed). For instance, the following matrices are diagonal:

$$\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & \pi \end{pmatrix}.$$

One feature of diagonal matrices is that it is very simple to calculate their products with vectors. For instance

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & \pi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2x \\ 4y \\ \pi z \end{pmatrix}.$$

A more serious reason. In general working with diagonal matrices simplifies calculations a lot. However this is not the only reason. In some situations we are able to solve problems *only if* we can deal with diagonal matrices. We now consider a system of first-order linear equations to illustrate this concept.

Consider two one-variable differentiable functions $x = x(t)$ and $y = y(t)$. These functions are functions of the variable t. You may think that $x(t)$ and $y(t)$ give the coordinates in the plane of a moving particle at time t. We denote by x' and y' the first derivatives of x and y with respect to t. Consider the following system of differential equations:

(1)
$$\begin{cases} x' &= x + 2y \\ y' &= 4x + 3y \end{cases}.$$

To solve this system it means that we are looking for all pairs of differentiable functions x and y such that themselves together with their derivatives x' and y' satisfy the above system. Solving the system (1) is a very difficult task, unless one has already taken a course in Ordinary Differential Equations. The main difficulty is that x' is expressed as a function of both x and y, and so is y'. We call this type of systems "mixed systems." In order to "unmix" the system we use the theory of change of basis and linear algebra.

First of all we rewrite the previous system in matrix notation:

(2)
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

where A is the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}.$$

The crucial observation is that A is similar to the following diagonal matrix

$$B = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}.$$

The matrix of change of basis is

$$S = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}.$$

Convince yourself that A and B are similar matrices via S by showing that $SB = SA$. *Question: How did you manage to see that A is similar to the diagonal matrix B? Answer: There is a general algebraic procedure to determine a diagonal matrix B and a matrix of change of basis S given a matrix A. This process is called diagonalization and we will learn this techniques towards the end of he course. Warning: There exist matrices that are not similar to any diagonal matrix! However this pathology does not occur in our situation as our matrix A is similar to the diagonal matrix B.*

As A and B are similar, they satisfy the relation $SB = AS$. Solving this equation for A we find $A = SBS^{-1}$. We plug this expression of A into equation (2) to find:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = SBS^{-1} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Multiply (on the left) the both sides of the previous equation by S^{-1} in order to find:

(3) $$S^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix} = BS^{-1} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Note that $S^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$ is a vector as S^{-1} is a 2×2-matrix. We introduce new variables u and v (note that these are still functions of t) defined as:

(4) $$\begin{pmatrix} u \\ v \end{pmatrix} = S^{-1} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Taking the derivatives of each side of the previous equation gives

(5) $$\begin{pmatrix} u' \\ v' \end{pmatrix} = S^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

At this point by using (4) and (5) we can rewrite equation (3) as

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = B \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

We rewrite the previous equation as a linear system:

(6) $$\begin{cases} u' & = 5u \\ & v' = -v \end{cases}.$$

Note that this system is "unmixed," namely each single equation depends on only one variable. Therefore we can solve each equation separately. The good news is that we are able to solve this type of equations! (I guess these equations are studied either in Calculus 2 or Calculus B-C). Via integration we can show that the solution of the first equation is $u(t) = C_1 e^{5t}$ where C_1 is an arbitrary constant (check that this is indeed the solution). Whereas the solution of the second equation is $v(t) = C_2 e^{-t}$ where C_2 is another arbitrary constant. To conclude we can write the solutions of the system (6) in vector form as:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} C_1 e^{5t} \\ C_2 e^{-t} \end{pmatrix}.$$

These are not yet the solutions of (1), but we are getting there. In view of equation (4) we have

$$S^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} C_1 e^{5t} \\ C_2 e^{-t} \end{pmatrix},$$

moreover by multiplying (on the left) both sides of the previous equation by S we obtain:

$$\begin{pmatrix} x \\ y \end{pmatrix} = S \begin{pmatrix} C_1 e^{5t} \\ C_2 e^{-t} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} C_1 e^{5t} \\ C_2 e^{-t} \end{pmatrix}.$$

Therefore the solutions x and y of (1) are

$$\begin{cases} x(t) & = C_1 e^{5t} + C_2 e^{-t} \\ & y(t) = 2C_1 e^{5t} - C_2 e^{-t} \end{cases}$$

where C_1 and C_2 are arbitrary real constants.

Exercise. Solve the following system of differential equations

$$\begin{cases} x' & = 5x - 4y \\ & y' = 3x - y \end{cases}.$$

Hint: The matrix $A = \begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$ is similar to the diagonal matrix $B = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$. The matrix of change of basis is $S = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Therefore the following relation holds $A = SBS^{-1}$.