Practice Midterm 1 Solutions

MAT 127

Spring 2002

Name:	**ID #:**	**Section:**

Please answer each question in the space provided. Show your work whenever possible. Unless otherwise marked, **answers without justification will get little or no partial credit**. Cross out anything the grader should ignore and circle or box the final answer. Graphing calculators are allowed but not required. You do not have to simplify numerical answers or write their approximate values: if the answer you got is $\sqrt{2}$, you should not replace it by 1.414.

The actual midterm will contain 5 problems, including one word problem. This practice test contains more problems to give you better practice.

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(1) Each of the following three functions solves one of the differential equations below. Which one? Justify your choices.

(a) $y(x) = x \sin x$ (b) $y = e^{2-x^2}$ (c) $y = \frac{x^2+1}{x-1}$

(I) $y' = -2xy$ (II) $xy + y''x = 2y' - 2\sin x$ (III) $1 - y' = \frac{2}{(x-1)^2}$

Solution: For each of the three given functions, let us calculate y', y'' and substitute in each of the three given equations. One of them should give a true identity. For example, for (a) we get

$$y(x) = x\sin x$$
$$y'(x) = \sin x + x\cos x$$
$$y''(x) = 2\cos x - x\sin x$$

Substituting this in (I), we get

$$\sin x + x\cos x = -2x \cdot x\sin x,$$

which is not true. Substituting in (II), we get

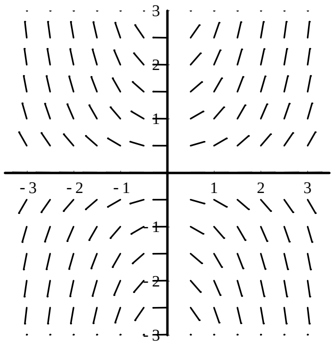
$$x(x\sin x) + (2\cos x - x\sin x)x = 2(\sin x + x\cos x) - 2\sin x$$

which is true; thus (a) is a solution of eqation (II). In a similar way, we find that (b) is a solution of (I) and (c) is a solution of (III).

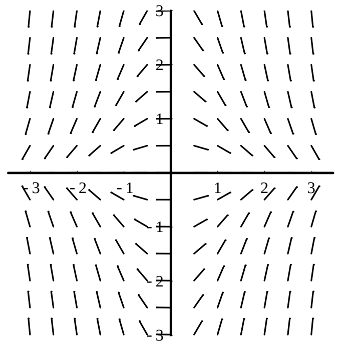
(2) Match each of the differential equations with their direction field below. (No justification required.)

(a) $y' = y(y-1.5)$ (b) $y' = xy$ (c) $y' = -xy$
(d) $y' = \frac{x}{y}$ (e) $y' = x - y$ (f) $y' = (y-1)(y+2)$

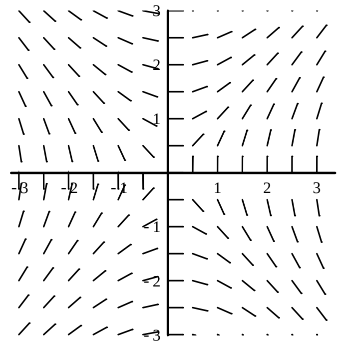
(1)



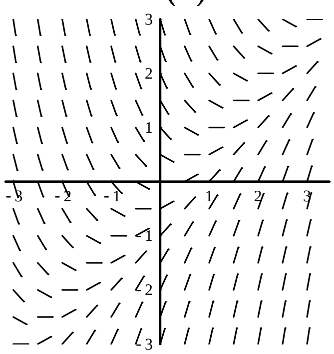
(2)



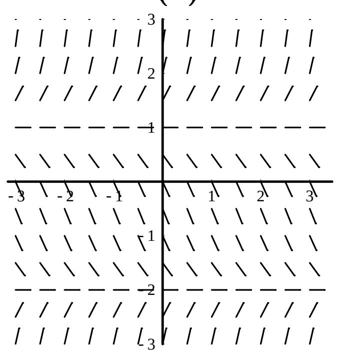
(3)



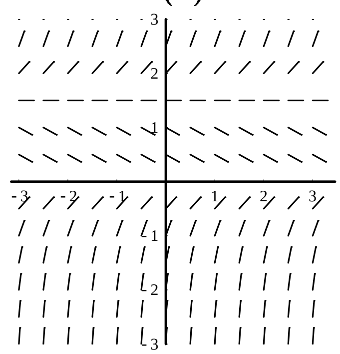
(4)



(5)



(6)



Solution: The answer is:

$(a) = (6)$	$(b) = (1)$	$(c) = (2)$
$(d) = (3)$	$(e) = (4)$	$(f) = (5)$

There are many ways to see this. One of the simplest ones: let us look for the points where $y' = 0$. For example, for $y' = y(y-1.5)$, one sees that $y' = 0$ for $y = 0$, $y = 1.5$; thus, on these two horizontal lines the direction field must have slope 0, i.e. be horizontal. Of all the figures, the only one which has this property is (6). Simialr consideration can be used in all other cases.

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(3) In problem 2 you found the direction field of the differential equation

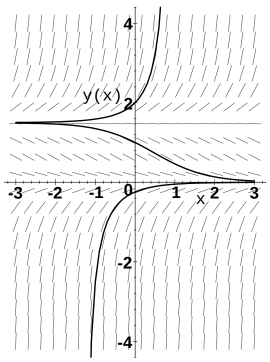
$$y' = y(y-1.5).$$

Use this to sketch the three graphs of the solutions to this differential equation that satisfy the initial conditions:

(a) $y(0) = 1$ (b) $y(0) = 2$ (c) $y(-1) = -2.5$

How does the limiting behavior of the solutions depend on the value of $y(0)$? What are the equilibrium solutions? Which one is stable? Which is unstable?

Solution: The graphs of solutions are shown below.



Limiting behavior: from the direction field, it looks like all solutions with $y(0) > 1.5$ will grow unboundedly; solutions with $0 < y(0) < 1.5$ will approach horizontal line $y = 0$ from above, and solutions with $y(0) < 0$ will approach horizontal line $y = 0$ form below. Equilibrium solutions are when $y' = 0$, which gives $y = 0$ or $y = 1.5$. From the direction field picture, $y = 1.5$ is unstable solution, and $y = 0$ is stable.

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(4) Find the solution of the differential equation

$$y' - 4y = -10\sin(2x) \tag{1}$$

which has the form $y(x) = A\sin(2x) + B\cos(2x)$.

Solution: Let $y(x) = A\sin(2x) + B\cos(2x)$. Then

$$y' = 2A\cos 2x - 2B\sin 2x.$$

Substituting this in (1), we get the following equation

$$\begin{aligned}(2A\cos 2x - 2B\sin 2x) - 4(A\sin 2x + B\cos 2x) &= -10\sin 2x\\ (2A - 4B)\cos 2x + (-2B - 4A)\sin 2x &= -10\sin 2x\end{aligned}$$

In order for this to be true for all values of x, the coefficients of $\sin 2x$ and $\cos 2x$ on both sides must be equal (the right hand side can be written as $0\cos 2x - 10\sin 2x$). Thus,

$$\begin{aligned}2A - 4B &= 0\\ -2B - 4A &= -10\end{aligned}$$

The first equation gives $A = 2B$; substituting it in the second equation, we get $-10B = -10, B = 1$. Thus, $A = 2, B = 1$ and

$$\boxed{y(x) = 2\sin 2x + \cos 2x}$$

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(5) (a) Find all solutions to the differential equation

$$y' = 3y + 15.$$

(b) Solve the initial value problem

$$y' = 3y + 15$$
$$y(0) = -1$$

Solution:

(a) Separating variables gives

$$\frac{dy}{3y+15} = dx$$

hence

$$\frac{1}{3}\ln|3y+15| = x + A;$$

simplifying and exponentiating yields $3y + 15 = \pm e^{3A}e^{3x}$, i.e.

$$y = -5 + \frac{\pm e^{3A}}{3}e^{3x}.$$

The expression $\frac{\pm e^{3A}}{3}$ is just a constant, so it's convenient to rename it C; thus we obtain $\boxed{y = -5 + Ce^{3x}, \text{ for any real number } C.}$ (The expression above, without the constant renamed, is also an acceptable answer.)

(b) From (a), we know that any function of the form $y(x) = Ce^{3x} - 5$ is a solution of the differential equation. Let us check if we can choose C so that $y(x) = Ce^{3x} - 5$ also satisfies the initial condition $y(0) = -1$:

$$Ce^{3\cdot 0} - 5 = -1$$
$$C \cdot 1 - 5 = -1$$
$$C = 5 - 1 = 4$$

Thus, $\boxed{y(x) = 4e^{3x} - 5}$ satisfies both the differential equation and the initial condition.

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(6) Find the orthogonal trajectories to the family of curves

$$y = \frac{1}{k+x}.$$

Solution: For the family of curves that we're given, we compute $y'(x) = -(k+x)^{-2}$. Now we must rewrite the right-hand side so as to eliminate k, i.e. just as a function of x and y. To do this, note that $y^2 = (k+x)^{-2}$, so that we can write our equation as $y' = -y^2$.

Now remember that to find the orthogonal family of trajectories, we wish to solve a DE for a family of functions $y(x)$ whose slopes are the *negative reciprocal* of those in the given family; in this case, that means we must solve

$$y'(x) = -\frac{1}{-y^2} = y^{-2}.$$

This is a separable differential equation, which we may integrate by rewriting it as

$$y^2 dy = dx,$$

i.e. integration gives $\boxed{\frac{y^3}{3} = x + C.}$

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(7) A tank of water initially contains 10 grams of salt dissolved in 10 liters of water. Water is drained from the tank at a rate of 5 liters per hour. Simultaneously, pure water (containing no salt) is added to the tank at a rate of 5 liters per hour. The water in the tank is kept thoroughly mixed, so the salt present is evenly distributed throughout the tank.

(a) Let $y(t)$ be the amount of salt (in grams) in the tank at time t. Write a differential equation for $y(t)$. (*Hint*: what is the concentration of time t?) What are the initial conditions?

(b) Find a solution of the initial value problem of part (a) using the fact (which follows from the general theory we will learn later) that this solution must be of the form $y(t) = Ae^{-rt}$ for some A, r.

(c) How long will it take for the salt concentration to drop to 0.1 gr/liter?

Solution:

(a) The only cause for the change of amount of salt in the tank is the loss of salt with the water going out of the tank. Since every hour 5 liters of salt water flows out of the tank, and concentration of salt at time t is $y(t)/volume = y(t)/10$ gr/liter, we see that the rate of salt loss is $5\frac{y(t)}{10} = y(t)/2$ gr/hour. This gives the following differential equation:

$$\frac{dy}{dt} = -\frac{y(t)}{2}.$$

The initial condition is $y(0) = 10$.

(b) If $y(t) = Ae^{-rt}$, then $y' = -rAe^{-rt}$. Substituting this in the equation, we get:

$$\begin{aligned} -rAe^{-rt} &= -\frac{Ae^{-rt}}{2} \\ -r &= -1/2 \\ r &= 1/2 \end{aligned}$$

Thus, to satisfy the equation, we must have $r = -1/2$, $y(t) = Ae^{-t/2}$. Now, to satisfy the initial condition, we must have

$$\begin{aligned} y(0) &= 10 \\ Ae^{-0/2} &= 10 \\ A &= 10. \end{aligned}$$

Thus, the solution of the initial value problem is $\boxed{y(t) = 10e^{-t/2}}$.

(c) SInce concentration is $c(t) = y(t)/10$, concentration of 0.1 gr/liter corresponds to $y(t) = 10 \cdot 0.1 = 1$ gr. Using the formula from part (b) for $y(t)$, we get the following equation which we need to solve for t:

$$\begin{aligned} 10e^{-t/2} &= 1 \\ e^{-t/2} &= 0.1 \\ -t/2 &= \ln 0.1 \\ t &= -2\ln 0.1 \text{ hours} \end{aligned}$$

(In fact, using $\ln(1/a) = -\ln a$, this can be rewrtitten as follows: $t = 2\ln 10$ hours, which is approximately 4.6 hours, or 4 hours and 36 minutes. But you can leave the answer in the form $t = -2\ln 0.1$ hours.)