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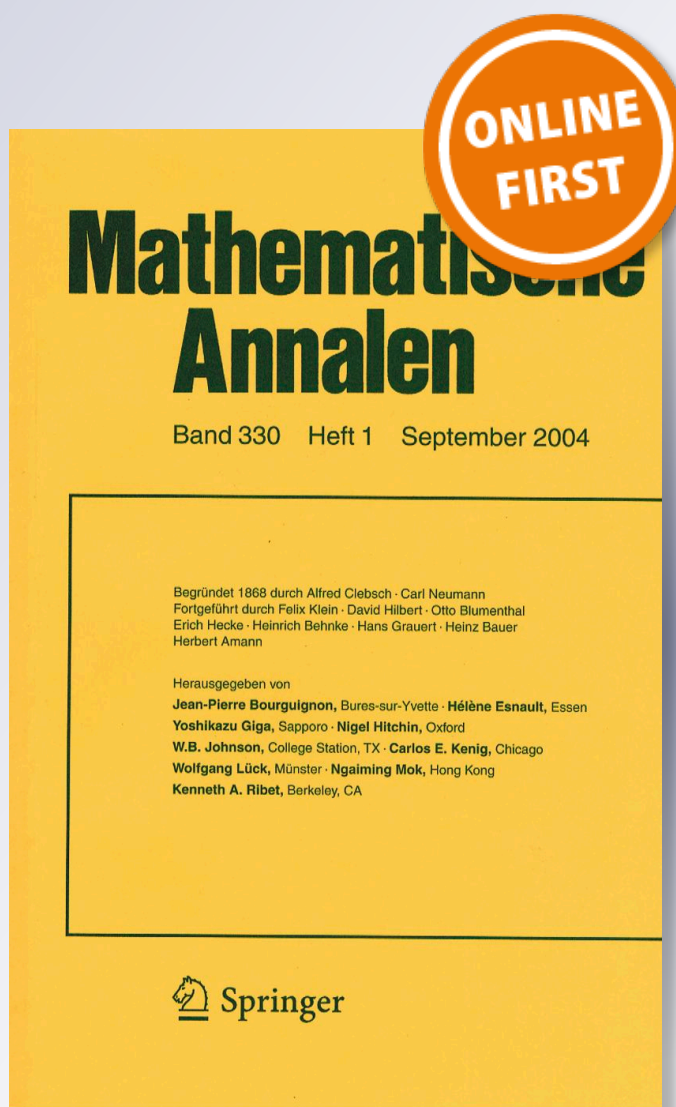
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On Kawai theorem for orbifold Riemann surfaces

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Abstract

We prove a generalization of Kawai theorem for the case of orbifold Riemann surface. The computation is based on a formula for the differential of a holomorphic map from the cotangent bundle of the Teichmüller space to the $\mathrm{PSL}(2, \mathbb{C})$ -character variety, which allows to evaluate explicitly the pullback of Goldman symplectic form in the spirit of Riemann bilinear relations. As a corollary, we obtain a generalization of Goldman's theorem that the pullback of Goldman symplectic form on the $\mathrm{PSL}(2, \mathbb{R})$ -character variety is a symplectic form of the Weil–Petersson metric on the Teichmüller space.

1 Introduction

The deformation space of complex projective structures on a closed oriented genus $g \geq 2$ surface is a holomorphic affine bundle over the corresponding Teichmüller space. The choice of a Bers section identifies the deformation space with the holomorphic cotangent bundle of the Teichmüller space, a complex manifold with a complex symplectic form. Kawai's theorem [16] asserts that symplectic form on the cotangent bundle is a pullback under the monodromy map of Goldman's complex symplectic form on the corresponding $\mathrm{PSL}(2, \mathbb{C})$ -character variety.

However, Kawai's proof is not very insightful. In fact, he does not use Goldman symplectic form as defined in [6], but rather uses a symplectic form on the moduli space of special rank 2 vector bundles on a Riemann surface associated with projective structures, as it is defined in [8]. The computation is highly technical and algebraic topology nature of the result gets obscured. Recently a shorter proof, relying on theorems of other authors, was given in [18]. Also in paper [4] it is proved, using special

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homological coordinates, that canonical Poisson structure on the cotangent bundle of the Teichmüller space induces the Goldman bracket on the character variety.

Here we prove a generalization of Kawai theorem for the case of orbifold Riemann surface. The computation is based on a formula for the differential of a holomorphic map from the cotangent bundle to the $\mathrm{PSL}(2, \mathbb{C})$ -character variety, which allows to evaluate explicitly the pullback of Goldman symplectic form in the spirit of Riemann bilinear relations. As a corollary, we obtain a generalization of Goldman's theorem that the pullback of Goldman symplectic form on $\mathrm{PSL}(2, \mathbb{R})$ -character variety is a symplectic form of the Weil–Petersson metric on the Teichmüller space.

The paper is organized as follows. In Sect. 2.1 we recall basic facts from the complex-analytic theory of Teichmüller space $\mathcal{T} = T(\Gamma)$, where Γ is a Fuchsian group of the first kind, and in Sect. 2.2 we define the holomorphic symplectic form ω on the cotangent bundle $\mathcal{M} = T^*\mathcal{T}$. In Sect. 2.3 we introduce the $\mathrm{PSL}(2, \mathbb{C})$ -character variety \mathcal{H} associated with the Fuchsian group Γ , and its holomorphic tangent space at $[\rho] \in \mathcal{H}$, the parabolic Eichler cohomology group $H_{\mathrm{par}}^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\rho})$. The Goldman symplectic form ω_G on \mathcal{H} is introduced in Sect. 2.4, and the holomorphic mapping $Q : \mathcal{M} \rightarrow \mathcal{H}$, as well as the map $\mathcal{F} : \mathcal{T} \rightarrow \mathcal{H}_{\mathbb{R}}$, are defined in Sect. 2.5. In Sect. 3 we explicitly compute the differential of the map Q in the fiber over the origin in \mathcal{T} . Lemma 1 neatly summarizes variational theory of the developing map in terms of the so-called Λ -operator, the classical third-order linear differential operator

$$\Lambda_q = \frac{d^3}{dz^3} + 2q(z) \frac{d}{dz} + q'(z),$$

associated with the second-order differential equation

$$\frac{d^2\psi}{dz^2} + \frac{1}{2}q(z)\psi = 0,$$

where q is a cusp form of weight 4 for Γ . Its properties are presented in **A1–A5** (see also, **B1–B3**).

The main result, Theorem 1,

$$\omega = -\sqrt{-1}Q^*(\omega_G),$$

is proved in Sect. 4. The proof uses Proposition 1 and explicit description of a canonical fundamental domain for Γ in Sect. 4.1. From here we obtain (see, Corollary 3)

$$\omega_{\mathrm{WP}} = \mathcal{F}^*(\omega_G),$$

which is a generalization of Goldman theorem for orbifold Riemann surfaces.

2 The basic facts

2.1 Teichmüller space of a Fuchsian group

Here we recall the necessary basic facts from the complex-analytic theory of Teichmüller spaces (see, classic paper [1] and book [2], and also [19,23]).

2.1.1. Let Γ be, in classical terminology, a Fuchsian group of the first kind with signature $(g; n, e_1, \dots, e_m)$, satisfying

$$2g - 2 + n + \sum_{i=1}^m \left(1 - \frac{1}{e_i}\right) > 0.$$

By definition, Γ is a finitely generated cofinite discrete subgroup of $\text{PSL}(2, \mathbb{R})$, acting on the Lobachevsky (hyperbolic) plane, the upper half-plane

$$\mathbb{H} = \{z = x + \sqrt{-1}y : y > 0\}.$$

The group Γ has a standard presentation with $2g$ hyperbolic generators $a_1, b_1, \dots, a_g, b_g$, m elliptic generators c_1, \dots, c_m of orders e_1, \dots, e_m , and n parabolic generators c_{m+1}, \dots, c_{m+n} satisfying the relation

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} c_1 \cdots c_{m+n} = 1.$$

The group Γ can be thought of as a fundamental group of the corresponding orbifold Riemann surface $X \simeq \Gamma \backslash \mathbb{H}$.

2.1.2. Let $\mathcal{A}^{-1,1}(\mathbb{H}, \Gamma)$ be the space of Beltrami differentials for Γ —a complex Banach space of $\mu \in L^\infty(\mathbb{H})$ satisfying

$$\mu(\gamma z) \frac{\overline{\gamma'(z)}}{\gamma'(z)} = \mu(z) \quad \text{for all } \gamma \in \Gamma,$$

with the norm

$$\|\mu\|_\infty = \sup_{z \in \mathbb{H}} |\mu(z)|.$$

For a Beltrami coefficient for Γ , $\mu \in \mathcal{A}^{-1,1}(\mathbb{H}, \Gamma)$ with $\|\mu\|_\infty < 1$, denote by w^μ the solution of the Beltrami equation

$$\begin{aligned} w_z^\mu &= \mu w_z^\mu, & z \in \mathbb{H}, \\ w_z^\mu &= 0, & z \in \mathbb{C} \setminus \mathbb{H}, \end{aligned}$$

that fixes $0, 1, \infty$, and put $\mathbb{H}^\mu = w^\mu(\mathbb{H})$, $\Gamma^\mu = w^\mu \circ \Gamma \circ (w^\mu)^{-1}$. The Teichmüller space $T(\Gamma)$ of a Fuchsian group Γ is defined by

$$T(\Gamma) = \{\mu \in \mathcal{A}^{-1,1}(\mathbb{H}, \Gamma) : \|\mu\|_\infty < 1\} / \sim,$$

where $\mu \sim \nu$ if and only if $w^\mu|_{\mathbb{R}} = w^\nu|_{\mathbb{R}}$. Equivalently, $\mu \sim \nu$ if and only if $w_\mu|_{\mathbb{R}} = w_\nu|_{\mathbb{R}}$, where w_μ is a q.c. homeomorphism of \mathbb{H} satisfying the Beltrami equation

$$(w_\mu)_{\bar{z}} = \mu(w_\mu)_z, \quad z \in \mathbb{H}.$$

We denote by $[\mu]$ the equivalence class of a Beltrami coefficient μ .

Teichmüller space $T(\Gamma)$ is a complex manifold of complex dimension

$$d = 3g - 3 + m + n.$$

The holomorphic tangent and cotangent spaces $T_0T(\Gamma)$ and $T_0^*T(\Gamma)$ at the base point, the origin $[0] \in T(\Gamma)$, are identified, respectively, with $\Omega^{-1,1}(\mathbb{H}, \Gamma)$ —the vector space of harmonic Beltrami differentials for Γ , and with $\Omega^2(\mathbb{H}, \Gamma)$ —the vector space of cusp forms of weight 4 for Γ . The corresponding pairing $T_0^*T(\Gamma) \otimes T_0T(\Gamma) \rightarrow \mathbb{C}$ is given by the absolutely convergent integral

$$\iint_F \mu(z)q(z)dx dy,$$

where F is a fundamental domain for Γ . There is a complex anti-linear isomorphism $\Omega^2(\mathbb{H}, \Gamma) \xrightarrow{\sim} \Omega^{-1,1}(\mathbb{H}, \Gamma)$ given by $q(z) \mapsto \mu(z) = y^2\overline{q(z)}$. Together with the pairing, it defines the Petersson inner product in $T_0T(\Gamma)$,

$$(\mu_1, \mu_2)_{\text{WP}} = \iint_F \mu_1(z)\overline{\mu_2(z)}y^{-2}dx dy.$$

There is a natural isomorphism between the Teichmüller spaces $T(\Gamma)$ and $T(\Gamma_\mu)$, where $\Gamma_\mu = w_\mu \circ \Gamma \circ w_\mu^{-1}$ is a Fuchsian group. For every $[\mu] \in T(\Gamma)$ it allows us to identify $T_{[\mu]}T(\Gamma)$ with $\Omega^{-1,1}(\mathbb{H}, \Gamma_\mu)$ and $T_{[\mu]}^*T(\Gamma)$ with $\Omega^2(\mathbb{H}, \Gamma_\mu)$. The conformal mapping

$$h_\mu = w_\mu \circ (w^\mu)^{-1} : \mathbb{H}^\mu \rightarrow \mathbb{H},$$

establishes natural isomorphisms

$$\Omega^{-1,1}(\mathbb{H}, \Gamma_\mu) \xrightarrow{\sim} \Omega^{-1,1}(\mathbb{H}^\mu, \Gamma^\mu) \quad \text{and} \quad \Omega^2(\mathbb{H}, \Gamma_\mu) \xrightarrow{\sim} \Omega^2(\mathbb{H}^\mu, \Gamma^\mu).$$

According to the isomorphism $T(\Gamma) \simeq T(\Gamma_\mu)$, the choice of a base point is inessential and we will use the notation \mathcal{T} for $T(\Gamma)$.

The Petersson inner product in the tangent spaces determines the Weil–Petersson Kähler metric on \mathcal{T} . Its Kähler (1, 1)-form is a symplectic form ω_{WP} on \mathcal{T} ,

$$\omega_{\text{WP}}(\mu_1, \bar{\mu}_2) = \frac{\sqrt{-1}}{2} \iint_F \left(\mu_1(z) \overline{\mu_2(z)} - \overline{\mu_1(z)} \mu_2(z) \right) y^{-2} dx dy, \quad (1)$$

where $\mu_1, \mu_2 \in T_0\mathcal{T}$.

2.1.3. Explicitly the complex structure on \mathcal{T} is described as follows. Let μ_1, \dots, μ_d be a basis of $\Omega^{-1,1}(\mathbb{H}, \Gamma)$. Bers' coordinates $(\varepsilon_1, \dots, \varepsilon_d)$ in the neighborhood U of the origin in \mathcal{T} are defined by $\|\mu\|_\infty < 1$, where $\mu = \varepsilon_1\mu_1 + \dots + \varepsilon_d\mu_d$. For the corresponding vector fields we have

$$\frac{\partial}{\partial \varepsilon_i} \Big|_\mu = \mathbf{P}_{-1,1} \left(\left(\frac{\mu_i}{1 - |\mu|^2} \frac{w_z^\mu}{w_z^\mu} \right) \circ (w^\mu)^{-1} \right) \in \Omega^{-1,1}(\mathbb{H}^\mu, \Gamma^\mu),$$

where $\mathbf{P}_{-1,1}$ is a projection on the subspace of harmonic Beltrami differentials. Let p_1, \dots, p_d be the basis in $\Omega^2(\mathbb{H}, \Gamma)$, dual to the basis μ_1, \dots, μ_d for $\Omega^{-1,1}(\mathbb{H}, \Gamma)$. For the holomorphic 1-forms $d\varepsilon_i$, dual to the vector fields $\frac{\partial}{\partial \varepsilon_i}$ on U , we have $d\varepsilon_i|_\mu = p_i^\mu$, where the basis p_1^μ, \dots, p_d^μ in $\Omega^2(\mathbb{H}^\mu, \Gamma^\mu)$ has the property

$$\mathbf{P}_2 \left(p_i^\mu \circ w^\mu (w_z^\mu)^2 \right) = p_i,$$

with \mathbf{P}_2 being a projection on $\Omega^2(\mathbb{H}, \Gamma)$.

2.2 Holomorphic symplectic form

Let $\mathcal{M} = T^*\mathcal{T}$ be the holomorphic cotangent bundle of \mathcal{T} with the canonical projection $\pi : \mathcal{M} \rightarrow \mathcal{T}$. It is a complex symplectic manifold with canonical $(2, 0)$ -holomorphic symplectic form $\omega = d\vartheta$, where ϑ is the Liouville 1-form (also called a tautological 1-form). At a point $(q, [\mu]) \in \mathcal{M}$ it is defined as follows (e.g., see, [3])

$$\vartheta(v) = q(\pi_*v), \quad v \in T_{(q, [\mu])}\mathcal{M}.$$

For the points in the fiber $\pi^{-1}(0)$ the symplectic form ω is given explicitly by

$$\omega((q_1, \mu_1), (q_2, \mu_2)) = \iint_F (q_1(z)\mu_2(z) - q_2(z)\mu_1(z)) dx dy, \quad (2)$$

where $(q_1, \mu_1), (q_2, \mu_2) \in T_{(q,0)}\mathcal{M} \simeq T_0^*\mathcal{T} \oplus T_0\mathcal{T}$.

2.2.1. Let $\theta(t)$ be a smooth curve in \mathcal{M} starting at $(q, 0) \in \mathcal{M}$ and lying in T^*U , where U is a Bers neighborhood of the origin in \mathcal{T} . Correspondingly, $\mu(t) = \pi(\theta(t))$ is a smooth curve in U satisfying $\mu(0) = 0$, and without changing the tangent vector to $\theta(t)$ at $t = 0$ we can assume that $\mu(t) = t\mu$ for some $\mu \in \Omega^{-1,1}(\mathbb{H}, \Gamma)$. We have

$$\theta(t) = \sum_{i=1}^d u^i(t) d\varepsilon_i|_{t\mu},$$

for small t and

$$\theta(0) = \sum_{i=1}^d u^i(0) p_i = q \in \Omega^2(\mathbb{H}, \Gamma).$$

The tangent vector to $\theta(t)$ at $t = 0$ is $(\dot{\theta}, \mu) \in T_{(q,0)}\mathcal{M}$, where

$$\dot{\theta} = \sum_{i=1}^d \dot{u}^i(0) p_i.$$

Here and in what follows the ‘over-dot’ denotes the derivative with respect to t at $t = 0$.

Equivalently, the curve $\theta(t)$ is given by the smooth family $q^t \in \Omega^2(\mathbb{H}^{t\mu}, \Gamma^{t\mu})$ with $q^0 = q$, and so

$$u^i(t) = \left(q^t, \left. \frac{\partial}{\partial \varepsilon_i} \right|_{t\mu} \right) = \iint_F q(t) \mu_i \, dx dy,$$

where

$$q(t) = q^t \circ w^{t\mu} (w_z^{t\mu})^2, \tag{3}$$

is a pull-back of the cusp form q^t on $\mathbb{H}^{t\mu}$ to \mathbb{H} by the map $w^{t\mu}$. It is a smooth family of forms of weight 4 for Γ and

$$\dot{u}^i(0) = \iint_F \dot{q} \mu_i \, dx dy, \quad i = 1, \dots, d,$$

so that

$$\dot{\theta} = \mathbf{P}_2(\dot{q}).$$

2.2.2. To summarize, the value of the symplectic form (2) on tangent vectors $(\dot{\theta}_1, \mu_1)$ and $(\dot{\theta}_2, \mu_2)$ to the curves $\theta_1(t)$ and $\theta_2(t)$ at $t = 0$, is given by the following expression

$$\omega((\dot{\theta}_1, \mu_1), (\dot{\theta}_2, \mu_2)) = \iint_F (\dot{q}_1 \mu_2 - \dot{q}_2 \mu_1) dx dy. \tag{4}$$

Remark 1 Though \dot{q} is a non-holomorphic form of weight 4 for Γ , it decays exponentially at the cusps. Indeed, by conjugation it is sufficient to consider the cusp ∞ . Since $w^{t\mu}(z + 1) = w^{t\mu}(z) + c(t)$, we have $q^t(z + c(t)) = q^t(z)$ and

$$q(t)(z) = \sum_{n=1}^{\infty} a_n(t) e^{2\pi\sqrt{-1}n w^{t\mu}(z)/c(t)} w_z^{t\mu}(z)^2,$$

where $a_n(t)$ are corresponding Fourier coefficients of $q^t(z)$. Therefore

$$\dot{q}(z) = \sum_{n=1}^{\infty} \dot{a}_n e^{2\pi\sqrt{-1}nz} + 2q(z)\dot{w}_z^\mu + q'(z)(\dot{w}^\mu(z) - \dot{c}),$$

where prime always denotes the derivative with respect to z . Since $q(z)$ and $q'(z)$ decay exponentially as $y \rightarrow \infty$, we obtain

$$\dot{q}(z) = O(e^{-\pi y}) \text{ as } y \rightarrow \infty.$$

2.3 The character variety

Here we recall necessary basic facts on the $\text{PSL}(2, \mathbb{C})$ -character variety for the fundamental group of the orbifold Riemann surface $X \simeq \Gamma \backslash \mathbb{H}$.

2.3.1. Let \mathbf{G} be a Lie group $\text{PSL}(2, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ be its Lie algebra. As in [6, §2.3], we identify \mathfrak{g} with the Lie algebra of vector fields $P(z)\frac{\partial}{\partial z}$ on \mathbb{H} , where $P(z) \in \mathcal{P}_2$ is a quadratic polynomial. Explicitly,

$$\mathfrak{g} \ni \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto (cz^2 - 2az - b)\frac{\partial}{\partial z} \in \mathcal{P}_2 \frac{\partial}{\partial z}.$$

Let $\langle \cdot, \cdot \rangle$ denote a 1/4 of the Killing form¹ of \mathfrak{g} . In terms of the standard basis $\{1, z, z^2\}$ of \mathcal{P}_2 the Killing form $\langle \cdot, \cdot \rangle$ is given by the matrix

$$C = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1/2 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

where $C_{ij} = \langle z^{i-1}, z^{j-1} \rangle, i, j = 1, 2, 3$. In general, for $P_1, P_2 \in \mathcal{P}_2$

$$\langle P_1, P_2 \rangle = -\frac{1}{2}B_0[P_1, P_2](z), \tag{5}$$

where for arbitrary smooth functions F and G ,

$$B_0[F, G] = F_{zz}G + FG_{zz} - F_zG_z. \tag{6}$$

Note that the right hand side of (5) does not depend on z .

2.3.2. As in [6,7], let \mathcal{X} be the \mathbf{G} -character variety of an orbifold Riemann surface X ,

$$\mathcal{X} = \text{Hom}_0(\Gamma, \mathbf{G})/\mathbf{G},$$

¹ Representing \mathfrak{g} by 2×2 traceless matrices over \mathbb{C} gives $\langle x, y \rangle = \text{tr } xy$.

which consists of irreducible homomorphisms $\rho : \Gamma \rightarrow \mathbf{G}$, modulo conjugation, that preserve traces of parabolic and elliptic generators of Γ . The character variety \mathcal{X} is a complex manifold of complex dimension $2d = 6g - 6 + 2m + 2n$, and the holomorphic tangent space $T_{[\rho]}\mathcal{X}$ at $[\rho]$ is naturally identified with the parabolic Eichler cohomology group

$$H_{\text{par}}^1(\Gamma, \mathfrak{g}_{\text{Ad}\rho}) = Z_{\text{par}}^1(\Gamma, \mathfrak{g}_{\text{Ad}\rho})/B^1(\Gamma, \mathfrak{g}_{\text{Ad}\rho}).$$

Here \mathfrak{g} is understood as a left Γ -module with respect to the action $\text{Ad}\rho$, and a 1-cocycle $\chi \in Z^1(\Gamma, \mathfrak{g}_{\text{Ad}\rho})$ is a map $\chi : \Gamma \rightarrow \mathcal{P}_2$ satisfying

$$\chi(\gamma_1\gamma_2) = \chi(\gamma_1) + \rho(\gamma_1) \cdot \chi(\gamma_2), \quad \gamma_1, \gamma_2 \in \Gamma, \tag{7}$$

where dot stands for the adjoint action of \mathbf{G} on $\mathfrak{g} \simeq \mathcal{P}_2 \frac{\partial}{\partial z}$,

$$(g \cdot P)(z) = \frac{P(g^{-1}(z))}{(g^{-1})'(z)}, \quad g \in \mathbf{G}, \quad P \in \mathcal{P}_2. \tag{8}$$

The parabolic condition, introduced in [21], means that the restriction of a 1-cocycle $\chi \in Z^1(\Gamma, \mathfrak{g}_{\text{Ad}\rho})$ to a parabolic subgroup Γ_α of Γ —the stabilizer of a cusp α for Γ —is a coboundary: there is some $P_\alpha(z) \in \mathcal{P}_2$ such that

$$\chi(\gamma) = \rho(\gamma) \cdot P_\alpha - P_\alpha, \quad \gamma \in \Gamma_\alpha.$$

We denote by $[\chi]$ the cohomology class of a 1-cocycle χ .

Remark 2 It is well-known (see, [21]) that the restriction of χ to a finite cyclic subgroup of Γ is a coboundary. Indeed, if $\gamma^n = 1$, then it follows from (7) that

$$0 = \chi(\gamma^n) = (1 + \rho(\gamma) + \dots + \rho(\gamma^{n-1})) \cdot \chi(\gamma). \tag{9}$$

Using the unit disk model of the Lobachevsky plane, we can assume that $\gamma(u) = \zeta u$, where $\zeta^n = 1$ and $|u| < 1$. It follows from (8) and (9) that

$$\chi(\gamma)(u) = au^2 + b,$$

and there is $P \in \mathcal{P}_2$ with the property

$$\chi(\gamma)(u) = \zeta P(u/\zeta) - P(u).$$

2.4 The Goldman symplectic form

2.4.1. In case $X \simeq \Gamma \backslash \mathbb{H}$ is a compact Riemann surface (the case $m = n = 0$), Goldman [6] introduced a complex symplectic form on the character variety \mathcal{X} . At a point $[\rho] \in \mathcal{X}$ it is defined as

$$\omega_G([\chi_1], [\chi_2]) = \langle [\chi_1] \cup [\chi_2] \rangle ([X]), \quad \text{where } [\chi_1], [\chi_2] \in T_{[\rho]}\mathcal{K}. \quad (10)$$

Here $[X]$ is the fundamental class of X under the isomorphism $H_2(X, \mathbb{Z}) \simeq H_2(\Gamma, \mathbb{Z})$, and $\langle [\chi_1] \cup [\chi_2] \rangle \in H^2(\Gamma, \mathbb{R})$ is a composition of the cup product in cohomology and of the Killing form. At a cocycle level it is given explicitly by

$$\langle \chi_1 \cup \chi_2 \rangle (\gamma_1, \gamma_2) = \langle \chi_1(\gamma_1), \text{Ad}\rho(\gamma_1) \cdot \chi(\gamma_2) \rangle, \quad \gamma_1, \gamma_2 \in \Gamma.$$

Since the right-hand side in (10) does not depend on the choice of representatives $\chi_1, \chi_2 \in Z^1(\Gamma, \mathfrak{g}_{\text{Ad}\rho})$ of the cohomology classes $[\chi_1], [\chi_2] \in H^1(\Gamma, \mathfrak{g}_{\text{Ad}\rho})$, we will use the notation $\omega_G(\chi_1, \chi_2)$.

According to [6, Proposition 3.9],² the fundamental class $[X]$ in terms of the group homology is realized by the following 2-cycle

$$c = \sum_{k=1}^g \left\{ \left(\frac{\partial R}{\partial a_k}, a_k \right) + \left(\frac{\partial R}{\partial b_k}, b_k \right) \right\} \in H_2(\Gamma, \mathbb{Z}), \quad (11)$$

where $R = R_g$,

$$R_k = \prod_{i=1}^k a_i b_i a_i^{-1} b_i^{-1}, \quad k = 1, \dots, g,$$

and by the Fox free differential calculus

$$\frac{\partial R}{\partial a_k} = R_{k-1} - R_k b_k, \quad \frac{\partial R}{\partial b_k} = R_{k-1} a_k - R_k. \quad (12)$$

In these notations (10) takes the form

$$\omega_G(\chi_1, \chi_2) = - \sum_{k=1}^g \left\langle \chi_1 \left(\# \frac{\partial R}{\partial a_k} \right), \chi_2(a_k) \right\rangle + \left\langle \chi_1 \left(\# \frac{\partial R}{\partial b_k} \right), \chi_2(b_k) \right\rangle, \quad (13)$$

where a cocycle χ extends from a map on Γ to a linear map defined on the integral group ring $\mathbb{Z}[\Gamma]$, and $\#$ denotes the natural anti-involution on $\mathbb{Z}[\Gamma]$,

$$\# \left(\sum n_j \gamma_j \right) = \sum n_j \gamma_j^{-1}.$$

Remark 3 We have

$$\# \frac{\partial R}{\partial a_k} = R_{k-1}^{-1} (1 - \alpha_k) \quad \text{and} \quad \# \frac{\partial R}{\partial b_k} = R_k^{-1} (1 - \beta_k),$$

² See also, exercises 4(b) and 4(c) on p. 46 in [5].

where $\alpha_k = R_k b_k^{-1} R_k^{-1}$ and $\beta_k = R_k a_k^{-1} R_{k-1}^{-1}$, are dual generators of the group Γ (see, Sect. 4.1.1), and expression (13) takes the form

$$\omega_G(\chi_1, \chi_2) = - \sum_{k=1}^g \langle \chi_1(\alpha_k), \rho(R_{k-1}) \cdot \chi_2(a_k) \rangle + \langle \chi_1(\beta_k), \rho(R_k) \cdot \chi_2(b_k) \rangle.$$

2.4.2. In case $m + n > 0$, we define $R_k, k = 1, \dots, g$, as before and put

$$R_{g+i} = R_g c_1 \cdots c_i, \quad i = 1, \dots, m + n; \quad R = R_{g+m+n}.$$

According to [10,11,14,17], the Goldman symplectic form ω_G on the character variety \mathcal{H} associated with the fundamental group of an orbifold Riemann surface is defined as follows

$$\begin{aligned} \omega_G(\chi_1, \chi_2) = & - \sum_{k=1}^g \left\langle \chi_1 \left(\# \frac{\partial R}{\partial a_k} \right), \chi_2(a_k) \right\rangle + \left\langle \chi_1 \left(\# \frac{\partial R}{\partial b_k} \right), \chi_2(b_k) \right\rangle \\ & - \sum_{i=1}^{m+n} \left\langle \chi_1 \left(\# \frac{\partial R}{\partial c_i} \right), \chi_2(c_i) \right\rangle - \sum_{i=1}^{m+n} \langle \chi_1(c_i^{-1}), P_{2i} \rangle, \end{aligned} \quad (14)$$

where

$$\frac{\partial R}{\partial c_i} = R_{g+i-1}, \quad (15)$$

and $P_{2i} \in \mathcal{P}_2$ are given by

$$\chi_2(\gamma) = \rho(\gamma) \cdot P_{2i} - P_{2i}, \quad \gamma \in \Gamma_i = \langle c_i \rangle, \quad i = 1, \dots, m + n.$$

As in the previous case, the right-hand side of (14) depends only on cohomology classes $[\chi_1], [\chi_2] \in H_{\text{par}}^1(\Gamma, \mathfrak{g}_{\text{Ad}\rho})$. For details and the proof that it defines a symplectic form on \mathcal{H} we refer to [10,11,14,17].

2.5 The holomorphic map $\mathcal{Q} : \mathcal{M} \rightarrow \mathcal{H}$

The holomorphic map $\mathcal{Q} : \mathcal{M} \rightarrow \mathcal{H}$ is defined as follows. Let $(q, [\mu]) \in \mathcal{M}$, where $q \in \Omega^2(\mathbb{H}^\mu, \Gamma^\mu)$. On $\mathbb{H}^\mu = w^\mu(\mathbb{H})$ consider the Schwarz equation

$$\mathcal{S}(f) = q,$$

where \mathcal{S} stands for the Schwarzian derivative,

$$\mathcal{S}(f) = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2.$$

Its solution, the developing map $f : \mathbb{H}^\mu \rightarrow \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, satisfies

$$f \circ \gamma^\mu = \rho(\gamma) \circ f \quad \text{for all } \gamma^\mu = w^\mu \circ \gamma \circ (w^\mu)^{-1} \in \Gamma^\mu,$$

and determines $[\rho] \in \text{Hom}_0(\Gamma, \mathbf{G})/\mathbf{G}$.

Indeed, f can be obtained as a ratio of two linearly independent solutions of the differential equation

$$\psi'' + \frac{1}{2}q(z)\psi = 0. \tag{16}$$

Since q is a cusp form of weight 4 for Γ^μ , a simple application of the Frobenius method (e.g., see, [15]) to (16) at cusps and elliptic fixed points shows that ρ preserves traces of parabolic and elliptic generators of Γ . Namely, the substitution $\zeta = e^{2\pi\sqrt{-1}z}$ sends the cusp ∞ to $\zeta = 0$ and transforms (16) to a second order linear differential equation with regular singular point at $\zeta = 0$. The characteristic equation has a double root $r = 0$, which corresponds to a parabolic monodromy, and similar analysis applies to elliptic fixed points.

Since the representation ρ is irreducible [9,20], we have $[\rho] \in \mathcal{X}$, which allows us to define the holomorphic map \mathcal{Q} by

$$\mathcal{M} \ni (q, [\mu]) \mapsto \mathcal{Q}(q, [\mu]) = [\rho] \in \mathcal{X}.$$

Remark 4 Besides the holomorphic embedding $\mathcal{T} \hookrightarrow \mathcal{M}$ given by the zero section, there is a smooth non-holomorphic embedding $\iota : \mathcal{T} \rightarrow \mathcal{M}$, given by

$$\mathcal{T} \ni [\mu] \mapsto (\mathcal{S}(h_\mu), [\mu]) \in \mathcal{M},$$

where $h_\mu = w_\mu \circ (w^\mu)^{-1}$ (see, Sect. 2.1.2). The image of the smooth curve $\{[t\mu]\}$ on \mathcal{T} under the map $\mathcal{F} = \mathcal{Q} \circ \iota$ —the curve $\{\Gamma_{t\mu}\}$ on \mathcal{X} —lies in the real subvariety $\mathcal{X}_\mathbb{R}$ of \mathcal{X} , the character variety for $\mathbf{G}_\mathbb{R} = \text{PSL}(2, \mathbb{R})$.

3 Differential of the map \mathcal{Q}

3.1 The set-up

Consider a smooth curve $\theta(t)$ on \mathcal{M} , defined in Sect. 2.2.1. Its image under the map \mathcal{Q} is a smooth curve on \mathcal{X} , given by the family $\{[\rho^t]\}$, where $[\rho^0] = [\rho] = \mathcal{Q}(q, 0) \in \mathcal{X}$. According to Sect. 2.5,

$$\rho^t(\gamma) = f^t \circ \gamma^{t\mu} \circ (f^t)^{-1} \quad \text{for all } \gamma^{t\mu} \in \Gamma^{t\mu}.$$

The maps $f^t : \mathbb{H}^{t\mu} \rightarrow \mathbb{P}^1$ are defined by

$$\mathcal{S}(f^t) = q^t, \tag{17}$$

where $f^0 = f : \mathbb{H} \rightarrow \mathbb{P}^1$ satisfies

$$\mathcal{S}(f) = q$$

and

$$f \circ \gamma = \rho(\gamma) \circ f \quad \text{for all } \gamma \in \Gamma.$$

Put $g^t = f^t \circ w^{t\mu} : \mathbb{H} \rightarrow \mathbb{P}^1$. It follows from (17) that

$$\mathcal{S}(g^t) = \mathcal{S}(f^t) \circ w^{t\mu} (w_z^{t\mu})^2 + \mathcal{S}(w^{t\mu}) = q(t) + \mathcal{S}(w^{t\mu}), \quad (18)$$

where $q(t)$ is a non-holomorphic form of weight 4 for Γ , given by (3). Differentiating with respect to t at $t = 0$ the equation

$$g^t \circ \gamma = \rho^t(\gamma) \circ g^t,$$

we get

$$\dot{g} \circ \gamma = \dot{\rho}(\gamma) \circ f + \rho(\gamma)' \circ f \dot{g},$$

and using the equation

$$\rho(\gamma)' \circ f f' = f' \circ \gamma \gamma',$$

we obtain

$$\frac{1}{\gamma'} \frac{\dot{g}}{f'} \circ \gamma = \frac{\dot{g}}{f'} + \frac{1}{f'} \frac{\dot{\rho}(\gamma)}{\rho(\gamma)'} \circ f.$$

For the corresponding cocycle χ , representing a tangent vector to the curve $[\rho^t]$ at $t = 0$, we have

$$\chi(\gamma) = \dot{\rho}(\gamma) \circ \rho(\gamma)^{-1} = -\frac{\dot{\rho}(\gamma^{-1})}{(\rho(\gamma^{-1})^{-1})'}$$

so that

$$\frac{1}{f'} \chi(\gamma^{-1}) \circ f = \frac{\dot{g}}{f'} - \frac{1}{\gamma'} \frac{\dot{g}}{f'} \circ \gamma. \quad (19)$$

Indeed, it immediately follows from (19) that $\chi \in Z^1(\Gamma, \mathfrak{g}_{\text{Ad}\rho})$. To show that χ is a parabolic cocycle, it is sufficient to check it for the subgroup Γ_∞ generated by $\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which corresponds to the cusp at ∞ . We can assume that the maps f^t fix ∞ , so that the maps $g^t = f^t \circ w^{t\mu}$ also have this property,

$$g^t(z + 1) = g^t(z) + c(t).$$

Thus $\dot{g}(z + 1) = \dot{g}(z) + \dot{c}$ and $\chi(\tau) = \dot{c}$. Whence there is $P \in \mathcal{P}_2$ such that $\chi(\tau) = P \circ \tau - P$.

3.2 Differential equation and the Λ -operator

From (18) it is easy to obtain a differential equation for \dot{g} . Namely, differentiate equation (18) with respect to t at $t = 0$. Using $g^0 = f$ and $\dot{w}_{zzz}^\mu = 0$ for $\mu \in \Omega^{-1,1}(\mathbb{H}, \Gamma)$, which follows from classic Ahlfors' formula in [1], we get

$$\dot{q} = \left. \frac{d}{dt} \right|_{t=0} \mathcal{S}(g^t) = \frac{\dot{g}_{zzz}}{f'} - 3 \frac{f''}{f'^2} \dot{g}_{zz} + \left(3 \frac{f''^2}{f'^3} - \frac{f'''}{f'^2} \right) \dot{g}_z.$$

Since $q = \mathcal{S}(f)$, a simple computation shows that this equation can be written neatly as follows

$$\Lambda_q \left(\frac{\dot{g}}{f'} \right) = \dot{q}, \tag{20}$$

where Λ_q is the following linear differential operator of the third order,

$$\Lambda_q(F)(z) = F_{zzz} + 2q(z)F_z + q'(z)F.$$

In case $q = 0$ the operator Λ_0 is just a third derivative operator. The Λ -operator is classical and goes back to Appell (see, [22, Example 10 in Sect. 14.7]). Its basic properties are summarized below.

A1. If ψ_1 and ψ_2 are solutions of the ordinary differential Eq. (16), then

$$\Lambda_q(\psi_1\psi_2) = 0.$$

Since for $q = \mathcal{S}(f)$ one can always choose $\psi_1 = \frac{1}{\sqrt{f'}}$ and $\psi_2 = \frac{f}{\sqrt{f'}}$,

$$\Lambda_q \left(\frac{P \circ f}{f'} \right) = 0$$

for every $P \in \mathcal{P}_2$.

A2. If a function h satisfies $\Lambda_0(h) = p$ and f is holomorphic and locally schlicht, then $H = \frac{h \circ f}{f'}$ satisfies

$$\Lambda_q(H) = P,$$

where $q = \mathcal{S}(f)$ and $P = p \circ f(f')^2$.

A3. If $q \circ \gamma (\gamma')^2 = q$ for some $\gamma \in \mathbf{G}$, then

$$\Lambda_q \left(\frac{F \circ \gamma}{\gamma'} \right) = \Lambda_q(F) \circ \gamma (\gamma')^2.$$

A4. The general solution of the equation

$$\Lambda_q(G) = Q,$$

where $q = \mathcal{S}(f)$ and Q is holomorphic on \mathbb{H} , is given by

$$G(z) = \frac{1}{2} \int_{z_0}^z \frac{(f(z) - f(u))^2}{f'(z)f'(u)} Q(u) du + \frac{1}{f'(z)} (af(z)^2 + bf(z) + c),$$

where a, b, c are arbitrary anti-holomorphic functions of z .

A5.

$$\Lambda_q(F)G + F\Lambda_q(G) = (B_q[F, G])_z,$$

where the bilinear form B_q is given by

$$B_q[F, G] = F_{zz}G + FG_{zz} - F_zG_z + 2q(z)FG.$$

All these properties are well-known and can be verified by direct computation. In particular, property **A4**, according to **A2**, follows from case $q = 0$, when the equation $\Lambda_0(G) = Q$ is readily solved by

$$G(z) = \frac{1}{2} \int_{z_0}^z (z - u)^2 Q(u) du + az^2 + bz + c.$$

Bilinear form B_q , introduced in **A5**, will play an important role in our approach. It has the following properties.

B1. We have

$$B_q \left[\frac{F \circ f}{f'}, \frac{G \circ f}{f'} \right] = B_0[F, G] \circ f,$$

where $q = \mathcal{S}(f)$. In general,

$$(B_{\mathcal{S}(f_1)}[F, G]) \circ f_2 = B_{\mathcal{S}(f_1 \circ f_2)} \left[\frac{F \circ f_2}{f'_2}, \frac{G \circ f_2}{f'_2} \right].$$

B2. If $q \circ \gamma (\gamma')^2 = q$ for some $\gamma \in \mathbf{G}$, then

$$B_q[F, G] \circ \gamma = B_q \left[\frac{F \circ \gamma}{\gamma'}, \frac{G \circ \gamma}{\gamma'} \right].$$

B3. If $(F \circ \gamma) \frac{\overline{\gamma'}}{\gamma'} = F$ for some $\gamma \in \mathbf{G}$, then

$$B_q[F, G] - B_q[F, G] \circ \gamma \frac{\overline{\gamma'}}{\gamma'} = B_q[F, H], \quad \text{where } H = G - \frac{G \circ \gamma}{\gamma'}.$$

3.3 The differential

We summarize the obtained results in the following statement.

Lemma 1 *Let $(\dot{\theta}, \mu) \in T_{(q,0)}\mathcal{M}$, where $\dot{\theta} = P_2(\dot{q})$, be a tangent vector corresponding to a curve $\{\gamma^t\}$. For a representative χ of the cohomology class*

$$[\chi] = d\mathcal{Q}|_{(q,0)}(\dot{\theta}, \mu) \in H_{\text{par}}^1(\Gamma, \mathfrak{g}_{\text{Ad}\rho}),$$

we have

$$\frac{1}{f'} \chi(\gamma^{-1}) \circ f = \frac{\dot{g}}{f'} - \frac{1}{\gamma'} \frac{\dot{g}}{f'} \circ \gamma,$$

where $\frac{\dot{g}}{f'}$ satisfies

$$\Lambda_q \left(\frac{\dot{g}}{f'} \right) = \dot{q}, \quad \frac{\partial}{\partial \bar{z}} \left(\frac{\dot{g}}{f'} \right) = \mu.$$

Proof It remains only to check the last equation. Since $g^t = f^t \circ w^{t\mu}$, it follows from the Beltrami equation for $w^{t\mu}$ that on \mathbb{H} the function g^t satisfies

$$g_{\bar{z}}^t = t\mu g_z^t,$$

and therefore

$$\dot{g}_{\bar{z}} = \mu f',$$

i.e.,

$$\frac{\partial}{\partial \bar{z}} \left(\frac{\dot{g}}{f'} \right) = \mu. \tag{21}$$

□

Remark 5 We have

$$\Lambda_q(\mu) = \dot{q}_{\bar{z}},$$

which is a compatibility condition of Eqs. (20) and (21). It can be also verified directly by differentiating the equation

$$\left(\frac{\partial}{\partial \bar{z}} - t\mu \frac{\partial}{\partial z} - 2t\mu_z \right) q(t) = 0$$

at $t = 0$,

$$\dot{q}_{\bar{z}} = 2q\mu_z + q'\mu = \Lambda_q(\mu).$$

Corollary 1 The function $\frac{\dot{g}}{f'}$ is given by the following formula

$$\frac{\dot{g}(z)}{f'(z)} = \dot{w}(z) + \frac{1}{2} \int_{z_0}^z \frac{(f(z) - f(u))^2}{f'(z)f'(u)} \tilde{q}(u) du + \frac{P(f(z))}{f'(z)},$$

where $P \in \mathcal{P}_2$ and $\tilde{q} = \dot{q} - \Lambda_q(\dot{w}) = \dot{q} - 2q\dot{w}_z - q'\dot{w}$.

Proof It follows from properties **A1** and **A4**, since the holomorphic function $\frac{\dot{g}}{f'} - \dot{w}$ satisfies

$$\Lambda_q \left(\frac{\dot{g}}{f'} - \dot{w} \right) = \tilde{q}.$$

□

Remark 6 Similarly to Wolpert's formulas [24] for Bers and Eichler–Shimura cocycles, from Corollary 1 one can obtain an explicit formula for the parabolic cocycle $\chi \in Z_{\text{par}}^1(\Gamma, \mathfrak{g}_{\text{Ad}\rho})$.

Corollary 2 For every cusp α for Γ there is $P_\alpha \in \mathcal{P}_2$ such that

$$\frac{\dot{g}(z)}{f'(z)} = \frac{P_\alpha(f(z))}{f'(z)} + O(e^{-c_\alpha \text{Im } \sigma_\alpha z}) \text{ as } \text{Im } \sigma_\alpha z \rightarrow \infty,$$

where $\sigma_\alpha \in \text{PSL}(2, \mathbb{R})$ is such that $\sigma_\alpha(\alpha) = \infty$ and $c_\alpha > 0$.

Proof It follows from Remark 1 and Lemma 1 (or from Corollary 1). □

Remark 7 For the family $q^t = \mathcal{S}(h_{t\mu})$, introduced in Remark 4, we have $g^t = w_{t\mu}$ and $\dot{q} = \dot{g}_{zzz}$. It follows from classic Ahlfors' formula in [1] that

$$\dot{q} = -\frac{1}{2}q, \text{ where } \mu = y^2\tilde{q}.$$

Thus

$$d\iota|_0(\mu) = (-\frac{1}{2}q, \mu) \in T_0\mathcal{M},$$

and it follows from (1) that

$$\iota^*(\omega) = \sqrt{-1} \omega_{\text{WP}}.$$

4 Computation of the symplectic form

4.1 The fundamental domain

Here we recall the definition of a canonical fundamental domain for the Fuchsian group Γ (see, [13] and references therein).

4.1.1. In case $m = n = 0$ choose $z_0 \in \mathbb{H}$ and standard generators $a_k, b_k, k = 1, \dots, g$. The oriented canonical fundamental domain F with the base point z_0 is a topological $4g$ -gon whose ordered vertices are given by the consecutive quadruples

$$(R_k z_0, R_k a_{k+1} z_0, R_k a_{k+1} b_{k+1} z_0, R_k a_{k+1} b_{k+1} a_{k+1}^{-1} z_0), \quad k = 0, \dots, g - 1.$$

Corresponding A and B edges of F are analytic arcs $A_k = (R_{k-1} z_0, R_{k-1} a_k z_0)$ and $B_k = (R_k z_0, R_k b_k z_0), k = 1, \dots, g$, and corresponding dual edges are $A'_k = (R_k b_k z_0, R_k b_k a_k z_0)$ and $B'_k = (R_{k-1} a_k z_0, R_{k-1} b_k a_k z_0)$ (see, Fig. 1 for a typical fundamental domain for a group Γ of genus 2).

We have

$$\partial F = \sum_{k=1}^g (A_k - B_k - A'_k + B'_k).$$

Here

$$A_k = \alpha_k(A'_k) \quad \text{and} \quad B_k = \beta_k(B'_k),$$

where $\alpha_k = R_{k-1} b_k^{-1} R_k^{-1}$ and $\beta_k = R_k a_k^{-1} R_{k-1}^{-1}$. They satisfy

$$[\alpha_k, \beta_k] = R_{k-1} R_k^{-1},$$

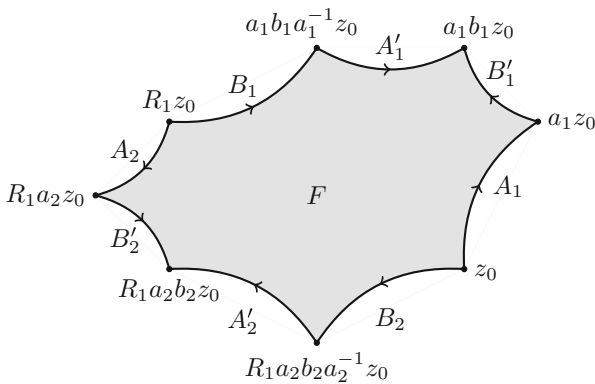


Fig. 1 Fundamental domain for a group Γ of genus 2

so that

$$\mathcal{R}_k = \prod_{i=1}^k [\alpha_i, \beta_i] = R_k^{-1} \quad \text{and} \quad \prod_{k=1}^g \alpha_k \beta_k \alpha_k^{-1} \beta_k^{-1} = 1.$$

The generators $\alpha_k, \beta_k, k = 1, \dots, g$, are dual generators of Γ , introduced by A. Weil [21] (see also, [12]), and

$$a_k^{-1} = \mathcal{R}_k \beta_k \mathcal{R}_k^{-1}, \quad b_k^{-1} = \mathcal{R}_{k-1} \alpha_k \mathcal{R}_{k-1}^{-1}.$$

We have $A_k = (\mathcal{R}_{k-1}^{-1} z_0, \beta_k^{-1} \mathcal{R}_k^{-1} z_0)$, $B_k = (\mathcal{R}_k^{-1} z_0, \alpha_k^{-1} \mathcal{R}_{k-1}^{-1} z_0)$ and

$$\partial F = \sum_{i=1}^{2g} (S_i - \lambda_i(S_i)),$$

where $S_k = A_k, S_{k+g} = -B_k$ and $\lambda_k = \alpha_k^{-1}, \lambda_{k+g} = \beta_k^{-1}, k = 1, \dots, g$.

Remark 8 The ordering of vertices of F for the dual generators corresponds to the opposite orientation, so that (cf. (11))

$$c = - \sum_{k=1}^g \left\{ \left(\frac{\partial \mathcal{R}}{\partial \alpha_k}, \alpha_k \right) + \left(\frac{\partial \mathcal{R}}{\partial \beta_k}, \beta_k \right) \right\}.$$

4.1.2. In general case $m + n > 0$, oriented canonical fundamental domain F with the base point z_0 is a $(4g + 2m + 2n)$ -gon whose ordered vertices are given by the consecutive quadruples

$$(R_k z_0, R_k a_{k+1} z_0, R_k a_{k+1} b_{k+1} z_0, R_k a_{k+1} b_{k+1} a_{k+1}^{-1} z_0), \quad k = 0, \dots, g - 1,$$

followed by the consecutive triples $(R_{g+i-1} z_0, z_i, R_{g+i} z_0), i = 1, \dots, m + n$. Here $z_i \in \mathbb{H}, i = 1, \dots, m$, are fixed points of the elliptic elements

$$\gamma_i = R_{g+i-1} c_i^{-1} R_{g+i-1}^{-1},$$

and $z_{m+j} \in \mathbb{R}, j = 1, \dots, n$, are fixed points of the parabolic elements

$$\gamma_{m+j} = R_{g+m+j-1} c_{m+j}^{-1} R_{g+m+j-1}^{-1}$$

(see, Fig. 2 for a typical fundamental domain of group Γ of signature $(1; 1, 6)$, where z_1 is elliptic fixed point of order 6 and z_2 is a cusp).

We have

$$\partial F = \sum_{k=1}^g (A_k - B_k - A'_k + B'_k) + \sum_{i=1}^{m+n} (C_i - C'_i),$$

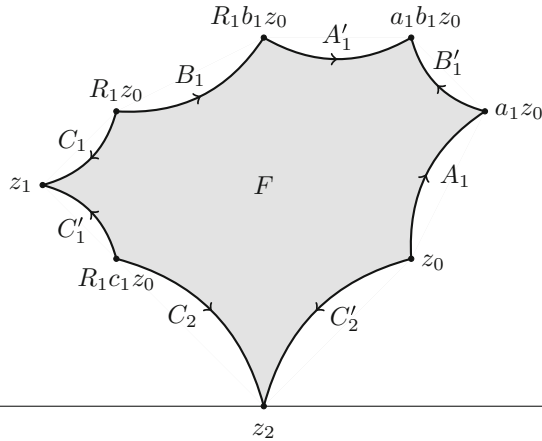


Fig. 2 Fundamental domain for a group Γ of signature $(1;1,6)$

where

$$C_i = (R_{g+i-1}z_0, z_i), \quad C'_i = (R_{g+i}z_0, z_i), \quad C_i = \gamma_i(C'_i), \quad i = 1, \dots, m+n.$$

The generators $\alpha_k, \beta_k, k = 1, \dots, g$, and $\gamma_i, i = 1, \dots, m+n$, are dual generators of Γ satisfying

$$\mathcal{R}_g \gamma_1 \cdots \gamma_{m+n} = 1.$$

We have $C_i = (\mathcal{R}_{g+i-1}^{-1}z_0, z_i)$ and

$$\partial F = \sum_{k=1}^N (S_k - \lambda_k(S_k)), \quad N = 2g + m + n, \tag{22}$$

where $S_{2g+i} = C_i, \lambda_{2g+i} = \gamma_i^{-1}, i = 1, \dots, m+n$.

4.2 The main formula

Here we obtain another representation for the symplectic form ω . Put $F^Y = \{z \in F : \text{Im}(\sigma_j^{-1}) \leq Y, j = 1, \dots, n\}$, where $\sigma_j^{-1}(x_j) = \infty$, and denote by $H_j(Y)$ corresponding horocycles in F . We have

$$\omega((\dot{\theta}_1, \mu_1), (\dot{\theta}_2, \mu_2)) = \frac{\sqrt{-1}}{2} \lim_{Y \rightarrow \infty} \int_{F^Y} (\dot{q}_1 \mu_2 - \dot{q}_2 \mu_1) dz \wedge d\bar{z}.$$

Lemma 2 *The symplectic form ω , evaluated on two tangent vectors $(\dot{\theta}_1, \mu_1)$ and $(\dot{\theta}_2, \mu_2)$ corresponding to the curves $\theta_1(t)$ and $\theta_2(t)$, is given by*

$$\begin{aligned} &\omega((\dot{\theta}_1, \mu_1), (\dot{\theta}_2, \mu_2)) \\ &= \frac{\sqrt{-1}}{4} \int_{\partial F} \left\{ \left(\dot{q}_2 \frac{\dot{g}_1}{f'} - \dot{q}_1 \frac{\dot{g}_2}{f'} \right) dz + \left(B_q \left[\mu_2, \frac{\dot{g}_1}{f'} \right] - B_q \left[\mu_1, \frac{\dot{g}_2}{f'} \right] \right) d\bar{z} \right\}. \end{aligned}$$

Proof Denote the 1-form under the integral by ϑ . We have, using Lemma 1,

$$\begin{aligned} d\vartheta &= \left(\dot{q}_2 \bar{z} \frac{\dot{g}_1}{f'} + \dot{q}_2 \left(\frac{\dot{g}_1}{f'} \right)_{\bar{z}} - \dot{q}_1 \bar{z} \frac{\dot{g}_2}{f'} - \dot{q}_1 \left(\frac{\dot{g}_2}{f'} \right)_{\bar{z}} \right) d\bar{z} \wedge dz \\ &\quad + \left(\Lambda_q(\mu_2) \frac{\dot{g}_1}{f'} + \mu_2 \Lambda_q \left(\frac{\dot{g}_1}{f'} \right) - \Lambda_q(\mu_1) \frac{\dot{g}_2}{f'} - \mu_1 \Lambda_q \left(\frac{\dot{g}_2}{f'} \right) \right) dz \wedge d\bar{z} \\ &= \left(\dot{q}_2 \bar{z} \frac{\dot{g}_1}{f'} + \dot{q}_2 \mu_1 - \dot{q}_1 \bar{z} \frac{\dot{g}_2}{f'} - \dot{q}_1 \mu_2 \right) d\bar{z} \wedge dz \\ &\quad + \left(\dot{q}_2 \bar{z} \frac{\dot{g}_1}{f'} + \mu_2 \dot{q}_1 - \dot{q}_1 \bar{z} \frac{\dot{g}_2}{f'} - \mu_1 \dot{q}_2 \right) dz \wedge d\bar{z} \\ &= 2(\dot{q}_1 \mu_2 - \dot{q}_2 \mu_1) dz \wedge d\bar{z}. \end{aligned}$$

Since due to exponential decay of \dot{q}_1, \dot{q}_2 and μ_1, μ_2 at the cusps the integrals over horocycles $H_j(Y)$ tend to 0 as $Y \rightarrow \infty$, by Stokes' theorem we get (4). \square

The line integral over ∂F in Lemma 2 can be evaluated explicitly.

Proposition 1 *We have*

$$\begin{aligned} &\omega((\dot{\theta}_1, \mu_1), (\dot{\theta}_2, \mu_2)) \\ &= \frac{\sqrt{-1}}{4} \sum_{i=1}^N \left(B_q \left[\frac{\dot{g}_2}{f'}, \frac{1}{f'} \chi_1(\lambda_i^{-1}) \circ f \right] - B_q \left[\frac{\dot{g}_1}{f'}, \frac{1}{f'} \chi_2(\lambda_i^{-1}) \circ f \right] \right) \Big|_{\partial S_i(0)}^{\partial S_i(1)}. \end{aligned}$$

Proof Using Lemma 2, formula (22), Lemma 1 and property B3, we get

$$\begin{aligned} &\frac{4}{\sqrt{-1}} \omega((\dot{\theta}_1, \mu_1), (\dot{\theta}_2, \mu_2)) \\ &= \sum_{i=1}^N \left(\int_{S_i} - \int_{\lambda_i(S_i)} \right) \left\{ \left(\dot{q}_2 \frac{\dot{g}_1}{f'} - \dot{q}_1 \frac{\dot{g}_2}{f'} \right) dz + \left(B_q \left[\mu_2, \frac{\dot{g}_1}{f'} \right] - B_q \left[\mu_1, \frac{\dot{g}_2}{f'} \right] \right) d\bar{z} \right\} \\ &= \sum_{i=1}^N \int_{S_i} \left\{ \left(\dot{q}_2 \frac{1}{f'} \chi_1(\lambda_i^{-1}) \circ f - \dot{q}_1 \frac{1}{f'} \chi_2(\lambda_i^{-1}) \circ f \right) dz \right. \\ &\quad \left. + \left(B_q \left[\mu_2, \frac{1}{f'} \chi_1(\lambda_i^{-1}) \circ f \right] - B_q \left[\mu_1, \frac{1}{f'} \chi_2(\lambda_i^{-1}) \circ f \right] \right) d\bar{z} \right\}. \end{aligned}$$

Using Lemma 1 and properties A1 and A5, we obtain

$$B_q \left[\mu, \frac{1}{f'} \chi(\lambda_i^{-1}) \circ f \right] = \frac{\partial}{\partial \bar{z}} B_q \left[\frac{\dot{g}}{f'}, \frac{1}{f'} \chi(\lambda_i^{-1}) \circ f \right]$$

and

$$\frac{\partial}{\partial z} B_q \left[\frac{\dot{g}}{f'}, \frac{1}{f'} \chi(\lambda_i^{-1}) \circ f \right] = \Lambda_q \left(\frac{\dot{g}}{f'} \right) \frac{1}{f'} \chi(\lambda_i^{-1}) \circ f = \dot{q} \frac{1}{f'} \chi(\lambda_i^{-1}) \circ f.$$

Since

$$\Phi_{\bar{z}} d\bar{z} = d\Phi - \Phi_z dz,$$

we finally get (note how the signs match)

$$\begin{aligned} & \frac{4}{\sqrt{-1}} \omega((\dot{\theta}_1, \mu_1), (\dot{\theta}_2, \mu_2)) \\ &= \sum_{i=1}^N \int_{S_i} \left(dB_q \left[\frac{\dot{g}_2}{f'}, \frac{1}{f'} \chi_1(\lambda_i^{-1}) \circ f \right] - dB_q \left[\frac{\dot{g}_1}{f'}, \frac{1}{f'} \chi_2(\lambda_i^{-1}) \circ f \right] \right) \\ &= \sum_{i=1}^N \left(B_q \left[\frac{\dot{g}_2}{f'}, \frac{1}{f'} \chi_1(\lambda_i^{-1}) \circ f \right] - B_q \left[\frac{\dot{g}_1}{f'}, \frac{1}{f'} \chi_2(\lambda_i^{-1}) \circ f \right] \right) \Big|_{\partial S_i(0)}^{\partial S_i(1)}. \end{aligned}$$

According to Corollary 2, $B_q \left[\frac{\dot{g}}{f'}, \frac{1}{f'} \chi(\lambda_i^{-1}) \circ f \right] (z)$ has a limit as z approaches the cusps for Γ . □

4.3 Main result

Theorem 1 *The pull-back of the Goldman symplectic form on \mathcal{K} by the map Q is $\sqrt{-1}$ times canonical symplectic form on \mathcal{M} ,*

$$\omega = -\sqrt{-1} Q^*(\omega_G).$$

Proof Since the choice of a base point for \mathcal{T} is inessential (see, Sect. 2.1.2), it is sufficient to compute the pullback only for the points in $Q(q, 0)$. For the convenience of the reader, consider first the case $m = n = 0$, when $N = 2g$. Using property **B2** and Eqs. (7)–(8), we have for arbitrary $\alpha, \beta \in \Gamma$,

$$\begin{aligned} & B_q \left[\frac{\dot{g}_1}{f'}, \frac{1}{f'} \chi_2(\alpha) \circ f \right] (\beta z_0) = B_q \left[\frac{1}{\beta'} \left(\frac{\dot{g}_1}{f'} \right) \circ \beta, \frac{1}{\beta'} \left(\frac{1}{f'} \chi_2(\alpha) \circ f \right) \circ \beta \right] (z_0) \\ &= B_q \left[\frac{\dot{g}_1}{f'} - \frac{1}{f'} \chi_1(\beta^{-1}) \circ f, \frac{1}{f'} \chi_2(\beta^{-1} \alpha) \circ f - \frac{1}{f'} \chi_2(\beta^{-1}) \circ f \right] (z_0) \\ &= B_q \left[\frac{\dot{g}_1}{f'}, \frac{1}{f'} (\chi_2(\beta^{-1} \alpha) - \chi_2(\beta^{-1})) \circ f \right] (z_0) \\ & \quad + B_0 [\chi_1(\beta^{-1}), \chi_2(\beta^{-1}) - \chi_2(\beta^{-1} \alpha)] (z_0). \end{aligned}$$

Using (5), (7) and $\text{Ad}\rho$ invariance of the Killing form, we obtain

$$\begin{aligned} B_0[\chi_1(\beta^{-1}), \chi_2(\beta^{-1}) - \chi_2(\beta^{-1}\alpha)](z_0) &= 2\langle \chi_1(\beta^{-1}), \rho(\beta^{-1})\chi_2(\alpha) \rangle \\ &= -2\langle \chi_1(\beta), \chi_2(\alpha) \rangle, \end{aligned}$$

so that

$$\begin{aligned} B_q \left[\frac{\dot{g}_1}{f'}, \frac{1}{f'} \chi_2(\alpha) \circ f \right] (\beta z_0) \\ = B_q \left[\frac{\dot{g}_1}{f'}, \frac{1}{f'} (\chi_2(\beta^{-1}\alpha) - \chi_2(\beta^{-1})) \circ f \right] (z_0) - 2\langle \chi_1(\beta), \chi_2(\alpha) \rangle. \end{aligned} \tag{23}$$

Now for $i = k$ using (23) for $\alpha = \alpha_k, \beta = \beta_k^{-1}\mathcal{R}_k^{-1}$ and $\alpha = \alpha_k, \beta = \mathcal{R}_{k-1}^{-1}$, we obtain

$$\begin{aligned} B_q \left[\frac{\dot{g}_1}{f'}, \frac{1}{f'} \chi_2(\lambda_k^{-1}) \circ f \right] \Big|_{\partial S_k(0)}^{\partial S_k(1)} \\ = B_q \left[\frac{\dot{g}_1}{f'}, \frac{1}{f'} (\chi_2(\mathcal{R}_k\beta_k\alpha_k) - \chi_2(\mathcal{R}_k\beta_k) - \chi_2(\mathcal{R}_{k-1}\alpha_k) + \chi_2(\mathcal{R}_{k-1})) \circ f \right] (z_0) \\ - 2\langle \chi_1(\beta_k^{-1}\mathcal{R}_k^{-1}) - \chi_1(\mathcal{R}_{k-1}^{-1}), \chi_2(\alpha_k) \rangle. \end{aligned} \tag{24}$$

For $i = k + g$ we use $\alpha = \beta_k, \beta = \mathcal{R}_k^{-1}$ and $\alpha = \beta_k, \beta = \alpha_k^{-1}\mathcal{R}_{k-1}^{-1}$ to compute

$$\begin{aligned} B_q \left[\frac{\dot{g}_1}{f'}, \frac{1}{f'} \chi_2(\lambda_{i+k}^{-1}) \circ f \right] \Big|_{\partial S_{i+k}(0)}^{\partial S_{i+k}(1)} \\ = B_q \left[\frac{\dot{g}_1}{f'}, \frac{1}{f'} (\chi_2(\mathcal{R}_k\beta_k) - \chi_2(\mathcal{R}_k) - \chi_2(\mathcal{R}_{k-1}\alpha_k\beta_k) + \chi_2(\mathcal{R}_{k-1}\alpha_k)) \circ f \right] (z_0) \\ - 2\langle \chi_1(\mathcal{R}_k^{-1}) - \chi_1(\alpha_k^{-1}\mathcal{R}_{k-1}^{-1}), \chi_2(\beta_k) \rangle. \end{aligned} \tag{25}$$

Since $\mathcal{R}_{k-1}\alpha_k\beta_k = \mathcal{R}_k\beta_k\alpha_k$ and $\mathcal{R}_g = 1$, we see that the sum over k of terms in the second lines in Eqs. (24)–(25) vanishes. Using (12)–(13) and Remark 8, we get

$$\begin{aligned} \sum_{i=1}^{2g} B_q \left[\frac{\dot{g}_1}{f'}, \frac{1}{f'} \chi_2(\lambda_i^{-1}) \circ f \right] \Big|_{\partial S_i(0)}^{\partial S_i(1)} \\ = 2 \sum_{k=1}^g \left(\langle \chi_1(\mathcal{R}_{k-1}^{-1}) - \chi_1(\beta_k^{-1}\mathcal{R}_k^{-1}), \chi_2(\alpha_k) \rangle + \langle \chi_1(\alpha_k^{-1}\mathcal{R}_{k-1}^{-1}) - \chi_1(\mathcal{R}_k^{-1}), \chi_2(\beta_k) \rangle \right) \\ = 2\omega_G(\chi_1, \chi_2). \end{aligned}$$

Similarly,

$$\begin{aligned} \sum_{i=1}^{2g} B_q \left[\frac{\dot{g}_2}{f'}, \frac{1}{f'} \chi_1(\lambda_i^{-1}) \circ f \right] \Big|_{\partial S_i(0)}^{\partial S_i(1)} \\ = -2\omega_G(\chi_2, \chi_1) \end{aligned}$$

and we finally obtain

$$\omega((\dot{\theta}_1, \mu_1), (\dot{\theta}_2, \mu_2)) = -\sqrt{-1}\omega_G(\chi_1, \chi_2).$$

In general, assume that $m + n > 0$. In this case

$$\begin{aligned} \sum_{i=1}^{2g} B_q \left[\frac{\dot{g}_1}{f'}, \frac{1}{f'} \chi_2(\lambda_i^{-1}) \circ f \right] \Big|_{\partial S_i(0)}^{\partial S_i(1)} = -B_q \left[\frac{\dot{g}_1}{f'}, \frac{1}{f'} \chi_2(\mathcal{R}_g) \circ f \right] (z_0) \\ + 2 \sum_{k=1}^g \left(\langle \chi_1(\mathcal{R}_{k-1}^{-1}) - \chi_1(\beta_k^{-1}\mathcal{R}_k^{-1}), \chi_2(\alpha_k) \rangle + \langle \chi_1(\alpha_k^{-1}\mathcal{R}_{k-1}^{-1}) - \chi_1(\mathcal{R}_k^{-1}), \chi_2(\beta_k) \rangle \right), \end{aligned} \tag{26}$$

and we need to compute

$$\sum_{i=1}^{m+n} B_q \left[\frac{\dot{g}_1}{f'}, \frac{1}{f'} \chi_2(\gamma_i) \circ f \right] \Big|_{\mathcal{R}_{g+i-1}^{-1}z_0}^{z_i}.$$

Using (23) with $\alpha = \gamma_i$ and $\beta = \mathcal{R}_{g+i-1}^{-1}$, we get

$$\begin{aligned} B_q \left[\frac{\dot{g}_1}{f'}, \frac{1}{f'} \chi_2(\gamma_i) \circ f \right] (\mathcal{R}_{g+i-1}^{-1}z_0) \\ = B_q \left[\frac{\dot{g}_1}{f'}, \frac{1}{f'} (\chi_2(\mathcal{R}_{g+i}) - \chi_2(\mathcal{R}_{g+i-1})) \circ f \right] (z_0) + 2\langle \chi_1(\mathcal{R}_{g+i-1}^{-1}), \chi_2(\gamma_i) \rangle. \end{aligned}$$

Since restriction of χ_2 to the stabilizer $\Gamma_i = \langle \gamma_i \rangle$ of a fixed point z_i is a coboundary, there is $P_{2i} \in \mathcal{P}_2$ such that

$$\chi_2(\gamma_i) = \rho(\gamma_i)P_{2i} - P_{2i}.$$

Using property **B2**, $\gamma_i z_i = z_i$ and (5), we get

$$\begin{aligned} B_q \left[\frac{\dot{g}_1}{f'}, \frac{1}{f'} \chi_2(\gamma_i) \circ f \right] (z_i) = B_q \left[\frac{\dot{g}_1}{f'}, \frac{1}{(\gamma_i^{-1})'} \left(\frac{1}{f'} P_{2i} \circ f \right) \circ \gamma_i^{-1} - \frac{1}{f'} P_{2i} \circ f \right] (z_i) \\ = B_q \left[\frac{1}{\gamma_i'} \frac{\dot{g}_1}{f'} \circ \gamma_i - \frac{\dot{g}_1}{f'}, \frac{1}{f'} P_{2i} \circ f \right] (z_i) \\ = -B_0[\chi_1(\gamma_i^{-1}), P_{2i}](z_i) = 2\langle \chi_1(\gamma_i^{-1}), P_{2i} \rangle. \end{aligned}$$

Thus using $\mathcal{R}_{g+m+n} = 1$ we obtain

$$\begin{aligned} & \sum_{i=1}^{m+n} B_q \left[\frac{\dot{g}_1}{f'}, \frac{1}{f'} \chi_2(\gamma_i) \circ f \right] \Big|_{\mathcal{R}_{g+i-1} z_0}^{z_i} \\ &= B_q \left[\frac{\dot{g}_1}{f'}, \frac{1}{f'} \chi_2(\mathcal{R}_g) \circ f \right] (z_0) + 2 \sum_{i=1}^{m+n} \left(\langle \chi_1(\mathcal{R}_{g+i-1}^{-1}), \chi_2(\gamma_i) \rangle + \langle \chi_1(\gamma_i^{-1}), P_{2i} \rangle \right). \end{aligned} \tag{27}$$

Putting together formulas (26)–(27) and using (14)–(15), we finally obtain

$$\omega((\dot{\theta}_1, \mu_1), (\dot{\theta}_2, \mu_2)) = -\sqrt{-1} \omega_G(\chi_1, \chi_2).$$

□

Remark 9 The above computation is a non-abelian analog of Riemann bilinear relations, which arise from the isomorphism

$$\mathcal{H}^1(X, \mathbb{C}) / \mathcal{H}^1(X, \mathbb{Z}) \xrightarrow{\sim} \mathcal{K}_{\text{ab}},$$

where $\mathcal{H}^1(X, \mathbb{C})$ is the complex vector space of harmonic 1-forms on X and $\mathcal{K}_{\text{ab}} = (\mathbb{C}^*)^{2g}$ is the complex torus—a character variety for the abelian group $G = \mathbb{C}^*$.

Combing Theorem 1 and Remark 7, we get a generalization of Goldman’s theorem [6, Sect. 2.5] for the case of orbifold Riemann surfaces.

Corollary 3 *The pullback of the Goldman symplectic form on the character variety $\mathcal{K}_{\mathbb{R}}$ by the map \mathcal{F} is a symplectic form of the Weil–Petersson metric on \mathcal{T} ,*

$$\omega_{\text{WP}} = \mathcal{F}^*(\omega_G).$$

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