

Weil-Petersson Geometry of the Universal Teichmüller Space

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1. Introduction

The universal Teichmüller space $T(1)$ is the simplest Teichmüller space that bridges spaces of univalent functions and general Teichmüller spaces. It was introduced by Bers [Ber65, Ber72, Ber73] and it is an infinite-dimensional complex Banach manifold. The universal Teichmüller space $T(1)$ contains Teichmüller spaces of Riemann surfaces as complex submanifolds.

The universal Teichmüller space $T(1)$ plays an important role in one of the approaches to non-perturbative bosonic closed string field theory based on Kähler geometry. Namely, in the “old approach” to string field theory as the Kähler geometry of the loop space [BR87a, BR87b], the loop space $\mathcal{L}(\mathbb{R}^d)$ is the configuration space for the closed strings,

$$\mathcal{L}(\mathbb{R}^d) = \mathbb{R}^d \times \Omega(\mathbb{R}^d).$$

The space $\Omega(\mathbb{R}^d)$ of based loops has a natural structure of an infinite-dimensional Kähler manifold. The space of all complex structures of $\Omega(\mathbb{R}^d)$ is

$$\mathcal{M} = S^1 \backslash \text{Diff}_+(S^1).$$

The space \mathcal{M} parameterizes vacuum states for Faddeev-Popov ghosts in the string field theory. The “flag manifolds” \mathcal{M} and

$$\mathcal{N} = \text{Möb}(S^1) \backslash \text{Diff}_+(S^1)$$

are infinite-dimensional complex Fréchet manifolds carrying a natural Kähler metrics [BR87a, BR87b, Kir87, KY87]. These manifolds also have an interpretation as coadjoint orbits of the Bott-Virasoro group, and the corresponding Kähler forms coincide with Kirillov-Kostant symplectic forms [Kir87, KY87]. Ricci tensor for \mathcal{M}

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is related to the problem of constructing reparametrization-invariant vacuum for ghosts.

The natural inclusion $\mathcal{N} \hookrightarrow T(1)$ is holomorphic (\mathcal{N} is a leaf of a holomorphic foliation of $T(1)$), and the Kirillov-Kostant symplectic form at the origin of \mathcal{N} is a pull-back of a certain symplectic form on the subspace of the tangent space to $T(1)$ at the origin [NV90] (an avatar of the Weil-Petersson structure on $T(1)$).

2. Basic facts

2.1. Definitions

Let

$$\begin{aligned}\mathbb{D} &= \{z \in \mathbb{C} : |z| < 1\}, \\ \mathbb{D}^* &= \{z \in \mathbb{C} : |z| > 1\}.\end{aligned}$$

The complex Banach spaces $L^\infty(\mathbb{D}^*)$ and $L^\infty(\mathbb{D})$ are the spaces of bounded Beltrami differentials on \mathbb{D}^* and \mathbb{D} respectively. Let $L^\infty(\mathbb{D}^*)_1$ be the unit ball in $L^\infty(\mathbb{D}^*)$. Two classical models of Bers' universal Teichmüller space $T(1)$ are the following.

Model A. Extend every $\mu \in L^\infty(\mathbb{D}^*)_1$ to \mathbb{D} by the reflection

$$\mu(z) = \overline{\mu\left(\frac{1}{\bar{z}}\right)} \frac{z^2}{\bar{z}^2}, \quad z \in \mathbb{D},$$

and consider the unique quasiconformal mapping $w_\mu : \mathbb{C} \rightarrow \mathbb{C}$, which fixes $-1, -i$ and 1 , and satisfies the Beltrami equation

$$\frac{\partial w_\mu}{\partial \bar{z}} = \mu \frac{\partial w_\mu}{\partial z}.$$

The mapping w_μ satisfies

$$\frac{1}{w_\mu(z)} = \overline{w_\mu\left(\frac{1}{\bar{z}}\right)}$$

and fixes the domains \mathbb{D}, \mathbb{D}^* , and the unit circle S^1 . For $\mu, \nu \in L^\infty(\mathbb{D}^*)_1$ set $\mu \sim \nu$ if

$$w_\mu|_{S^1} = w_\nu|_{S^1}.$$

The universal Teichmüller space $T(1)$ is defined as the set of equivalence classes of the mappings w_μ ,

$$T(1) = L^\infty(\mathbb{D}^*)_1 / \sim.$$

Model B. Extend every $\mu \in L^\infty(\mathbb{D}^*)_1$ to be zero outside \mathbb{D}^* and consider the unique solution w^μ of the Beltrami equation

$$\frac{\partial w^\mu}{\partial \bar{z}} = \mu \frac{\partial w^\mu}{\partial z},$$

satisfying $f(0) = 0$, $f'(0) = 1$ and $f''(0) = 0$, where $f = w^\mu|_{\mathbb{D}}$ is holomorphic on \mathbb{D} . For $\mu, \nu \in L^\infty(\mathbb{D}^*)_1$ set $\mu \sim \nu$ if

$$w^\mu|_{\mathbb{D}} = w^\nu|_{\mathbb{D}}.$$

The universal Teichmüller space is defined as the set of equivalence classes of the mappings w^μ ,

$$T(1) = L^\infty(\mathbb{D}^*)_1 / \sim.$$

Since $w_\mu|_{S^1} = w_\nu|_{S^1}$ if and only if $w^\mu|_{\mathbb{D}} = w^\nu|_{\mathbb{D}}$, the two definitions of the universal Teichmüller space are equivalent. The set $T(1)$ is a topological space with the quotient topology induced from $L^\infty(\mathbb{D}^*)_1$.

2.2. Properties of $T(1)$

1. The universal Teichmüller space $T(1)$ has a unique structure of a complex Banach manifold such that the projection map

$$\Phi : L^\infty(\mathbb{D}^*)_1 \rightarrow T(1)$$

is a holomorphic submersion.

2. The holomorphic tangent space $T_0T(1)$ at the origin is identified with the Banach space $\Omega^{-1,1}(\mathbb{D}^*)$ of harmonic Beltrami differentials,

$$\begin{aligned} \Omega^{-1,1}(\mathbb{D}^*) &= \{\mu \in L^\infty(\mathbb{D}^*) : \\ &\mu(z) = (1 - |z|^2)^2 \overline{\phi(z)}, \phi \in A_\infty(\mathbb{D}^*)\}, \end{aligned}$$

where

$$\begin{aligned} A_\infty(\mathbb{D}^*) &= \{\phi \text{ holomorphic on } \mathbb{D}^* : \\ &\|\phi\|_\infty = \sup_{z \in \mathbb{D}^*} |(1 - |z|^2)^2 \phi(z)| < \infty\}. \end{aligned}$$

3. The universal Teichmüller space $T(1)$ is a group (not a topological group!) under the composition of the quasiconformal mappings. The group law on $L^\infty(\mathbb{D}^*)_1$

$$\lambda = \nu * \mu^{-1}$$

is defined through $w_\lambda = w_\nu \circ w_\mu^{-1}$ and projects to $T(1)$. Explicitly,

$$\lambda = \left(\frac{\nu - \mu}{1 - \bar{\mu}\nu} \frac{(w_\mu)_z}{(\bar{w}_\mu)_{\bar{z}}} \right) \circ w_\mu^{-1}.$$

For every $\mu \in L^\infty(\mathbb{D}^*)_1$ the right translations

$$R_{[\mu]} : T(1) \longrightarrow T(1), \quad [\lambda] \longmapsto [\lambda * \mu],$$

where $[\lambda] = \Phi(\lambda) \in T(1)$, are biholomorphic automorphisms of $T(1)$. The left translations, in general, are not even continuous mappings.

4. The group $T(1)$ is isomorphic to the subgroup of the group $\text{Homeo}_{qs}(S^1)$ of quasimetric homeomorphisms of S^1 fixing $-1, -i$ and 1 . By definition, $\gamma \in \text{Homeo}_{qs}(S^1)$ if it is orientation preserving and satisfies

$$\frac{1}{M} \leq \left| \frac{\gamma(e^{i(\theta+t)}) - \gamma(e^{i\theta})}{\gamma(e^{i\theta}) - \gamma(e^{i(\theta-t)})} \right| \leq M$$

for all θ and all $|t| \leq \pi/2$ with some constant $M > 0$.

Remark 1. The closure of \mathcal{N} in $T(1)$ is the subgroup of symmetric homeomorphisms in $\text{Möb}(S^1) \setminus \text{Homeo}_{qs}(S^1)$ satisfying the above inequality with M replaced by $1 + o(t)$ as $t \rightarrow 0$.

2.3. Bers embedding and the complex structure of $T(1)$

Let $A_\infty(\mathbb{D}) = \left\{ \phi \text{ holomorphic on } \mathbb{D} : \|\phi\|_\infty = \sup_{z \in \mathbb{D}} |(1 - |z|^2)^2 \phi(z)| < \infty \right\}$.

and let $\mathcal{S}(f)$ be the Schwarzian derivative,

$$\mathcal{S}(f) = \frac{f_{zzz}}{f_z} - \frac{3}{2} \left(\frac{f_{zz}}{f_z} \right)^2.$$

For every $\mu \in L^\infty(\mathbb{D}^*)_1$ the holomorphic function $\mathcal{S}(w^\mu)|_{\mathbb{D}} \in A_\infty(\mathbb{D})$ and, by Kraus-Nehari inequality, lies in the ball of radius 6. The Bers embedding $\beta : T(1) \hookrightarrow A_\infty(\mathbb{D})$ is defined by

$$\beta([\mu]) = \mathcal{S}(w^\mu)|_{\mathbb{D}},$$

and is a holomorphic map of complex Banach manifolds. Define the mapping $\Lambda : A_\infty(\mathbb{D}) \rightarrow \Omega^{-1,1}(\mathbb{D}^*)$ by

$$\Lambda(\phi)(z) = -\frac{1}{2} (1 - |z|^2)^2 \phi \left(\frac{1}{\bar{z}} \right) \frac{1}{\bar{z}^4}.$$

By Ahlfors-Weill theorem, the mapping Λ is inverse to the Bers embedding β over the ball of radius 2 in $A_\infty(\mathbb{D})$.

The complex structure of $T(1)$ is explicitly described as follows. For every $\mu \in L^\infty(\mathbb{D}^*)_1$ let $U_\mu \subset T(1)$ be the image of the ball of radius 2 in $A_\infty(\mathbb{D})$ under the map $h_\mu^{-1} = R_{[\mu]}^{-1} \circ \Lambda$. The inverse map $h_\mu = \beta \circ R_{[\mu]} : U_\mu \rightarrow A_\infty(\mathbb{D})$ and the maps $h_{\mu\nu} = h_\mu \circ h_\nu^{-1} : h_\mu(U_\mu) \cap h_\nu(U_\nu) \rightarrow h_\mu(U_\mu) \cap h_\nu(U_\nu)$ are biholomorphic (as functions in the Banach space $A_\infty(\mathbb{D})$). The open covering $T(1) = \bigcup_{\mu \in L^\infty(\mathbb{D}^*)_1} U_\mu$ with coordinate maps h_μ and transition maps $h_{\mu\nu}$ defines a complex-analytic atlas on $T(1)$ modelled on the Banach space $A_\infty(\mathbb{D})$.

The canonical projection $\Phi : L^\infty(\mathbb{D}^*)_1 \rightarrow T(1)$ is a holomorphic submersion and the Bers embedding $\beta : T(1) \rightarrow A_\infty(\mathbb{D})$ is a biholomorphic map with respect to this complex structure. Complex coordinates on $T(1)$ defined by the coordinate charts (U_μ, h_μ) are called Bers coordinates.

2.4. The universal Teichmüller curve

The universal Teichmüller curve $\mathcal{T}(1)$ is a complex fiber space over $T(1)$ with a holomorphic projection map

$$\pi : \mathcal{T}(1) \rightarrow T(1).$$

The fiber over each point $[\mu]$ is the quasi-disk $w^\mu(\mathbb{D}^*) \subset \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with the complex structure induced from $\hat{\mathbb{C}}$ and

$$\mathcal{T}(1) = \{([\mu], z) : [\mu] \in T(1), z \in w^\mu(\mathbb{D}^*)\}.$$

The fibration $\pi : \mathcal{T}(1) \rightarrow T(1)$ has a natural holomorphic section given by

$$T(1) \ni [\mu] \mapsto ([\mu], \infty) \in \mathcal{T}(1)$$

which defines the embedding $T(1) \hookrightarrow \mathcal{T}(1)$. The universal Teichmüller curve is a complex Banach manifold modelled on $A_\infty(\mathbb{D}) \oplus \mathbb{C}$.

2.5. Velling-Kirillov metric on $\mathcal{T}(1)$

The Velling-Kirillov metric at the origin of $\mathcal{T}(1)$ is defined by

$$\|v\|_{VK}^2 = \sum_{n=1}^{\infty} n|c_n|^2, \quad \text{where} \quad v = \sum_{n \neq 0} c_n e^{in\theta} \frac{\partial}{\partial \theta} \in T_0 S^1 \setminus \text{Homeo}_{qs}(S^1)$$

– the tangent space at the origin of a real Banach manifold $S^1 \setminus \text{Homeo}_{qs}(S^1)$. (The series in the definition of $\|v\|_{VK}^2$ is always convergent.) At other points the Velling-Kirillov metric is defined by the right translations. The Velling-Kirillov metric on $\mathcal{T}(1)$ is Kähler with symplectic form ω_{VK} .

Remark 2. For the space $S^1 \setminus \text{Diff}_+(S^1)$ this metric was introduced by Kirillov [Kir87] and has been studied by Kirillov-Yuriev [KY87]. Velling [Vel] introduced a Hermitian metric for $\mathcal{T}(1)$ using geometric theory of functions, and in [Teo02] the second author extended Kirillov's metric to $\mathcal{T}(1)$ and proved that it coincides with the metric introduced by Velling. The Velling-Kirillov metric is the unique Kähler metric on $\mathcal{T}(1)$ invariant under the right translations [Kir87, Teo02].

3. Weil-Petersson metric on $T(1)$

As a Banach manifold, the universal Teichmüller space does not carry a natural Hermitian metric. However, it is possible (see [TT03] for detailed construction and proofs) to introduce a new Hilbert manifold structure on $T(1)$ such that it has a natural Hermitian metric. Namely, define the Hilbert space of harmonic Beltrami differentials on \mathbb{D}^* by

$$H^{-1,1}(\mathbb{D}^*) = \left\{ \mu = \rho^{-1} \bar{\phi}, \phi \text{ holomorphic on } \mathbb{D}^* : \|\mu\|_2^2 = \iint_{\mathbb{D}^*} |\mu|^2 \rho(z) d^2 z < \infty \right\},$$

where

$$\rho(z) = \frac{4}{(1 - |z|^2)^2}$$

is the density of the hyperbolic metric on \mathbb{D}^* .

The natural inclusion map $H^{-1,1}(\mathbb{D}^*) \hookrightarrow \Omega^{-1,1}(\mathbb{D}^*)$ is bounded, and it can be shown that the family \mathfrak{D} , defined by

$$T(1) \ni [\mu] \mapsto D_0 R_{[\mu]}(H^{-1,1}(\mathbb{D}^*)) \subset T_{[\mu]}T(1),$$

is an integrable distribution on $T(1)$. Integral manifolds of the distribution \mathfrak{D} are Hilbert manifolds modelled on the Hilbert space $H^{-1,1}(\mathbb{D}^*)$. Thus the universal Teichmüller space $T(1)$ carries a new structure of a Hilbert manifold. Similarly to the Banach manifold structure, the Hilbert manifold structure can be also described by a complex-analytic atlas. Let $T_0(1)$ be the component of origin of the Hilbert manifold $T(1)$, $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1) \subset T_0(1)$.

As a Hilbert manifold, the universal Teichmüller space $T(1)$ has a natural Hermitian metric, defined by the Hilbert space inner product on tangent spaces. Thus the Weil-Petersson metric is a right-invariant metric on $T(1)$, defined at the origin of $T(1)$ by

$$g_{\mu\bar{\nu}} = \langle \mu, \nu \rangle = \iint_{\mathbb{D}^*} \mu\bar{\nu}\rho(z)d^2z, \quad \mu, \nu \in H^{-1,1}(\mathbb{D}^*) = T_0T(1).$$

If

$$v = \sum_{n \neq -1, 0, 1} c_n e^{in\theta} \frac{\partial}{\partial \theta} \in T_0 \text{Möb}(S^1) \setminus \text{Homeo}_{qs}(S^1)$$

– the tangent space to a real Hilbert manifold $\text{Möb}(S^1) \setminus \text{Homeo}_{qs}(S^1)$ at the origin – then

$$\|v\|_{WP}^2 = \sum_{n=2}^{\infty} (n^3 - n)|c_n|^2,$$

The Weil-Petersson metric on $T(1)$ is Kähler with symplectic form ω_{WP} .

4. Riemann tensor of the Weil-Petersson metric

Let $G = \frac{1}{2}(\Delta_0 + \frac{1}{2})^{-1}$ be (the one-half of) the resolvent kernel of the Laplace-Beltrami operator of the hyperbolic metric on \mathbb{D}^* (acting on functions) at $\lambda = \frac{1}{2}$. Explicitly

$$G(z, w) = \frac{2u + 1}{2\pi} \log \frac{u + 1}{u} - \frac{1}{\pi}, \quad \text{where} \quad u(z, w) = \frac{|z - w|^2}{(1 - |z|^2)(1 - |w|^2)}.$$

Set

$$G(f)(z) = \iint_{\mathbb{D}^*} G(z, w)f(w)\rho(w)d^2w.$$

Theorem A.

- (i) *The Weil-Petersson metric is a Kähler metric on a Hilbert manifold $T(1)$, and the Bers coordinates are geodesic coordinates at the origin of $T(1)$.*
- (ii) *Let $\mu_\alpha, \mu_\beta, \mu_\gamma, \mu_\delta \in H^{-1,1}(\mathbb{D}^*) \simeq T_0T(1)$ be orthonormal tangent vectors. Then the Riemann tensor at the origin of $T(1)$ is given by*

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = -\frac{\partial^2 g_{\alpha\bar{\beta}}}{\partial t_\gamma \partial \bar{t}_\delta} = -\langle G(\mu_\alpha \bar{\mu}_\delta), \mu_\beta \bar{\mu}_\gamma \rangle - \langle \mu_\alpha \bar{\mu}_\beta, G(\bar{\mu}_\gamma \mu_\delta) \rangle.$$

- (iii) *The Hilbert manifold $T_0(1)$ is Kähler-Einstein with the negative definite Ricci tensor,*

$$Ric_{WP} = -\frac{13}{12\pi} \omega_{WP}.$$

5. Characteristic forms of $\mathcal{T}(1)$

Let $V = T_v\mathcal{T}(1)$ be the vertical tangent bundle of the fibration

$$\pi : \mathcal{T}(1) \rightarrow T(1).$$

The hyperbolic metric on $w^\mu(\mathbb{D}^*)$ defines a Hermitian metric on V , defining the first Chern form $c_1(V)$ – a $(1, 1)$ -form on $\mathcal{T}(1)$.

Mumford-Morita-Miller characteristic forms (“ κ -forms”) are (n, n) -forms on the Hilbert manifold $T(1)$, defined by

$$\kappa_n = (-1)^{n+1} \pi_* (c_1(V)^{n+1}),$$

where $\pi_* : \Omega^*(\mathcal{T}(1)) \rightarrow \Omega^{*-2}(T(1))$ is the operation of “integration over the fibers” of $\pi : \mathcal{T}(1) \rightarrow T(1)$, considered as a fibration of Hilbert manifolds.

Theorem B.

- (i) *On $\mathcal{T}(1)$, considered as a Banach manifold,*

$$c_1(V) = -\frac{2}{\pi} \omega_{VK}.$$

- (ii) *On $T(1)$, considered as a Hilbert manifold,*

$$\kappa_1 = \frac{1}{\pi^2} \omega_{WP}.$$

- (iii) *The characteristic forms κ_n are right-invariant on the Hilbert manifold $T(1)$ and for $\mu_1, \dots, \mu_n, \nu_1, \dots, \nu_n \in H^{-1,1}(\mathbb{D}^*) \simeq T_0T(1)$,*

$$\begin{aligned} &\kappa_n(\mu_1, \dots, \mu_n, \bar{\nu}_1, \dots, \bar{\nu}_n) \\ &= \frac{i^n (n+1)!}{(2\pi)^{n+1}} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \int \int_{\mathbb{D}^*} G(\mu_1 \bar{\nu}_{\sigma(1)}) \dots G(\mu_n \bar{\nu}_{\sigma(n)}) \rho(z) d^2z. \end{aligned}$$

6. Applications

The Weil-Petersson properties of the universal Teichmüller space $T(1)$ are “universal” in the sense that all curvature properties of finite-dimensional Teichmüller spaces can be deduced from them. In particular, Wolpert explicit formulas [Wol86] follow from Theorems **A** and **B** by using an “averaging procedure”, based on a uniform distribution of lattice points of a cofinite Fuchsian group in the hyperbolic plane (see [TT03] for details). There are also connections with Hilbert spaces of univalent functions and other related issues, which will be discussed elsewhere.

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