Solutions to the Practice Final Exam

1. (a)-(6), (b)-(1), (c)-(2), (d)-(3), (e)-(4), (f)-(5)

2. Plugging in the functions to the equations, we find the answer is (b) $y = 2e^x + e^{-x}$.

3. This equation is separable. Separate variables, we have

$$\frac{dy}{y^2} = \cos x dx$$

Integrate both sides,

$$-y^{-1} = \sin x + C$$

which we can rewrite as

$$y = \frac{-1}{\sin x + C}$$

where C is a constant. To determine it, we use the initial condition, $y(0) = 5$. Plug it in, we get $5 = \frac{-1}{\sin 0 + C}$, which solves $C = -1/5$. Therefore, the final solution to the initial value problem is

$$y(x) = \frac{-1}{\sin x - 1/5} \quad or \quad y(x) = \frac{5}{1 - 5\sin x}$$

4. Suppose the temperature of the turkey is $y(t)$. According to Newton's law, y obeys the following differential equation,

$$y'(t) = k(y - 20)$$

where k is a constant, and 20 is the room temperature in °C. The initial condition is given by $y(0) = 100$°C. The goal is to solve this initial value problem.

$\frac{dy}{dt} = k(y - 20)$ is a separable equation. Separate variables,

$$\frac{dy}{y - 20} = kdt$$

Integrate both sides, we get

$$\ln(y - 20) = kt + C$$

where C is the integration constant. Exponentiate both sides, we get

$$y = Ae^{kt} + 20$$

where $A = e^C$ is an unknown constant, which can be determined using the initial conditions $y(0) = 100°\text{C}$ Plug it in, $100 = Ae^0 + 20 \Longrightarrow A = 80$. Now we need to solve for k, using the other condition $y(1) = 60°\text{C}$. Plugging this into the above equation, we get $60 = 80e^k + 20$, or $k = \ln 1/2 = -\ln 2$. Put all this together,

$$y = 80e^{-t\ln 2} + 20$$

is the solution to the problem. The question asks us to find the time t at which y is 30°C.

$$30 = 80e^{-t\ln 2} + 20 \Longrightarrow e^{-t\ln 2} = 1/8 \Longrightarrow -t\ln 2 = \ln 1/8$$

If you know $\ln 1/8 = -3\ln 2$, you get the final answer $t = 3$ hours.

5. suppose the amount of salt in the tank is S(t). We have $\frac{dS}{dt}$ =rate in - rate out. Since only pure water comes into the tank, rate in=0. The brine is going out of the tank at a rate of 10(L/min). We need to figure out how much salt is there in this 10(L) of brine. Since the tank holds 1000(L) of brine, in which the amount of salt is S(kg). Therefore, in each liter of brine, there is $\frac{S}{1000}$(kg/L) salt. So 10(L) of brine contains $10\frac{S}{1000}$(kg) of salt. In another word, rate out = $10\frac{S}{1000}$. Put these informations in, we obtain a differential equation which describe the amount of salt in the tank.

$$\frac{dS}{dt} = 0 - 10\frac{S}{1000} \quad or \quad \frac{dS}{dt} = -\frac{S}{100}$$

Separate variables,

$$\frac{dS}{S} = -\frac{1}{100}dt$$

Integrate both sides,

$$\ln S = -\frac{1}{100}t + C$$

Exponentiate

$$S(t) = Ae^{-\frac{1}{100}t}$$

with $A = e^C$ a constant. From the initial condition $S(0) = 15$, we determine $A = 15$. So the function that describes the amount of salt in the tank is

$$S(t) = 15e^{-\frac{1}{100}t}$$

After 20 minutes,

$$S(20) = 15e^{-\frac{1}{100}20} = 15e^{-\frac{1}{5}}$$

6. (a) $\lim_{n\to\infty} \cos(n\pi/2)$ doesn't exist, so the series is divergent.

(b) First sol: When n is big enough ($n > 4$), $\frac{1}{n!} < \frac{1}{n^2}$. $\sum \frac{1}{n^2}$ is convergent, (p-series with $p = 2 > 1$). Therefore $\sum \frac{1}{n!}$ is convergent by comparison.

second sol: the ratio test.

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}| = \lim_{n\to\infty} |\frac{1}{(n+1)!}\frac{n!}{1}| = \lim_{n\to\infty} |\frac{1}{n+1}| = 0 < 1$$

so the series is convergent.

(c). First sol: $\frac{n}{2n^2+3} > \frac{n}{2n^2} = \frac{1}{2n}$. Since $\sum \frac{1}{2n}$ is divergent (harmonic series), $\sum \frac{n}{2n^2+3}$ diverges by comparison.

Second sol: use limit comparison test, with $a_n = \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{n}{2n^2+3}} = \lim_{n\to\infty} \frac{1}{n}\frac{2n^2+3}{n} = \lim_{n\to\infty} \frac{2n^2+3}{n^2} = 2 > 0$$

Since $\sum \frac{1}{n}$ diverges, so is $\sum \frac{n}{2n^2+3}$.

(d). $\lim_{n\to\infty} \cos e^{-n} = \cos 0 = 1 \neq 0$, so the series diverges.

(e). $\frac{1}{n3^n} < \frac{1}{3^n} = (\frac{1}{3})^n$. $\sum(\frac{1}{3})^n$ is convergent (geometric series with $r = 1/3$), so our series is convergent as well.

Note: you could also use ratio test.

(f). This is an alternating series, so let's try alternating series test. First

$$\lim_{n\to\infty} \frac{\ln(n^2)}{n} = \lim_{n\to\infty} \frac{2\ln n}{n} = 0$$

then

$$\frac{2\ln(n+1)}{n+1} < \frac{2\ln n}{n}$$

for $n \geq 3$. This needs some justification, $(\frac{\ln x}{x})' = \frac{1-\ln x}{x^2} < 0$ when $\ln x > 1$, i.e., $x > e$. So the associated function is decreasing when $x > e$, which implies the sequence is decreasing when $n \geq 3$. Therefore the series converges by the laternating series test.

7.(a). By the ratio test, we know when

$$\lim_{n\to\infty} |\frac{(x-5)^{n+1}}{(x-5)^n}| = \lim_{n\to\infty} |x-5| = |x-5| < 1$$

the power series is convergent. Therefore the radius of convergence is 1, and the series converges in $(4, 6)$. To find the interval of convergence, we need to check the end points, namely, $x = 4$ and $x = 6$.

When $x = 4$, the power series becomes $\sum_{n=0}^{\infty}(-1)^n$, which diverges. ($\lim_{n\to\infty}(-1)^n$ doesn't exist.)

when $x = 6$, the power series becomes $\sum_{n=0}^{\infty} 1^n = \sum_{n=0}^{\infty} 1$. It diverges because $\lim_{n\to\infty} 1 = 1 \neq 0$.

So the interval of convergence is $(4, 6)$.

(b). By the ratio test,

$$\lim_{n\to\infty} |\frac{3^{n+1}x^{n+1}}{3^n x^n}| = \lim_{n\to\infty} |3x| = |3x| < 1$$

so for the power series to converge, $|x| < \frac{1}{3} \Longrightarrow$ the radius of convergence is 1/3. Now check the end points, $-1/3$ and $1/3$.

when $x = -1/3$, the series becomes $\sum 3^n(-1/3)^n = \sum(-1)^n$, diverges.

when $x = 1/3$, the series becomes $\sum 3^n(1/3)^n = \sum 1^n$, again diverges.

So the interval of convergence is $(-\frac{1}{3}, \frac{1}{3})$.

(c).

$$\lim_{n\to\infty} \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty} |(n+1)x|$$

This converges only when $x = 0$. So the interval of convergence is a single point, 0. Note that in this case, there is no end points to be checked.

(d).

$$\lim_{n\to\infty} \left|\frac{(x+2)^{n+1}(n+1)\ln(n+1)}{\frac{(x+2)^n}{n\ln n}}\right| \lim_{n\to\infty} \left|\frac{n\ln n}{(n+1)\ln(n+1)}(x+2)\right| = |x+2|$$

Because when $n \to \infty$, $n+1 \sim n$. If $|x+2| < 1$, our series is convergent. So the radius of convergence is 1, and the series converges in $(-3, -1)$.

when $x = -3$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\ln n}$. This is convergent by the alternating series test. ($\lim_{n\to\infty} \frac{1}{n\ln n} = 0$ and $\frac{1}{n\ln n}$ is decreasing for n big enough, check them).

when $x = -1$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{n\ln n}$. It's divergent (Ex 15, Sec 8.3, You ought to use the integral test).

So the interval of convergence is $[-3, -1)$.

8.(a). see example 3, Sec 8.7, page 616.

(b).

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n$$

(By eq. 1 in the box on page 618). It converges when $|-x| < 1$, or $|x| < 1$.

(c).

$$\sin x^3 = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1)!}$$

By eq. 3 in the box on page 618.

(d). $(\frac{1}{(1-x)^2})' = \frac{-2}{(1-x)^3}$, so $f(x) = \frac{1}{(1-x)^3} = -\frac{1}{2}(\frac{1}{(1-x)^2})'$.

$$f(x) = -\frac{1}{2}(\frac{1}{(1-x)^2})' = -\frac{1}{2}(\sum_{n=0}^{\infty}(n+1)x^n)' = -\frac{1}{2}\sum_{n=0}^{\infty}((n+1)x^n)' = -\frac{1}{2}\sum_{n=0}^{\infty}(n+1)nx^{n-1}$$

For the series expansion of $\frac{1}{(1-x)^2}$, refer to example 5 in sec 8.6, page 608. So if you have to start scratch and use eq. 1 in the box on page 618, you need to differentiate it twice.

(e). $(x-1)^3$ is already of the desired form, so we only need to worry about $\ln x$.

Using Taylor's formula (or the method discussed in class, check your notes), we find

$$\ln x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(x-1)^{n+1}$$

Multiply $(x-1)^3$ in, we get the desired power series expansion

$$(x-1)^3 \ln x = (x-1)^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(x-1)^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(x-1)^{n+4}$$

9.(a). compare this expression with eq. 2 in the box on page 618, we see that it's just the right hand side with x replaced by $-\ln 3$. so it must equal to the left hand side with x replaced by $-\ln 3$, which is $e^{-\ln 3} = e^{\ln 3^{-1}} = 1/3$.

(b).

$$\sum_{n=0}^{\infty} \frac{3^n}{5^n n!} = \sum_{n=0}^{\infty} \frac{(3/5)^n}{n!}$$

which is the right hand side of eq. 2 in the box on page 618 with x replaced by 3/5, so must equal to $e^{3/5}$.

(c). compare to eq. 4 in the box on page 618, we see the series equals to $\cos \pi = -1$.

10.

$$\int \frac{\sin x}{x} dx = \int \frac{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}{x} dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!} dx = \sum_{n=0}^{\infty} (-1)^n \int \frac{x^{2n}}{(2n+1)!} dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!}$$

$$\int_0^1 \frac{\sin x}{x} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!} |_0^1 = \sum_{n=0}^{\infty} (-1)^n \frac{1^{2n+1}}{(2n+1)(2n+1)!} - 0$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)(2n+1)!} = 1 - \frac{1}{18} + \frac{1}{600} - \frac{1}{29400} + \dots$$

So, up to the 3rd decimal point,

$$\int_0^1 \frac{\sin x}{x} dx \simeq 1 - \frac{1}{18} + \frac{1}{600} = \frac{1703}{1800}$$

by the error estimates of alternating series. We used eq. 3 in the box on page 618 to represent $\sin x$ as a power series.

11. By eq. 2 in the box on page 618, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we have

$$\frac{1}{\sqrt{e}} = e^{-1/2} = \sum_{n=0}^{\infty} \frac{(-1/2)^n}{n!} = 1 + (-\frac{1}{2}) + (-\frac{1}{2})^2 \frac{1}{2!} + (-\frac{1}{2})^3 \frac{1}{3!} + (-\frac{1}{2})^4 \frac{1}{4!} \dots\dots = 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} + \frac{1}{384} \dots\dots$$

$$\simeq 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} = \frac{77}{144}$$

the error is less than $\frac{1}{100}$ by the error estimates of alternating series.

12. (a) By eq. 3 in the box on page 618, $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$,

$$\sin 1 = \sum_{n=0}^{\infty} (-1)^n \frac{1^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} = 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040} \dots.$$

$$\simeq 1 - \frac{1}{6} + \frac{1}{120} = \frac{101}{120}$$

The error is less than $\frac{1}{1000}$ by error estimates on alternating series.

(b) By eq. 4 in the box on page 618, $\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$,

$$\cos 1/2 = \sum_{n=0}^{\infty}(-1)^n \frac{(1/2)^{2n}}{(2n)!} = 1 - (\frac{1}{2})^2\frac{1}{2!} + (\frac{1}{2})^4\frac{1}{4!} .... = 1 - \frac{1}{8} + \frac{1}{384} .... \simeq 1 - \frac{1}{8} = \frac{7}{8}$$

The error is less than $\frac{1}{100}$ by error estimates on alternating series.

13.(a). Suppose $y = \sum_{n=0}^{\infty} c_n x^n$ is a power series expansion of the differential equation $y' = xy$, which we rewrite as $y' - xy = 0$.

$$y' = (\sum_{n=0}^{\infty} c_n x^n)' = \sum_{n=0}^{\infty}(c_n x^n)' = \sum_{n=1}^{\infty} c_n n x^{n-1} = \sum_{m=0}^{\infty} c_{m+1}(m+1)x^m$$

We set $m = n - 1$ to convert the power series into the form $\sum ...x^m$.

$$xy = x\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+1} = \sum_{m=1}^{\infty} c_{m-1}x^m$$

We set $m = n + 1$ to convert the power series into the form $\sum ...x^m$. Plug these into the equation, we get

$$\sum_{m=0}^{\infty} c_{m+1}(m+1)x^m - \sum_{m=1}^{\infty} c_{m-1}x^m = 0 \Longrightarrow c_1 + \sum_{m=1}^{\infty}(c_{m+1}(m+1) - c_{m-1})x^m = 0$$

$$\Longrightarrow c_1 = 0 \quad and \quad c_{m+1}(m+1) - c_{m-1} = 0$$

so we obtain the recursive formula

$$c_{m+1} = \frac{c_{m-1}}{m+1}$$

when $m = 1$, it gives $c_2 = \frac{c_0}{2}$. $m = 2$, $c_3 = \frac{c_1}{3} = 0$, $m = 3$, $c_4 = \frac{c_2}{4} = \frac{c_0}{4 \cdot 2}$, (...you need to keep going for a while if you don't see the pattern). In general, $c_n = 0$ when n is odd, when n is even, the coeeficients are $\frac{1}{2^n (n)!}$. So the function is $y(x) = \sum_{n=0}^{\infty} \frac{c_0}{2^n (n)!}x^{2n}$. Plug in the initial value, we find $c_0 = 1$. The final solution to the initial value problem is $y(x) = \sum_{n=0}^{\infty} \frac{1}{2^n (n)!}x^{2n}$

(b). This equation is separable, separate variables, we get $\frac{dy}{y} = xdx$. Integrate both sides, $\ln y = \frac{1}{2}x^2 + C$, or $y = Ae^{1/2x^2}$, where $A = e^C$ is a constant. Plug in the initial value, we find A=1. Therefore, the functional solution to the initial value problem is $y = e^{\frac{1}{2}x^2}$.

(c). In (a) and (b), we found 2 solutions to the same initial value problem. Therefore, they must equal to each other. That is,

$$y = e^{\frac{1}{2}x^2} = \sum_{n=0}^{\infty} \frac{1}{(n)!} x^{2n}$$

which makes sense, because (using eq. 2 in the box on page 618)

$$e^{\frac{1}{2}x^2} = \sum_{n=0}^{\infty} \frac{1}{(n)!} (\frac{1}{2}x^2)^n = \sum_{n=0}^{\infty} \frac{1}{2^n (n)!} x^{2n}$$

14. (a)

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}{x} = \lim_{x\to 0} \frac{x - \frac{x^3}{3!} + ...}{x} = \lim_{x\to 0} (1 - \frac{x^2}{3!} + ...) = 1$$

Because all the terms other than the 1st contains x, therefore vanishes when taking the limit.

(b)

$$\lim_{x\to 0} \frac{1 - \cos x}{e^x - 1 - x} = \lim_{x\to 0} \frac{1 - (1 - \frac{x^2}{2!} + \frac{x^4}{4!} ...)}{(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...) - 1 - x} = \lim_{x\to 0} \frac{\frac{x^2}{2!} - \frac{x^4}{4!} ....}{\frac{x^2}{2!} + \frac{x^3}{3!} + ...} = 1$$

Here, you have a quotient of 2 power series, so you need to use long division. However, since we are looking for the limit, you just need 1 term. It turns out it's equivalent to just keep the leading terms in the numerator and the denominator.