LUDWIG DMITRIEVICH FADDEEV

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Elected ForMemRS 2010

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Ludwig Faddeev was a preeminent mathematical physicist of the second half of the twentieth and early twenty-first century. He made fundamental contributions to mathematics and theoretical physics. These included the solution of the quantum three-body scattering problem, groundbreaking work on the quantization of gauge theories, the formulation of Hamiltonian methods for classical integral models, and the creation, with colleagues, of the quantum inverse scattering method which led to the discovery of quantum groups. An incomplete list of important concepts and results named after him include the Faddeev equations, the Faddeev–Popov determinant, Faddeev–Popov ghosts, the Faddeev–Green function, the Gardner–Faddeev–Zakharov bracket, Faddeev–Zamolodchikov algebra and the Faddeev quantum dilogarithm. Faddeev was a charismatic teacher and a natural leader at the Steklov Institute of the Russian Academy of Sciences and the International Mathematical Union.

The person

Family background

Ludwig Dmitrievich Faddeev was born on 23 March 1934 in Leningrad in the Soviet Union (now Saint Petersburg, Russia). He had two siblings: an older sister, Maria, a chemist, and a younger brother, Mikhail, a mathematician. His father, Dmitry Konstantinovich Faddeev, was

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a well-known Soviet mathematician, the founder of the Leningrad modern algebra school, while his mother, Vera Nikolaevna Faddeeva, née Zamyatina, was also a mathematician. Vera Nikolaevna was a cousin of the Russian–Soviet author Evgeny Zamyatin, famous for his anti-utopia novel *We* (Zamyatin 1924).

Dmitry Konstantinovich studied for three years at the N. A. Rimsky-Korsakov Leningrad Conservatory, and was an accomplished pianist. Ludwig's parents dreamt of a musical career as a conductor for their son. According to family legend, the name Ludwig was homage to Ludwig van Beethoven. During the Second World War the Faddeev family was evacuated and mainly lived in Kazan, a city on the banks of the Volga river, 800 km east of Moscow. It had no music school, and so dreams of Ludwig becoming a conductor were not destined to be realized.

Early years

After the war, the Faddeev family returned to Leningrad, and Ludwig attended the boys' school No. 155; at that time school education in the Soviet Union was separate for boys and girls. He was good at mathematics, but had other interests: photography, detector radio modelling and skiing. His passion during his school years was reading rather than mathematics. In the fifth grade, for example, he learned the medieval history of England by reading Shakespeare's history plays. Faddeev graduated in 1951 and entered the Physics Department of Leningrad State University rather than the Mathematics Department where his father was Dean. In his 2005 press interview Faddeev remembered:

> It was in 1943. I saw my father was very excited, wandering around in our room. I asked, 'What happened?' He answered that he just discovered something very important.* I asked, 'How many people can understand this?' My father answered that only about five people . . . I decided that instead of becoming a mathematician, I would become a physicist.

His escape from mathematics was, however, not to last. In 1950 a brilliant young mathematician, Olga Aleksandrovna Ladyzhenskaya,† was appointed to the newly established chair of mathematical physics in the physics department. She was a charismatic teacher and responsible for all mathematics courses for physics students, and became Faddeev's mentor.

In 1951, when Faddeev entered the university, quantum mechanics was still a new field–only 25 years had passed since Schrödinger's equation was published. The remarkable achievements of quantum mechanics and quantum field theory attracted the attention of mathematicians, and Ladyzhenskaya organized a student seminar to study K. O. Friedrichs' monograph *Mathematical aspects of the quantum theory of fields* (Friedrichs 1953). Faddeev was fascinated by the mathematical beauty and intricacy of quantum field theory and was the main speaker at the seminar. He dreamt of being able to get a serious grasp of quantum field theory, which describes quantum systems with infinitely many degrees of freedom. Each subsequent seminar would begin with Ladyzhenskaya saying: 'Ludwig, remind us, please, the definition of creation and annihilation operators.' Faddeev was under the powerful influence of Vladimir Aleksandrovich Fock, one of the giants of twentieth-century theoretical physics,

* This refers to group cohomology, the foundation of homological algebra, discovered independently by D. K. Faddeev (Faddeev 1947) and by S. Eilenberg and S. MacLane (MacLane 1947).

† Ladyzhenskaya, one of the greatest women mathematicians of the twentieth century, is famous for her rigorous proof of the Fourier method for hyperbolic partial differential equations.

who also worked at the Leningrad State University. Faddeev regarded Fock, Hermann Weyl (ForMemRS 1936), Paul Dirac (FRS 1930) and Richard Feynman (ForMemRS 1965) as his 'spiritual teachers', whose works were a great influence on his thinking.

Life at the Steklov Institute

In 1956 Faddeev became a graduate student at the Leningrad branch of the V. A. Steklov Mathematical Institute of the USSR Academy of Sciences, commonly known as the Steklov Institute in Leningrad and abbreviated to LOMI, now POMI. Faddeev's scientific supervisor was Ladyzhenskaya, and his 1959 PhD thesis, '*Properties of the S-matrix for the scattering on the local potential*', is now a classic reference for the inverse problem of the one-dimensional Schrödinger equation (4)*. Faddeev gracefully acknowledged:

> It is interesting that Ladyzhenskaya's own interests were not close to this subject. However, it was she who made this choice for my education. The intuition of the scientific supervisor plays an important role for the development of the student. I am forever grateful to Ladyzhenskaya for the advice she gave me to develop my own independent scientific career.

Faddeev kept Ladyzhenskaya's portrait on the writing desk in his office at POMI, the institution in which he worked for the rest of his life.

By the end of 1950s, mathematical life in the USSR was blossoming and the young Faddeev participated in numerous conferences. He gave talks at Gel'fand's famous seminar (Israil Moiseevich Gel'fand (ForMemRS 1977)), attended the Rochester conference on theoretical physics in Kiev in 1959 and the International Congress of Mathematicians (ICM) in Stockholm in 1962, where he befriended Peter Lax, Jurgen Moser and Louis Nirenberg, whom he met again in Novosibirsk in 1963. This was a golden era of Soviet mathematics. Faddeev was friendly with many mathematicians of his generation, including V. I. Arnol'd (ForMemRS 1988), F. A. Berezin, A. A. Kirillov, Y. I. Manin, V. P. Maslov, R. A. Minlos, S. P. Novikov and Y. G. Sinai.

Pursuing his own path, Faddeev solved the quantum three-body scattering problem in just under three years. This monumental achievement was immediately recognized both at home and abroad. In 1963, at the young age of 29, Faddeev defended his doctoral dissertation 'Mathematical questions in the quantum theory of scattering for a three-particle system' (3).

Inspired by his success with a quantum system with three particles, Faddeev decided to work in quantum field theory, which describes systems with infinitely many particles. At that time quantum field theory went out of fashion because of the belief that quantum electrodynamics was self-contradictory, despite the fact that Feynman, Schwinger and Tomonaga were awarded the 1965 Nobel Prize in physics for their fundamental work on quantum electrodynamics. Faddeev was essentially studying on his own, reading classic papers by Feynman and Schwinger and, on the more mathematical side, by Irving Segal. He was seeking a theory in which interactions would not be introduced ad hoc, but rather described by some general and natural principle.

A typical example was Einstein's theory of general relativity (Albert Einstein ForMemRS 1921), whose quantization, the theory of quantum gravity, is still out of reach, a sort of 'Holy Grail' of theoretical physics. In his famous 1962 lectures in Poland, Feynman attempted to formulate such a theory (Feynman 1963). In preparation for entering the field, Faddeev studied

* Numbers in this form refer to the bibliography at the end of the text.

the book by A. Lichnerowicz, *Théories relativistes de la gravitation et de l'électromagnetisme* (Lichnerowicz 1955). In 1964, Faddeev accidentally came across a Russian translation of his other book, *Théorie globale des connexions et des groupes d'holonomie* (Lichnerowicz 1957) in a second-hand bookshop on Nevsky Prospect. He immediately realized that connections in principal and vector bundles from the Lichnerowicz book are precisely the Yang–Mills fields he knew from Feynman's lectures and other papers. At that time nobody had any appreciation of the significance of the Yang–Mills theory. For Feynman, the quantization of the theory was just an exercise, necessary for quantizing the theory of gravity. By the end of 1966 Faddeev and Victor Popov, his young colleague at LOMI, submitted their famous paper (6) on the quantization of the Yang–Mills theory to the new European journal *Physics Letters*. Faddeev remembered how carefully he chose the words in this paper, as if he knew that it would become a cornerstone of quantum field theory. Nowadays, Yang–Mills fields play a fundamental role in the Standard Model. However, it was the young Faddeev who first understood the importance and geometric origin of the Yang–Mills fields, before they became so popular in the 1970s.

At that time Faddeev was working in the Laboratory of Mathematical Physics, which meant partial differential equations, with Ladyzhenskaya as its head. Soon he became head of a small group called the Mathematical Problems of Physics, which initially consisted of himself, V. N. Popov and his first graduate student, P. P. Kulish. They were later joined by I. Ya. Arefyeva, his only female student (figure 1). In 1969 Faddeev's group became the Laboratory of Mathematical Problems of Physics, and started expanding by admitting students from Leningrad State University. During 1968 to 1973, he taught a course in quantum mechanics in the Mathematics Department at the Leningrad State University. The course notes were later published (22) and became a classic reference.

From the late 1960s Faddeev ran a seminar on Thursdays at LOMI, which became the weekly Faddeev seminar on quantum field theory. It covered a variety of topics in mathematics and theoretical physics, with many distinguished national and international speakers. Frequent speakers from Moscow included A. A. Belavin, A. M. Polyakov, P. B. Wiegmann and A. A. Zamolodchikov from the Landau Institute. Later, many Western mathematicians and physicists gave talks at the seminar, including M. F. Atiyah (FRS 1962, PRS 1990–1995), D. Gross, G. Segal (FRS 1982) and E. Witten (ForMemRS 1999).

Mathematical philosophy

Faddeev was not fond of long philosophical discussions and arguments, but he had firm beliefs. Unlike his friend Vladimir Arnol'd, who once famously said that 'mathematics is a part of physics', Faddeev believed that physics was the study of the fundamental structure of the matter, while mathematics was the universal language for presenting these truths. He kept saying that fundamental physics has a single problem whose ultimate solution means the 'end of physics', while mathematics has infinitely many problems and will continue forever. Faddeev believed that mathematics, and therefore theoretical physics, is expressed by formulas and the ultimate criterion of their truthfulness is the beauty of a formula.

Personal life

Faddeev had a happy family life. His wife, Anna Mikhailovna Veselova (figure 2), was a fellow student at the university. Her father, Mikhail Grirorievich Veselov, was a professor in the Physics Department specializing in quantum mechanics and quantum chemistry. Her mother, Elizaveta Nikolaevna Yustova, was also a physicist, working on colour science—her

Figure 1. In the Steklov Institute, Leningrad, 1978. First row, from left to right: Faddeev, Ladyzhenskaya, P. P. Kulish and I. Ya. Arefyeva. Photograph from the Faddeev family collection. Credit: Dr. Arthur Budagov. (Online version in colour.)

work was cited by Feynman in chapter 35 of his famous *Feynman lectures on physics, vol. 1* (Feynman 1962). Faddeev and Anna Veselova had two daughters, Elena and Maria, and four grandchildren.

Faddeev enjoyed sport—for seven years he was stroke of the rowing eight team of the Physics Department and a member of the cross country skiing team. Throughout his life he loved skiing and hiking. He also had a lifelong love of literature, Vladimir Nabokov's *Ada* being one of his favourite novels. In his later years he preferred Russian classics, especially works by the Russian nineteenth-century classical novelist Nikolai Leskov, as well as the contemporary Russian writer Victor Pelevin.

Faddeev's musical taste was impeccable. He was very proud of his music collection and audio system. Leaving aside Bach, whom he thought to be *hors de concours*, Faddeev's favourites were centred on the harmony of nineteenth-century music. His tastes varied from *Peer Gynt* by Edvard Grieg to *Der Rosenkavalier* by Richard Strauss. He also enjoyed jazz and popular music. He once said:

> Popular music ended for me with the Beatles. I listen to the Beatles and Rolling Stones, but cannot listen to Pink Floyd. It is fictional, empty music, like Rachmaninov. In classical music I stopped at Bartok, whom I do not understand, and Stravinsky, but I do like the music of Prokofiev and Shostakovich.

Figure 2. Faddeev with his wife Anna. Photograph from the Faddeev family collection. (Online version in colour.)

Public life

Faddeev did not shy away from public life in general nor from administrative duties. In 1976 he became a full member (Academician) of the USSR Academy of Sciences, and in the same year he was appointed director of LOMI—he remained its head until 2000. He was also founding director of the Euler International Mathematical Institute (IMU) from 1993 to 2017. During the same period, he was academician-secretary of the Mathematical Sciences Division of the Russian Academy of Sciences, comprising both mathematics and applied mathematics and informatics.

Faddeev also held important international positions. He was the vice-president of the International Mathematical Union (IMU) (1983–1986) and president (1986–1990). During his tenure at the IMU, Faddeev became friendly with Sir Michael Atiyah and Jean-Pierre Serre (ForMemRS 1974). He was well respected and very popular with his colleagues in Europe and in the USA, and travelled frequently; yet, when the Soviet Union collapsed in 1991, Faddeev, unlike many of his colleagues, did not accept a position abroad. He was disheartened by the decline of fundamental science in Russia during the 1990s and by the loss of respect for scientists. He remained on very friendly terms with his former students and colleagues working abroad, and regained his optimism when scientific life in Russia started to improve and the fundamental sciences gained more recognition. The devastating reform of the Russian Academy of Sciences in 2013 came as a deep shock to him.

Faddeev belonged to 'the generation of giants', mathematicians and physicists born in the 1930s who visibly changed and advanced the course of science and left scientific schools as their legacy. Our generation is lucky to have had scientists of the calibre of Faddeev and Atiyah.

Mathematics and theoretical physics*

Scattering theory

Scattering theory was Faddeev's first love. It plays a fundamental role in the description of quantum phenomena, describing changes in a quantum particle state as it passes through the potential centre. The mathematical formalism of scattering theory uses a Hamiltonian operator acting in Hilbert space and the *S*-matrix describing scattering by the potential centre.

In the 1950s and 1960s the mathematical formalism of quantum mechanics was a challenging source of important problems in the spectral theory of differential operators, as was attested in the classic monograph by E. C. Titchmarsh (FRS 1931) (Titchmarsh 1962). The major mathematical problem was the rigorous formulation of scattering theory for the multi-dimensional Schrödinger operator with a short-range potential; the corresponding problem for the Schrödinger operator with three interacting particles was considered to be out of reach.

Faddeev made several fundamental contributions to scattering theory. With Ladyzhenskaya (1), he studied the Friedrichs model and developed the perturbation theory of the continuous spectrum. The method, improved by Faddeev (5), was successfully used in various problems of scattering theory, from three-particle scattering (3) to the spectral theory of automorphic functions (8). The model, nowadays referred as the Friedrichs–Faddeev model, is fundamental for studying the perturbation of a continuous spectrum in functional analysis. Faddeev stated that Ladyzhenskaya 'helped me by establishing a splendid compactness criterion in the space of Hölder functions.'

Faddeev's best-known contribution to scattering theory is his solution of the quantum mechanical scattering problem for three particles (3), a truly groundbreaking result. The problem is very difficult because the total potential does not decrease in some spatial directions. Moreover, in the scattering process two particles can form a bound state, which is prohibited in the two-particle case by conservation of energy. In mathematical terms, this means that the continuous spectrum of a three-particle Hamiltonian for interacting particles is quite different from that of a free three-particle Hamiltonian, and so standard perturbation theory is not applicable.

Faddeev successfully overcame these difficulties. His main idea was to isolate and sum-up singular terms in the standard integral equations, using methods developed in his study of the Friedrichs model. As a result, a new set of integral equations, nowadays called Faddeev's equations, is obtained; the major advantage is that Faddeev's equations can be analysed using Fredholm theory. This analysis is rather complicated, and one can only admire Faddeev's technical power in carrying out these sophisticated calculations and his clever estimation of various singular integrals.

The result is an eigenfunction expansion theorem for a three-particle Hamiltonian operator and a detailed description of the block structure of the *S*-matrix. These blocks describe different physical processes and reflect the more versatile nature of three-particle scattering

* Technical details of the research described in this part can be found in the review by Takhtajan *et al.* (2017).

compared with the scattering by a central potential: the first represents multi-channel scattering processes and the second one-channel scattering.

Faddeev's equations provide a powerful mathematical tool for studying quantum three-body problems and are used in numerical calculations for processes of nuclear physics—they provide excellent accuracy in the calculation of scattering processes.

The inverse scattering problem

Another important problem in the spectral theory of differential operators with clear physical significance is the inverse scattering problem, meaning recovering the potential in the Schrödinger operator from a given scattering amplitude. It was originally formulated for the radial Schrödinger equation and was solved by Marchenko and Krein, and by Gel'fand and Levitan using a different approach. The equivalence of these approaches was proved by Faddeev (2). This survey and its continuation (15) became basic references for several generations of experts in scattering theory and mathematical physics in the USSR and worldwide. Faddeev remarked (2): 'It is interesting to note that in the USSR the inverse problem has been studied on the whole by mathematicians, whereas abroad it has been studied almost exclusively by physicists.'

The origin of the first paper (2) is of interest. In 1958, academician N. N. Bogolyubov invited Faddeev to give a talk on the inverse problem for the radial Schrödinger equation at a conference organized by the Dubna Laboratory of Theoretical Physics. Gel'fand, Krein, Levitan and Marchenko were among the audience. The talk was based on a paper that Faddeev had prepared for his postgraduate examination, and resulted in an invitation to the 25-year old author to submit a review paper (2) to the prestigious journal *Uspekhi Matematicheskikh Nauk*.

The corresponding problem for the one-dimensional Schrödinger equation, which is intermediate between the radial and three-dimensional cases, had not been rigorously analysed. Its complete solution was given in Faddeev's PhD thesis (4). Specifically, he formulated precise mathematical conditions on the potential so that the Hamiltonian had a two-fold absolutely continuous spectrum and a finite number of negative eigenvalues, completely solving the direct and inverse scattering problems. In this case the *S*-matrix is a 2×2 matrix, and Faddeev gave the necessary and sufficient conditions for such a matrix to be an *S*-matrix of the one-dimensional Schrödinger equation with decaying potential. He then derived integral equations to recover the potential from 'scattering data'. This result, Faddeev's theorem, is classic and a cornerstone for applications to the theory of integrable nonlinear evolution equations.

The inverse scattering problem for the three-dimensional Schrödinger operator is much more complicated than the previous two. The fundamental difference is that the *S*-matrix, which is uniquely determined by the local potential, seemed to depend on a larger number of parameters than the potential itself. A complete list of properties of the *S*-matrix, which followed from the locality of the potential in space, was not known—the problem was considered extremely difficult. It was solved by Faddeev in a series of papers, subsequently summarized (15). Faddeev himself regarded these as his most technically advanced papers.

Faddeev had discovered new types of Green functions for the multi-dimensional Schrödinger operator, now referred to as Faddeev–Green functions. These functions form a family, parameterized by points on a two-dimensional unit sphere, which represent all directions in three-dimensional space. Using these new functions, Faddeev was able to describe all analytic properties of the scattering amplitude, completely characterizing the *S*-matrix. He proved that these properties are also sufficient, and uniquely determine the local potential. The demonstration (15) is a remarkable *tour de force*.

The spectral theory of automorphic functions

In the nineteenth century, Felix Klein (ForMemRS 1885) and Henry Poincaré (ForMemRS 1894) discovered automorphic forms and functions as generalizations of trigonometric and elliptic functions. Nowadays, they play a fundamental role in the Langlands programme relating representation theory and geometry to number theory. In his classic paper, Selberg (Selberg 1956) introduced spectral methods into the theory of automorphic forms, which led to active developments worldwide. In particular, Gel'fand, Graev, Pyatetskii-Shapiro and Fomin established a connection between Selberg's approach and the theory of infinite-dimensional representations of semi-simple Lie groups. However, their methods, summarized by Gel'fand *et al.* (Gel'fand *et al*. 1966), did not lead to a complete spectral decomposition of the automorphic Laplace operator because of the presence of a continuous spectrum.

Faddeev (8) solved this problem by cleverly applying his perturbation theory of the continuous spectra developed for the Friedrichs model (5). Using these methods, Faddeev proved the analytic continuation of the eigenfunctions of the continuous spectrum, the Eisenstein–Maass series, and established the eigenfunction expansion theorem for the automorphic Laplace operator. In particular, Faddeev's result gives an operator-theoretical proof of the prime number theorem, the statement that the Riemann zeta function $\zeta(s)$ does not vanish on the line where the real part of s is 1.

Faddeev was the first to provide a rigorous proof to the results announced by Selberg (Selberg 1956).* Serge Lang (Lang 1975) wrote:

> The Faddeev paper on the spectral decomposition of the Laplace operator on the upper half-plane is an exceedingly good introduction to analysis, placing the latter in a nice geometric framework. . . . Faddeev's method comes from perturbation theory and scattering theory, and as such is interesting for its own sake, as well as to analysts who may know the analytic part and may want to see how it applies in the group-theoretic context.

In a further paper, co-authored with B. S. Pavlov (12), Faddeev applied Lax–Phillips scattering theory to the Laplace operator for the modular group, and gave a pure operator-theoretic reformulation of the Riemann hypothesis. This brilliant and unexpected result inspired Lax and Phillips to give a systematic exposition of the spectral theory of the Laplace operator using their approach. In the preface to their book (Lax & Phillips 1976), they wrote:

> Our interest in harmonic analysis of SL(2, $\mathbb{R}$) stems from the fascinating 1972 Faddeev–Pavlov paper (12) in which they showed that the Lax–Phillips scattering theory could be applied to the automorphic wave equation. After studying the paper (12) we decided to redo this development entirely within the framework of our theory.

These methods (8) could be used in a systematic derivation of the famous Selberg trace formula by Faddeev and his students A. B. Venkov and V. L. Kalinin (13). They showed that the function introduced by Selberg, now known as the Selberg zeta function, is a regularized characteristic determinant of the automorphic Laplace operator. Although Faddeev did not

* An approach given in Selberg's lectures at Göttingen University in 1954 was unpublished at the time.

publish more on this subject, he was always interested in it. The idea of a regularization, either of the Hilbert identity as an equation for the resolvent of a self-adjoint operator or of the definition of its trace and characteristic determinant, is a recurrent theme in all Faddeev's work, from the theory of the Schrödinger operator and automorphic Laplace operator to the quantum theory of gauge fields. In 1981 Faddeev gave a plenary lecture at the conference dedicated to the ninetieth birthday of I. M. Vinogradov (ForMemRS 1942). He talked about the universal role of determinants in mathematics and theoretical physics, from the Selberg zeta function to Faddeev–Popov ghost determinants in the theory of Yang–Mills fields. In his scientific autobiography (29), he wrote: 'If in a single term I had to characterize my technical means, it would be determinants.'

Classical and quantum integrable systems

At the beginning of a 1971 symposium in Novosibirsk, V. E. Zakharov told Faddeev about a remarkable paper by the US applied mathematicians C. S. Gardner, J. M. Greene, M. D. Kruskal (ForMemRS 1997) and R. Miura (Gardner *et al.* 1967) on the integration of the Korteweg–de Vries equation (KdV), a well-known nonlinear partial differential equation in the theory of shallow waves and plasma physics. Their discovery was striking—if at each time t one associates with a KdV solution a one-dimensional Schrödinger operator with a potential depending on t as a parameter, then the time evolution of its scattering data is given by surprisingly simple formulas. This was immediately thought of as a nonlinear generalization of the Fourier transform method, where the inverse scattering problem played the role of an inverse Fourier transform.

The magic of this result was explained by Lax (Lax 1968), who showed that the KdV equation is equivalent to a certain operator equation, nowadays known as the Lax equation. Soon after, it turned out that there is a wide class of nonlinear evolution equations integrable by this method, nowadays called the 'inverse scattering method' (ISM).

Faddeev's discussion of these results with Zakharov led to their joint paper (11), where the KdV equation was shown to be an infinite-dimensional completely integrable Hamiltonian system. This was truly a groundbreaking result—its fundamental role and impact cannot be overestimated. The notion of complete integrability goes back to the classical works of Leonhard Euler (FRS 1747), Joseph-Louis Lagrange (FRS 1791), Karl Gustav Jacob Jacobi (ForMemRS 1833) and Sofya Vasilyevna Kovalevskaya on rigid-body dynamics. In the mid twentieth century, however, integrability in Hamiltonian mechanics seemed to be a very rare phenomenon and there were no nontrivial integrable examples with infinitely many degrees of freedom. Faddeev and Zakharov's paper (11) for the first time gave such a nontrivial example, starting with the Hamiltonian theory of equations integrable by ISM. This topic became extremely popular and was further developed by mathematicians and physicists worldwide. Faddeev repeatedly emphasized that this paper with Zakharov combined in a miraculous way topics he had worked on quite independently—the inverse scattering problem for the one-dimensional Schrödinger equation, trace identities and Hamiltonian mechanics.

At the beginning of 1972 Faddeev visited the USA, where he gave a number of talks and, in particular, one about his new work with Zakharov. The US physicist J. R. Klauder, who attended the lecture, mentioned a nonlinear version of the Klein–Gordon equation, the sine-Gordon equation (SG), which originally appeared in the study of surfaces of constant negative curvature and then in nonlinear optics and in the theory of superconductivity in studies of the Josephson effect. This equation is relativistically invariant, which immediately interested

Faddeev. It may be regarded as an essentially nonlinear model of classical field theory in two-dimensional space-time.

In 1973 Faddeev, his new student L. A. Takhtajan (the present author) and V. E. Zakharov applied the Lax method to the SG equation. Faddeev and Takhtajan developed the ISM formalism and proved that the SG equation is a completely integrable infinite-dimensional Hamiltonian system (14). The explicit formulas for the Hamiltonian and total momentum of the SG model show that, in addition to classical relativistic massive particles, there are solitons, massive particles with different masses and nontrivial topological charges, as well as particles representing the bound states of solitons and antisolitons. Faddeev used to say that at the level of classical spectra, this was the first example that one non-linear self-interacting field generates several kinds of particles, realizing Einstein's dream in his pursuit of the elusive Unified Field Theory.

The Hamiltonian formalism was Faddeev's passion, which he shared with his students. The Hamiltonian approach to integrable equations solvable by ISM was worked out in the Laboratory of Mathematical Problems of Physics at LOMI by Faddeev, Kulish, N. Y. Reshetikhin, A. G. Reyman, M. A. Semenov-Tian-Shansky, E. K. Sklyanin (FRS 2008) and Takhtajan, work summarized in a monograph by Faddeev and Takhtajan (41).

Faddeev was, however, always interested in quantum field theory, and from the very beginning realized that integrable equations have rich potential for quantization (figure 3). As he wrote (29): 'If in a single word I had to focus the sphere of my scientific interests, it would be quantization.' The most interesting equation from this point of view was the SG equation. Faddeev expected that the quantum version would result in an example of a relativistic quantum field theory with a rich spectrum of particles, such as the principal particles, solitons and their bound states, all generated by a single field. It was natural to expect that, as in the classical SG model, the quantum SG model would be exactly soluble. Specifically, the infinitely many conservation laws obtainable from the SG equation imply that the number of particles of each kind and their momenta are conserved during interactions.

In physics seminars, however, Faddeev was fiercely attacked by V. N. Gribov and his school, who asserted that quantum corrections destroy completely the classical integrability of the SG model. Still, Faddeev was certain that integrability remains after quantization: there is no multiple creation of quantum particles in the SG model, and the number of particles of each kind and their momenta are conserved. This result was proved within the framework of perturbation theory by I. Ya. Arefyeva and Faddeev's student V. E. Korepin (Arefyeva & Korepin 1974). It led to the remarkable conclusion that the scattering is factorizable: all many-particle processes are reduced to two-particle interactions. Subsequently, in 1975, Faddeev, Korepin and Kulish performed a systematic quantization of the SG model by means of functional integration (17).

Successful semiclassical quantization of the SG model raised hopes that it could be quantized exactly, and at the beginning of 1978 Faddeev posed the problem of extending the ISM to the quantum realm. Combining the classical ISM with Baxter's method for solving the eight-vertex model in statistical mechanics (Baxter 1972), and using earlier work by Sklyanin and Faddeev (18), such a method was formulated by Faddeev, Sklyanin and Takhtajan (20), who used it to solve the quantum SG model exactly. Called the quantum inverse scattering method (QISM), it is a method for exact solution of quantum models corresponding to equations solvable by the classical ISM technique.

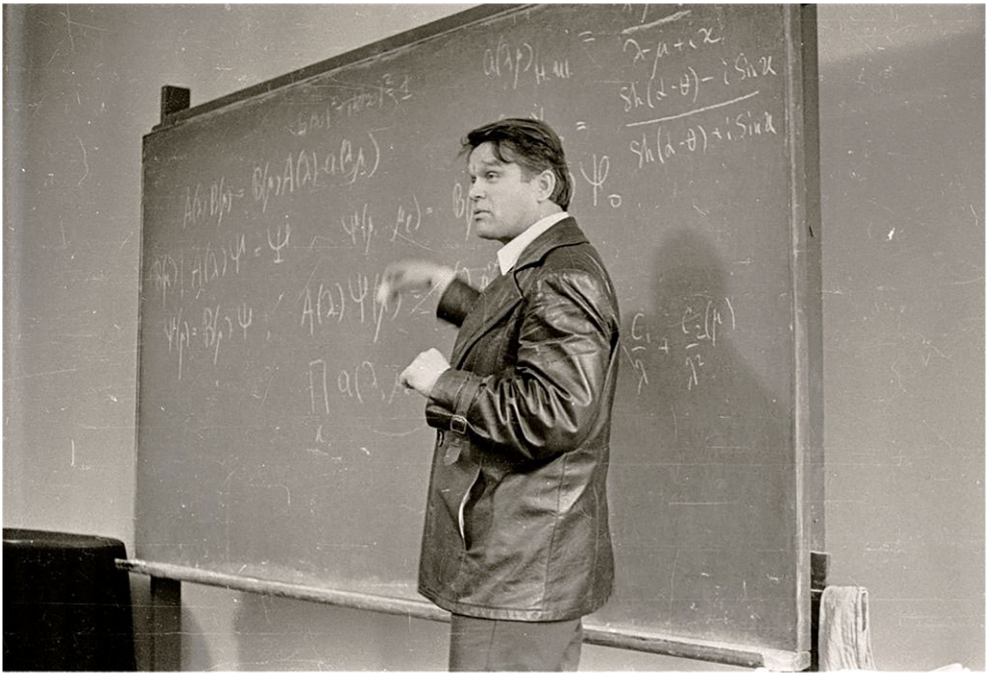

Figure 3. Faddeev giving a talk on the quantization of solitons. Photograph from the Faddeev family collection. Credit: Dr. Arthur Budagov. (Online version in colour.)

The key ingredients of the method are the now famous Yang–Baxter equation (figure 4) and the algebraic Bethe ansatz, as formulated by Faddeev, Sklyanin and Takhtajan (20) and emphasized by Faddeev and Takhtajan (21). Besides the interest of these pure mathematical developments, the QISM technique has applications to realistic physical models. Thus, Faddeev and Takhtajan applied QISM to the isotropic antiferromagnetic Heisenberg spin chain of spins $\frac{1}{2}$, studied by H. Bethe (ForMemRS 1957) in 1931 (Bethe 1931), and discovered (23, 24) that elementary excitations, spinons, have spin $\frac{1}{2}$ and not 1, as was commonly believed in the condensed matter community. The paper (23) attracted a lot of attention, and these spin $\frac{1}{2}$ spinons are considered prototypical fractionalized excitations.

The QISM and its ingredients, the Yang–Baxter equation and the algebraic Bethe ansatz, became quite popular and there are numerous papers on this subject. Faddeev very much liked to survey QISM in his lectures and pedagogical reviews (34, 36). He returned to the topic of complete integrability in 2013 in his last review paper (43).

Quantum groups

The ISM served as a basis for new mathematical structures and concepts. Thus, the concept of the classical r-matrix, introduced by Sklyanin, led V. G. Drinfeld to introduce Poisson–Lie groups, Lie groups endowed with Poisson structures compatible with group operation. In a similar way, quantum Lie groups and Lie algebras were introduced by Drinfel'd and by M. Jimbo as abstractions of concrete algebraic constructions arising within QISM (Drinfeld 1986; Jimbo 1985).

Figure 4. Seoul, 1997: Faddeev with Baxter and Yang. From the C.N. Yang archive, used with permission. (Online version in colour.)

The basic algebraic formalism of QISM consists of the Yang–Baxter equation and commutation relations for the quantum monodromy matrix. In joint papers with Reshetikhin and Takhtajan, subsequently summarized (30), this formalism was taken as a basis for systematic definition of quantum Lie groups and Lie algebras. This approach, known as the Faddeev–Reshetikhin–Takhtajan formalism, is nowadays widely used in the theory of quantum groups and its applications. As explained (30), it is based on the prior works by Kulish & Reshetikhin (Kulish & Reshetikhin 1983), Sklyanin (Sklyanin 1982) and Faddeev & Takhtajan (27).

Faddeev liked the commutation relations for the quantum monodromy matrix in QISM and often returned to it in his subsequent papers. He used it to describe the exchange algebra in the Wess–Zumino–Novikov–Witten model (31), in a joint paper with A. Y. Alekseev to define the quantum cotangent bundles of Lie groups (32), and to construct the lattice analogues of Kac–Moody algebras, jointly with Alekseev and Semenov-Tian-Shansky (33). This relation was also the basis for joint papers with R. M. Kashaev and A.Yu. Volkov devoted to Faddeev's favourite quantum lattice Liouville model (39, 40).

Special functions, including the q-products, had been studied by mathematicians since Euler, who discovered in 1748 what is now called Euler's q-dilogarithm. In 1995 Faddeev suggested considering a special ratio of these functions, and introduced a new q-function, nowadays called the Faddeev quantum dilogarithm (35). This function has remarkable analytic properties and can also be expressed as a certain ratio of double gamma functions, first introduced by V. P. Alekseevskii in 1889 and systematically studied by E. W. Barnes (FRS

1909) in 1899. Surprisingly, the Alekseevskii–Barnes function also occurs in the expressions for the *S*-matrix and the form-factors of the quantum SG model obtained, respectively, by A. A. Zamolodchikov and Al. A. Zamolodchikov and by F. A. Smirnov.

Faddeev's most remarkable discovery is his invention of the modular double of the quantum group, and his explanation of the fundamental role of the quantum dilogarithm. His papers on these topics (38, 42) became widely used both in pure mathematics and in applications to conformal field theory.

Quantum field theory

Faddeev made fundamental contributions to the quantum theory of Yang–Mills fields, upon which the Standard Model of particle physics is constructed. He eloquently described its history and development in his address 'My life amid quantum fields' (44) on the occasion of receiving the M. V. Lomonosov Medal, the highest award of the Russian Academy of Sciences.

In the mid 1950s, after the great success of quantum electrodynamics (QED), quantum field theory encountered serious difficulties and went out of fashion for a long time. The final *coup de grăce* was the discovery of the 'zero-charge paradox' by L. D. Landau (ForMemRS 1960) and I. J. Pomeranchuk (Pomeranchuk 1955), also known as the Landau pole problem. This was quite surprising, since QED enabled many subtle effects to be calculated with unprecedented precision. Nevertheless, the result of intricate calculations by Landau and Pomeranchuk seemed to reveal a direct logical contradiction in the foundations of QED.

Physicists tried to overcome these difficulties, on the one hand, by recovering the scattering amplitude from its postulated analytic properties and, on the other hand, by developing an abstract axiomatic approach. Neither of these approaches attracted Faddeev, who held that mathematical and, in particular, geometric beauty should be a crucial feature of a physical theory, free from the difficulties that undermined trust in the very existence of quantum field theory. Working on quantum field theory at that time meant working alone, against the mainstream.

In his last short paper, *Fundamental problems*, written just before the tragic car accident that cut short his scientific career, Landau (Landau 1960) wrote:

> The Hamiltonian method for strong interactions is dead and must be buried, although of course with all deserved honours. The brevity of life does not allow us the luxury of spending time on problems which will lead to no new results.

In the early 1960s Landau's followers treated these words as the Master's testament. When Faddeev and Popov made a decisive step forward in quantum Yang–Mills theory in 1966, no leading physics journal in the USSR would publish their paper, nor could it be published abroad since permission from the Nuclear Physics Division of the USSR Academy of Sciences was required. As a result, Faddeev and Popov were only able to send a short two-page text to the new European journal *Physics Letters* (6) and to publish it as a preprint (7) of the Kiev Theoretical Physics Institute.

The Yang–Mills theory was formulated by C.-N. Yang (ForMemRS 1992) and R. L. Mills (Yang & Mills 1954) in their classic paper as a field theory where interactions are invariant under independent rotations of the isotopic spin at all space-time points. From the very beginning it received devastating criticism from W. E. Pauli (ForMemRS 1953), since, in its primitive form, it predicted the existence of a multiplet of massless charged particles which were not observed experimentally. The geometric origin and beauty of the Yang–Mills theory was not immediately recognized. Consequently, the Yang–Mills theory was little known and poorly understood in the 1950s and the problem of its quantization was not solved. It was believed that the model was not renormalizable, which was a 'death sentence' for a physical theory.

The first attempt to construct a quantum Yang–Mills theory was made by Feynman in the early 1960s. Like Faddeev several years later, Feynman aimed to quantize the theory of gravity, but, because of the cumbersome calculations involved, he decided, on the suggestion of Gell-Mann (ForMemRS 1978), to start with the technically simpler Yang–Mills theory. Applying the standard methods of perturbation theory, Feynman realized that the naive diagrammatic approach gave a nonunitary answer in the one-loop approximation. The unitarity-restoring correction could be interpreted as the contribution of an additional scalar particle. This fictitious particle behaved like a fermion, thus breaking the usual relation between spin and statistics.

Faddeev learned about Feynman's results from the text of his talk at the 1962 Warsaw conference on gravitation (Feynman 1963). Outstanding problems were: to explain these results outside perturbation theory and to calculate corrections to the naive theory beyond the one-loop approximation. To solve them, Faddeev and Popov used the technique of functional integration, proposed by Feynman himself. These problems were also studied by Bryce DeWitt in the mid 1960s, who constructed a correct quantization procedure for Yang–Mills fields and Einstein's gravitation theory but did not introduce the celebrated Faddeev–Popov ghosts. The brilliant idea of introducing ghosts, as well as Faddeev's deep understanding of the structure of quantum Yang–Mills theory, continue to influence the development of this important topic.

As in QED, obtaining the diagram expansion is only a first step in the construction of a correct quantum theory. The second, no less important, step is the proof of renormalizability of the theory, and the construction of renormalized coupling constants and renormalized perturbation series. Renormalizability of the theory depends crucially on the explicit use of its gauge invariance. The renormalization problem can be combined with the mechanism of spontaneous symmetry breaking, the 'Higgs mechanism' proposed by P. Higgs (FRS 1983) and independently by Brout and Englert in 1964, by means of which quanta of the Yang–Mills theory can acquire mass. The gauge model of electromagnetic and weak interactions based on the Higgs mechanism was proposed by S. Weinberg (ForMemRS 1981) in 1967 and completely changed theorists' attitude to gauge theories. As a result, Faddeev and Popov's paper, published in the same year after a year-long delay, immediately became foundational for the further development of gauge theories (6). G. 't Hooft and M. J. G. Veltman verified that Yang–Mills theory with spontaneous gauge symmetry breaking is renormalizable.* A key discovery of Gross, F. Wilczek and D. Politzer in the early 1970s showed that this theory is free from Landau poles. This made it possible to extend gauge theories to strong interactions. The result of this unparalleled development was the construction of the Standard Model of particle physics, resulting in several Nobel Prizes, to S. L. Glashow, Abdus Salam (FRS 1959) and S. Weinberg in 1979, to 't Hooft and Veltman in 1999, to Gross, Politzer and Wilczek in 2004, and to Englert and Higgs in 2013.

Faddeev's friend, the great physicist Chen-Ning Yang, wrote in the foreword to Faddeev's selected works (2016):

* Renormalizability of the massless Yang–Mills theory was proved by A. A. Slavnov (Slavnov 1972) slightly earlier.

> Many people, including myself, felt that Faddeev should have shared the Nobel Prize of 1999 with 't Hooft and Veltman. There is a strange cultural phenomenon among theoretical physicists in the 20th century: downplaying the importance of mathematics.... It seems that with the exuberance of the youthful Heisenberg and Pauli, there began the idea that mathematics is at best detrimental to originality in physics.... Although Heisenberg in his old age changed his views about mathematics, American hubris seemed to have taken over, to perpetuate the cultural phenomenon of downplaying the importance of mathematics. I speculate that may have been part of the reason that Faddeev was not included in the 1999 Prize.

The quantization of the Yang–Mills theory by the Feynman path integral uses the 'integration' over the infinite-dimensional space of all Yang–Mills connections. The symmetry group *G* gives rise to the infinite-dimensional group of gauge transformations acting on this space and the gauge equivalent connections correspond to the same physical action. Therefore, the naive path integral is proportional to the volume of the gauge group and trivially diverges. Hence, one has to integrate over the orbits of the gauge group on the space of all connections, and it is this problem that was solved by Faddeev and Popov.

They invented what is nowadays called the 'Faddeev–Popov trick', which allows integration over the space of all connections using the Dirac delta function of the gauge fixing condition. However, it requires the insertion of what is now called the 'Faddeev–Popov determinant', which leads to particles with apparently the wrong statistics. Faddeev was familiar with the work of F. A. Berezin on integration with respect to anticommuting variables and realized that this determinant can be rewritten as a path integral over such variables if one introduces fictitious anticommuting variables with values in the adjoint representation of the Lie algebra of the Lie group *G*—these are the famous Faddeev–Popov ghosts. When the symmetry group *G* is Abelian, the ghosts do not interact with the gauge field and are not needed for constructing perturbation theory. However, in non-Abelian theories, the Faddeev–Popov ghosts play a key role. In the Kiev preprint (7), Faddeev and Popov formulated the corresponding diagram technique for perturbation theory in the Yang–Mills theory, along with propagators of gauge fields and ghosts and all interaction vertices. Nowadays this method is explained in all textbooks on quantum field theory.

Another nontrivial problem is the choice of asymptotic conditions required to obtain the correct path integral for the matrix elements of the scattering matrix. The correct definition for these conditions on the path integral for the *S*-matrix was given in Faddeev's lectures (16) at the physics school in Les Houches, France.

After Faddeev and Popov's paper (6) in which the Yang–Mills theory was quantized using the Lagrangian formalism of functional integration, theoretical physicists naturally asked whether the proposed formalism was unitary. Faddeev was well aware that using the Hamiltonian formalism instead of the Lagrangian approach to functional integration guarantees unitarity. But there was a difficulty because, in the Hamiltonian approach, the classical dynamics of Yang–Mills fields is described by a generalized Hamiltonian system with constraints. Although Dirac (Dirac 1958) proposed a general method for the classical description of constrained systems and their quantization in the operator formalism, the problem of quantization in the Feynman integral formalism in the Hamiltonian approach remained open. Faddeev gave an elegant mathematical interpretation of Dirac's formalism for the first-class constraints, and used it to solve the quantization problem for such systems using functional integration (9). In the context of Yang–Mills theory, this proves the unitarity of Faddeev–Popov quantization by using a manifestly unitary Hamiltonian approach.

Faddeev's paper (9), which opened the first issue of the new journal *Teoreticheskaya i Matematicheskaya Fizika*, founded by Bogolyubov in 1969, immediately became a classic. It had a huge influence not only on the development of the theory of gauge fields but also on theoretical physics as a whole. In particular, Faddeev suggested a very elegant geometric interpretation of Dirac's formalism and showed that in a special case it produces the method of Hamiltonian reduction, widely used in modern mathematics. The reduced phase space in Dirac's formalism is called the symplectic, or Marsden–Weinstein, quotient by mathematicians. Faddeev's contributions are summarized in a monograph co-authored with A. A. Slavnov entitled *Introduction to the quantum theory of gauge fields* (19), which remains a classic text to the present day.

Another topic that attracted Faddeev's attention was quantum anomalies in four-dimensional quantum field theory. They were discovered at the end of the 1960s by Adler (Adler 1969), and by Bell & Jackiw (Bell & Jackiw 1969). The presence of an anomaly means that the gauge symmetry of classical theory is not preserved under quantization, being destroyed by quantum corrections. In the mid 1980s Faddeev and his student S. L. Shatashvili studied the mathematical aspects of anomalies in gauge theories. Their first paper on this subject already contained a key observation that the well-known Wess–Zumino action is actually a 1-cocycle for the group of gauge transformations that acts on functions on the space of Yang–Mills fields (25). They also found the 2-cocycle corresponding to the Abelian extension of the infinite-dimensional Lie algebra of equal-time commutation relations for the quantum version of Gauss's Law in four-dimensional space-time. Faddeev was very proud of the fact that he used the results of his father, D. K. Faddeev, who discovered group cohomology in the 1940s, in quantum field theory.

The 'descent method' for calculating cocycles in the gauge group (25) became very popular. Faddeev emphasized that the corresponding 2-cocycle in Gauss's Law (25) should also arise in the Hamiltonian approach to the full quantum theory, where the gauge field is dynamical (26). In a subsequent paper, Faddeev and Shatashvili developed an effective method for quantizing systems with second-class constraints, applicable to the quantization of anomalous gauge theories, now known as Faddeev–Shatashvili quantization (28).

Throughout his scientific activities, Faddeev thought about how to deal with the inevitable divergences in the standard approach to quantum field theory. He cited Van Hove, who argued that renormalizations in field theories arise because the eigenfunction expansion of a free Hamiltonian is used. In a joint paper with Kulish, Faddeev showed that infrared divergences in QED, the so-called infrared catastrophe, can be avoided by modifying simultaneously the space of asymptotic states and the definition of the scattering operator by properly taking into account the asymptotic dynamics (10). The method of Faddeev and Kulish is now used in various areas of theoretical physics.

Mathematical physics

In his 1999 essay (37) Faddeev described his views on the nature of modern mathematical physics:

> When somebody asks me what I do in science, I call myself a specialist in mathematical physics. As I have been there for more than 40 years, I have some definite interpretation of this combination of words 'mathematical physics'.

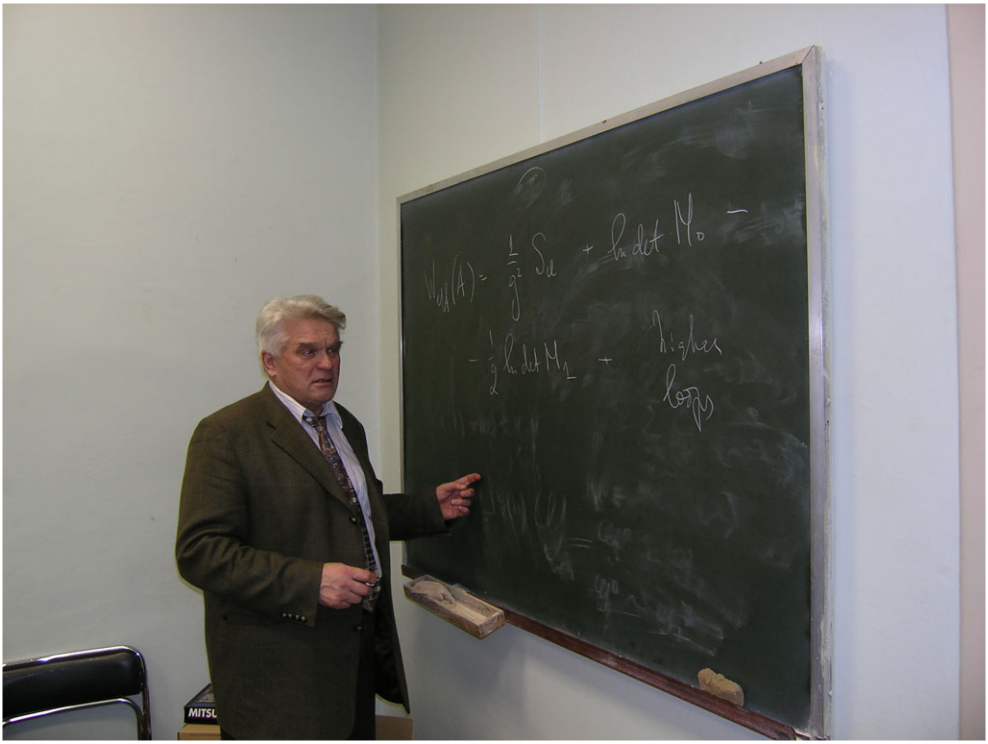

Figure 5. Faddeev at the Steklov Institute, Saint Petersburg, *ca* 2013. Copyright unknown. (Online version in colour.)

He explained that the meaning of the term ‘mathematical physics’ (MP) changes with time, place and person. Thus at the beginning of the twentieth century, the term MP was effectively equivalent to the concept of theoretical physics. Not only Henri Poincaré but also Albert Einstein were called mathematical physicists. It follows from the documents in the archives of the Nobel Committee that MP had a right to appear in both the nominations and discussion of the candidates for the Nobel Prize in physics, so the term MP was practically equivalent to the concept of theoretical physics. In the mathematical interpretation, MP appeared as the theory of partial differential equations and calculus of variations, and had its sources in differential geometry, classical electrodynamics, hydrodynamics and elasticity theory. Since the 1960s MP encompasses parts of functional analysis used in quantum theory, including the spectral theory of operators in Hilbert space, the mathematical theory of scattering and representation theory of Lie groups. The main goal was the quest for rigorous mathematical theorems on results that were understood by physicists in their own way.

In Faddeev’s definition, the main goal of MP is the use of mathematical intuition for the derivation of original results in fundamental physics. His papers on quantum field theory prepared the way to a revolution of gauge theories of elementary particle physics. Faddeev also recognized the importance of deep ideas coming from physics for the development of pure mathematics. One can say that perhaps Faddeev was the first great mathematician whose work was completely based on the ideas of quantum theory. His technical arsenal included both the methods of functional analysis, developed in the first half of the twentieth century and forming the mathematical toolbox of quantum mechanics, and more recent methods of symplectic geometry, representation theory, algebraic analysis and quantum field theory. Faddeev's ideas continue to play a definitive role in MP. They live on in the work of his students and will live further in the work of their students and their descendants (figure 5).

Awards and recognition

1971	USSR State Prize
1974	Heinemann Prize in mathematical physics
1990	Dirac Medal
1995	State Prize of the Russian Federation (jointly with A. A. Slavnov)
1996	Max Planck Medal
2002	Pomeranchuk Prize
2002	Demidov Prize
2002	Euler Medal
2005	State Prize of the Russian Federation
2006	Poincaré Prize
2008	Shaw Prize (jointly with V. I. Arnol'd)
2013	Lomonosov Medal of the Russian Academy of Sciences
2016	Establishment of the international Faddeev Medal at the 23rd International Conference on Many-Particle Quantum Theory

Acknowledgements

In the preparation of this memoir I extensively used the review paper Takhtajan *et al.* (2017), and I am grateful to all its authors. I am also very grateful to the Faddeev family, especially to A. Yu. Volkov, for helpful comments and suggestions. The frontispiece portrait is from the Royal Society archives, copyright unknown, photos 1 and 3 were taken by A. S. Budagov in 1978, photo 4 is from C.-N. Yang personal archive, and photos 2 and 5 are from Faddeev's family archive.

Author profile

Leon Takhtajan

Leon Takhtajan studied mathematics at the Leningrad State University. He obtained his PhD in 1975 at the Steklov Institute in Leningrad (LOMI) under the guidance of Faddeev, and defended his doctoral dissertation in 1982. Currently he is SUNY distinguished professor in the Mathematics Department of the State University of New York (SUNY) at Stony Brook, USA, and a leading researcher at the Euler Mathematical Institute in Saint Petersburg, Russia. His research interests are in mathematical physics with the application of quantum theory to geometry, algebra and analysis. Faddeev and Takhtajan co-authored many papers, and their book *Hamiltonian methods in the theory of solitons* (41) was reprinted in 2007 in the Springer series 'Classics in mathematics'.

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