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On the spectral theory of a functional-difference operator in conformal field theory

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Abstract. We consider the functional-difference operator $H = U + U^{-1} + V$, where U and V are the Weyl self-adjoint operators satisfying the relation $UV = q^2VU$, $q = e^{\pi i\tau}$, $\tau > 0$. The operator H has applications in the conformal field theory and representation theory of quantum groups. Using the modular quantum dilogarithm (a q-deformation of the Euler dilogarithm), we define the scattering solution and Jost solutions, derive an explicit formula for the resolvent of the self-adjoint operator H on the Hilbert space $L^2(\mathbb{R})$, and prove the eigenfunction expansion theorem. This theorem is a q-deformation of the well-known Kontorovich–Lebedev transform in the theory of special functions. We also present a formulation of the scattering theory for H.

Keywords: modular quantum dilogarithm, Weyl operators, functional-difference operator, Schrödinger operator, Fourier transform, Casorati determinant, Sokhotski–Plemelj formula, scattering solution, Jost solutions, resolvent of an operator, eigenfunction expansion, Kontorovich–Lebedev transform, scattering theory, scattering operator.

§ 1. Introduction

Quantum mechanics gave a powerful impetus to the development of the spectral theory of differential operators. In particular, various spectral problems for the Schrödinger operator were studied very extensively. The direct and inverse scattering problems for the Schrödinger operator were studied in the classical papers of Gel'fand, Krein, Levitan, Marchenko and Povzner in the 1950s (see the surveys [1], [2] and references therein). The fundamental role of these papers in the development of the theory of classical integrable systems is well known.

Since its formulation in the 1980s, two-dimensional conformal field theory [3] stimulated further the development of the representation theory of infinite-dimensional Lie groups and Lie algebras. One of the fundamental models in conformal field theory is the quantum Liouville model, whose discrete version was considered by us more than 30 years ago (see the lecture [4] published in 1986). It is in the explicit construction of the L-operator in [4] that the quantum group $\mathrm{SL}_q(2,\mathbb{R})$ was first

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introduced. The matrix trace of the L-operator is a functional-difference operator H which plays an important role in the quantization of the Teichmüller space [5], [6] and representation theory of the non-compact quantum group $\mathrm{SL}_q(2,\mathbb{R})$ [7]. In the notation of § 2.1 this operator is of the form $H = U + U^{-1} + V$. It acts on the functions $\psi(x)$ on the real line by the formula

$$(H\psi)(x) = \psi(x + 2\omega') + \psi(x - 2\omega') + e^{\frac{\pi i x}{\omega}}\psi(x).$$

Here ω and ω' are pure imaginary with positive imaginary parts, and the function $\psi(x)$ is assumed to be analytic in the strip $|\operatorname{Im} z| \leqslant 2|\omega'|$, $z = x + iy$ (see § 2.1 and § 4 for precise definitions). The operator H is closely related to the representation theory of the quantum group $\mathrm{SL}_q(2,\mathbb{R})$ with $q = e^{\pi i \tau}$, where $\tau = \omega'/\omega > 0$ (see [7], [8]).

The eigenvalue problem for H has the form

$$\psi(x + 2\omega', \lambda) + \psi(x - 2\omega', \lambda) + e^{\frac{\pi i x}{\omega}}\psi(x, \lambda) = \lambda\psi(x, \lambda) \tag{1.1}$$

and is a functional-difference analogue of the Schrödinger operator whose potential decays exponentially as $x \to -\infty$ and grows exponentially as $x \to \infty$. The continuous limit of (1.1) is the equation

$$-\widetilde{\psi}''(x, \lambda) + e^{2x}\widetilde{\psi}(x, \lambda) = \lambda\widetilde{\psi}(x, \lambda) \tag{1.2}$$

for the modified Bessel functions of e^x.

In [6], the eigenfunction expansion theorem for H in the momentum representation was stated as formal completeness and orthogonality relations in the sense of distributions. A detailed derivation of these relations using properties of the modular quantum dilogarithm (see § 2.2) was given in [8]. Nevertheless, the spectral theory of H as an unbounded self-adjoint operator on the Hilbert space $L^2(\mathbb{R})$ has not been previously considered.

In the present paper we fill this gap and give a complete analytic study of the functional-difference operator H. Namely, we define the scattering solution and Jost solutions of (1.1), present an explicit formula for the resolvent of the self-adjoint operator H on $L^2(\mathbb{R})$, and prove the eigenfunction expansion theorem. We also give a formulation of the scattering theory for H.

We now discuss the content of the paper in more detail. In § 2 we collect all necessary concepts and notation. Specifically, in § 2.1 we define a Weyl pair U, V of unbounded self-adjoint operators on $L^2(\mathbb{R})$ satisfying the relation $UV = q^2 VU$, and § 2.2 contains the properties of the modular quantum dilogarithm $\gamma(z)$, which is a q-deformation of the Euler dilogarithm and is expressed in terms of the ratio of the Alekseevskii–Barnes double gamma functions.

In § 3 we investigate the *free operator* H_0, formally given by the expression

$$(H_0\psi)(x) = \psi(x + 2\omega') + \psi(x - 2\omega').$$

Thus, in § 3.1 we define H_0 as an unbounded self-adjoint operator on $L^2(\mathbb{R})$ with domain $D(H_0)$ and with absolutely continuous spectrum of multiplicity 2 filling $[2, \infty)$. In § 3.2 we use the Fourier transform to give an explicit expression for the resolvent $R_0(\lambda)$ of H_0 in the form of an integral operator with kernel $R_0(x-y;\lambda)$; see (3.5). Note that in contrast to the case of the Schrödinger operator, where the resolvent kernel is obtained by the method of variation of parameters using the simple formula $\theta'(x)=\delta(x)$ ($\theta(x)$ is the Heaviside function), the main equation

$$R_0(x+2\omega';\lambda)+R_0(x-2\omega';\lambda)-\lambda R_0(x;\lambda)=\delta(x)$$

for the resolvent of H_0 holds because of the Sokhotski–Plemelj formula

$$\frac{1}{2\pi i}\biggl(\frac{1}{x-i0}-\frac{1}{x+i0}\biggr)=\delta(x)$$

from the theory of distributions.

In § 4 we study the operator H. Following [6], we consider the Fourier transform of (1.1) (this is the functional-difference equation (4.2) of the first order) and its special solution $\widehat{\varphi}(p,k)$ in § 4.1. This solution is expressed in terms of the modular quantum dilogarithm using the convenient parametrization $\lambda=2\cosh\bigl(\frac{\pi ik}{\omega}\bigr)$. In § 4.2 we define a solution $\varphi(x,k)$ of (1.1) as the inverse Fourier transform of $\widehat{\varphi}(p,k)$. The necessary properties of $\varphi(x,k)$ are collected in Lemma 4.1. They show that $\varphi(x,k)$ plays the role of the scattering solution of (1.1). In particular, for real x and k the solution $\varphi(x,k)$ decays exponentially as $x\to+\infty$ and oscillates as $x\to-\infty$. Moreover, $\varphi(x,k)$ is an entire function of x and analytically continues to the strip $0<\operatorname{Im}k\leqslant|\omega|$, which corresponds to the values $\lambda\in\mathbb{C}\setminus[2,\infty)$. In § 4.3 we introduce the Jost solutions $f_\pm(x,k)$ of (1.1) as the solutions with the following asymptotic behaviour for real k:

$$f_\pm(x,k)=e^{\pm 2\pi ikx}+o(1)\quad\text{as}\ \ x\to-\infty.$$

We note that unlike the differential equation (1.2), which has two linearly independent solutions, the functional-difference equation (1.1) has an infinite-dimensional space of solutions since one can multiply a solution by a *quasi-constant*, a holomorphic $2\omega'$-periodic function of x. Therefore determining the Jost solutions is a non-trivial problem. Using the analogy between (1.1) and (1.2) along with the properties of the solutions of (1.2) collected in § 6, we define the Jost solutions $f_\pm(x,k)$ by the integral representation (4.12). The properties of the solutions $f_\pm(x,k)$ are contained in Lemma 4.2. In particular, they admit analytic continuation to the strip $0<\operatorname{Im}k\leqslant|\omega|$, and

$$\varphi(x,k)=M(k)f_+(x,k)+M(-k)f_-(x,k),$$

where the function $M(k)$ is analytic in the strip $0\leqslant\operatorname{Im}k\leqslant|\omega|$ and is explicitly expressed in terms of the modular quantum dilogarithm (see Lemma 4.1).

The analytic properties of the solutions $\varphi(x,k)$ and $f_\pm(x,k)$ are used in § 5.1 to show that the resolvent $R_\lambda(H)=(H-\lambda I)^{-1}$ of H is defined for $\lambda\notin[2,\infty)$ and is a bounded integral operator on $L^2(\mathbb{R})$ whose integral kernel $R(x,y;\lambda)$ is given by the explicit formula (5.1) (see Proposition 5.1). In § 5.2 we prove the eigenfunction expansion theorem for scattering solutions of H. Namely, computing the jump of the resolvent across the continuous spectrum, we prove (see Theorem 5.1) that the operator

$$(\mathscr{U}\psi)(k)=\int_{-\infty}^{+\infty}\psi(x)\varphi(x,k)\,dx$$

is an isometric isomorphism between the Hilbert spaces $L^2(\mathbb{R})$ and $\mathscr{H}_0 = L^2([0,\infty),\rho(k)\,dk)$, where

$$\rho(k)=\frac{1}{M(k)M(-k)}=4\sinh\biggl(\frac{\pi ik}{\omega}\biggr)\sinh\biggl(\frac{\pi ik}{\omega'}\biggr)$$

is the spectral function of H. Moreover, $\mathscr{U}H\mathscr{U}^{-1}$ is the operator of multiplication by $\lambda=2\cosh\bigl(\frac{\pi ik}{\omega}\bigr)$ on $\mathscr{H}_0$. Hence H has a simple absolutely continuous spectrum filling $[2,\infty)$.

Comparison of the equations (1.1) and (1.2) shows that the eigenfunction expansion for H is a q-analogue of the well-known Kontorovich–Lebedev transform in the theory of Bessel functions (see Remark 5.4). In § 5.3 we give a formulation of the scattering theory for H and show that the scattering operator is the operator of multiplication by the function

$$S(k)=\frac{M(-k)}{M(k)}.$$

The appendix (§ 6) contains known properties of the solutions of (1.2).

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§ 2. Basic concepts and notation

2.1. Weyl's operators. Let $L^2(\mathbb{R})$ be the Hilbert space of functions that are square integrable over the real axis with respect to the Lebesgue measure. The Weyl operators in quantum mechanics are the following unitary operators $U(u)$ and $V(v)$ on $L^2(\mathbb{R})$, where $u,v\in\mathbb{R}$:

$$(U(u)\psi)(x)=\psi(x-u),\quad (V(v)\psi)(x)=e^{-ivx}\psi(x),\qquad \psi\in L^2(\mathbb{R})$$

(see, for example, [9], Ch. 2, where Planck's constant $\hbar$ is chosen to be 1). The operators $U(u)$ and $V(v)$ satisfy Weyl's commutation relations

$$U(u)V(v)=e^{iuv}V(v)U(u).$$

In the representation theory of the quantum group $\mathrm{SL}_q(2,\mathbb{R})$ it is necessary to use complex u and v. Then the Weyl operators $U(u)$ and $V(v)$ become unbounded self-adjoint operators on $L^2(\mathbb{R})$. Namely, using the classical Weierstrass notation, we write 2ω, $2\omega'$ for the generators of a lattice in $\mathbb{C}$ such that $\operatorname{Im}\tau>0$, where $\tau=\frac{\omega'}{\omega}$, and put $q=e^{\pi i\tau}$. The key role in the representation theory of $\mathrm{SL}_q(2,\mathbb{R})$ is played by operators U and V which are formally defined by

$$(U\psi)(x)=\psi(x+2\omega'),\qquad (V\psi)(x)=e^{\frac{\pi ix}{\omega}}\psi(x) \tag{2.1}$$

and satisfy the relation

$$UV = q^2 VU \tag{2.2}$$

on the common domain of U and V. Especially interesting is the representation theory of real forms of $\mathrm{SL}_q(2,\mathbb{R})$ that correspond to the cases $q \in \mathbb{R}$ and $|q| = 1$. In the first case we have $\omega' \in i\mathbb{R}$, $\omega \in \mathbb{R}$ and $0 < q < 1$, which corresponds to a rectangular period lattice. In the second case the half-periods ω, ω' are pure imaginary and the theory of elliptic functions is inapplicable.

It is the latter case that arises in applications to conformal field theory. We shall consider the Weyl pair U, V with $|q| = 1$. This corresponds to the case when the half-periods ω and ω' are pure imaginary with positive imaginary parts. It is convenient to use the normalization

$$\omega\omega' = -\frac{1}{4}, \tag{2.3}$$

which is assumed to hold throughout the paper. In papers on quantum Liouville theory it is customary to use the parametrization $\omega = \frac{i}{2b}$, $\omega' = \frac{ib}{2}$ (where $\tau = b^2$ and $b > 0$) as is done in [10].

The operators U and V defined by (2.1) are unbounded self-adjoint operators on $L^2(\mathbb{R})$. This follows from von Neumann's general spectral theorem since they are real-valued functions of the self-adjoint operators P and Q in quantum mechanics: $U = e^{2i\omega' P}$ and $V = e^{\frac{\pi i Q}{\omega}}$, where $P = -i\frac{d}{dx}$ and Q is the operator of multiplication by the independent variable x.

The Weyl operators U and V can also be defined directly. Namely, U is a self-adjoint operator on $L^2(\mathbb{R})$ with domain

$$D(U) = \left\{\psi(x) \in L^2(\mathbb{R})\colon e^{-\frac{\pi i p}{\omega}}\widehat{\psi}(p) \in L^2(\mathbb{R})\right\},$$

where

$$\widehat{\psi}(p) = \mathscr{F}(\psi)(p) = \int_{-\infty}^{+\infty} \psi(x) e^{-2\pi i p x}\, dx$$

is the Fourier transform[1] on $L^2(\mathbb{R})$. Equivalently, $D(U)$ consists of those functions $\psi(x)$ which admit analytic continuation to the strip $\{z = x + iy \in \mathbb{C}\colon 0 < y < 2|\omega'|\}$ such that $\psi(x + iy) \in L^2(\mathbb{R})$ for all $0 \leqslant y < 2|\omega'|$ and there is a limit

$$\psi(x + 2\omega' - i0) = \lim_{\varepsilon \to 0^+} \psi(x + 2\omega' - i\varepsilon)$$

in the sense of convergence in $L^2(\mathbb{R})$. We have $(U\psi)(x) = \psi(x + 2\omega' - i0)$ for $\psi \in D(U)$. The domain $D(U^{-1})$ of the inverse operator U^{-1} is defined in a similar way and $(U^{-1}\psi)(x) = \psi(x - 2\omega' + i0)$. The domain $D(V)$ of the self-adjoint operator V consists of functions $\psi(x) \in L^2(\mathbb{R})$ such that $e^{\frac{i\pi x}{\omega}}\psi(x) \in L^2(\mathbb{R})$. Thus we have

$$U^{-1} = \mathscr{F}^{-1} V \mathscr{F},$$

where the inverse Fourier transform is given by

$$\psi(x) = \int_{-\infty}^{+\infty} \widehat{\psi}(p) e^{2\pi i p x}\, dp.$$

[1] We use the normalization of the Fourier transform adopted in analytic number theory.

Remark 2.1. An important role in representation theory is played by the modular double of $\mathrm{SL}_q(2,\mathbb{R})$ introduced in [11]. Its principal series representations are realized in $L^2(\mathbb{R})$ and use the dual Weyl operators $\check{U}$ and $\check{V}$ along with U and V. They satisfy the relation dual to (2.2):

$$\check{U}\check{V} = \check{q}^2\check{V}\check{U}, \qquad \check{q} = e^{\pi i/\tau}$$

and are given by the formulae

$$(\check{U}\psi)(x) = \psi(x+2\omega), \qquad (\check{V}\psi)(x) = e^{\frac{\pi i x}{\omega'}}\psi(x),$$

which are obtained from (2.1) by interchanging the half-periods ω and ω'.

2.2. The modular quantum dilogarithm. We put

$$\gamma(z) = \exp\biggl\{-\frac{1}{4}\int_{-\infty}^{+\infty}\frac{e^{itz}}{\sin(\omega t)\sin(\omega' t)}\,\frac{dt}{t}\biggr\}, \tag{2.4}$$

where $|\operatorname{Im} z| < |\omega| + |\omega'|$ and the contour of integration passes above the singularity at $t = 0$. The function $\gamma(z)$ plays a fundamental role in the definition of the modular double of $\mathrm{SL}_q(2,\mathbb{R})$ given by the second author in [11]. It was later called the *modular quantum logarithm*. Here the adjective 'modular' accounts for the invariance of $\gamma(z)$ under the interchange of ω and ω', that is, of τ and $1/\tau$. The words 'quantum dilogarithm' refer to the asymptotics of $\gamma(z)$ as $\tau = b^2 \to 0$, which is easily obtained for real z from the representation (2.4):

$$\gamma\biggl(\frac{z}{b}\biggr) = \exp\biggl\{\frac{1}{2\pi\tau}\mathrm{Li}_2(-e^{-2\pi z}) + O(1)\biggr\} \quad \text{as} \quad \tau \to 0,$$

where

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^2}$$

is the Euler dilogarithm.

Remark 2.2. The function $\gamma(z)$ has an interesting history. It appears in number theory under the name of *double sine* [12], [13] and in the theory of quantum integrable systems of Calogero–Moser type under the name of *hyperbolic gamma function* [14]. It also plays the role of S-matrix in the quantum non-linear σ-model [15] and occurs in form-factors for the quantum sine-Gordon model [16]. The function $\gamma(z)$ is expressed in terms of the ratio of the second-order gamma functions, which were introduced by Barnes [17] in 1899 and earlier investigated in Alekseevskii's thesis [18] in 1889.

We shall use the following remarkable properties of the modular quantum dilogarithm (see [6], [19]).

Proposition 2.1. 1) *The function $\gamma(z)$ admits a meromorphic continuation to the whole complex z-plane with poles at $z = -(2m+1)\omega - (2n+1)\omega'$ for integer $m, n \geqslant 0$. Moreover,*

$$\gamma(z - \omega'') = \frac{c}{z} + O(1), \qquad z \to 0,$$

where $\omega'' = \omega + \omega'$ *and*

$$c = \frac{e^{i(\frac{\pi}{4}-\beta)}}{2\pi}, \qquad \beta = \frac{\pi}{12}\left(\tau + \frac{1}{\tau}\right).$$

2) *The function* $\gamma(z)$ *satisfies the difference equations*

$$\gamma(z+\omega) = (1 + e^{-\frac{\pi i z}{\omega'}})\gamma(z-\omega), \qquad \gamma(z+\omega') = (1 + e^{-\frac{\pi i z}{\omega}})\gamma(z-\omega').$$

3) *The following reflection formula holds*:

$$\gamma(z)\gamma(-z) = e^{i\beta + i\pi z^2}.$$

Hence $\gamma(z)$ *has zeros at* $z = (2m+1)\omega + (2n+1)\omega'$ *for integer* $m, n \geqslant 0$.

4) *The following reality property holds*:

$$\overline{\gamma(z)} = \frac{1}{\gamma(\bar{z})}.$$

5) *The function* $\gamma(z)$ *has the following asymptotics as* $z \to \infty$ *with* $|\arg z| < \frac{\pi}{2} - \delta$:

$$\gamma(z) = 1 + o(1)$$

uniformly in z *for every* δ, $0 < \delta < \frac{\pi}{2}$.

§ 3. The operator H_0

Here we consider the free operator $H_0 = U + U^{-1}$. Formally, it acts on the functions $\psi(x)$ on the real line by the formula

$$(H_0\psi)(x) = \psi(x + 2\omega') + \psi(x - 2\omega'), \tag{3.1}$$

where $\psi(x)$ is assumed to be analytic in the strip $|\operatorname{Im} z| \leqslant 2|\omega'|$, $z = x + iy$. Clearly, the operator $b^{-2}(H_0 - 2I)$ turns into $-\frac{d^2}{dx^2}$ as $2\omega' = ib \to 0$.

3.1. The domain. The formula (3.1) determines an unbounded self-adjoint operator H_0 on $L^2(\mathbb{R})$. Its domain $D(H_0)$ consists of those functions $\psi(x)$ which admit an analytic continuation to the strip $\{z = x + iy \in \mathbb{C}\colon |y| < 2|\omega'|\}$ such that $\psi(x + iy) \in L^2(\mathbb{R})$ for all y, $|y| < 2|\omega'|$, and the limits

$$\psi(x + 2\omega' - i0) = \lim_{\varepsilon \to 0^+} \psi(x + 2\omega' - i\varepsilon), \qquad \psi(x - 2\omega' + i0) = \lim_{\varepsilon \to 0^+} \psi(x - 2\omega' + i\varepsilon)$$

exist in the sense of convergence in $L^2(\mathbb{R})$. For $\psi \in D(H_0)$, the formula (3.1) is understood as $(H_0\psi)(x) = \psi(x + 2\omega' - i0) + \psi(x - 2\omega' + i0)$.

In the *momentum representation*, the operator $\widehat{H}_0 = \mathscr{F} H_0 \mathscr{F}^{-1}$ is the operator of multiplication by the function $2\cosh\left(\frac{\pi i p}{\omega}\right)$ and is naturally self-adjoint. Thus $D(H_0)$ can equivalently be defined as

$$D(H_0) = \left\{\psi(x) \in L^2(\mathbb{R})\colon \int_{-\infty}^{+\infty} \cosh^2\left(\frac{\pi i p}{\omega}\right)|\widehat{\psi}(p)|^2\,dp < \infty\right\}.$$

This is a *hyperbolic analogue* of the Sobolev space $W^{2,2}(\mathbb{R})$.

3.2. The resolvent of H_0. In the momentum representation, the operator

$$R_0(\lambda) = (H_0 - \lambda I)^{-1}$$

is the operator of multiplication by $(2\cosh(\frac{\pi ip}{\omega}) - \lambda)^{-1}$. It is a bounded operator on $L^2(\mathbb{R})$ for $\lambda \in \mathbb{C} \setminus [2,\infty)$. Since the function $2\cosh(\frac{\pi ip}{\omega})$ is a two-to-one map of the real axis $-\infty < p < +\infty$ onto $[2,\infty)$, the spectrum of H_0 is absolutely continuous and fills the semi-infinite interval $[2,\infty)$ with multiplicity 2.

In the *coordinate representation*, the operator $R_0(\lambda)$ for $\lambda \in \mathbb{C} \setminus [2,\infty)$ is an integral operator on $L^2(\mathbb{R})$ with kernel depending on the difference of the arguments:

$$(R_0(\lambda)\psi)(x) = \int_{-\infty}^{+\infty} R_0(x-y;\lambda)\psi(y)\,dy, \tag{3.2}$$

where

$$R_0(x;\lambda) = \int_{-\infty}^{+\infty} \frac{e^{2\pi ipx}}{2\cosh(\frac{\pi ip}{\omega}) - \lambda}\,dp. \tag{3.3}$$

In what follows we use the convenient parametrization

$$\lambda = 2\cosh\left(\frac{\pi ik}{\omega}\right), \tag{3.4}$$

mapping the resolvent set $\mathbb{C}\setminus[2,\infty)$ onto the *physical sheet* (the strip $0 < \operatorname{Im} k \leqslant |\omega|$) and covering the continuous spectrum $[2,\infty)$ twice by the real axis $-\infty < k < +\infty$. In the parametrization (3.4) we easily calculate the integral (3.3) using the residue theorem and obtain

$$R_0(x;\lambda) = \frac{\omega}{\sinh(\frac{\pi ik}{\omega})}\left(\frac{e^{-2\pi ikx}}{1-e^{-4\pi i\omega x}} + \frac{e^{2\pi ikx}}{1-e^{4\pi i\omega x}}\right). \tag{3.5}$$

Note that the function $R_0(x;\lambda)$ is regular at $x=0$. We immediately conclude from (3.5) that the following estimate holds for $0 < \operatorname{Im} k \leqslant |\omega|$:

$$|R_0(x;\lambda)| \leqslant Ce^{-2\pi \operatorname{Im} k|x|},$$

where $C > 0$ is a constant.[2] Hence the formulae (3.2) and (3.5) indeed determine a bounded operator on $L^2(\mathbb{R})$ for $\lambda \notin [2,\infty)$.

The eigenvalue equation

$$\psi(x+2\omega',k) + \psi(x-2\omega',k) = 2\cosh\left(\frac{\pi ik}{\omega}\right)\psi(x,k) \tag{3.6}$$

has solutions $f_-(x,k) = e^{-2\pi ikx}$ and $f_+(x,k) = e^{2\pi ikx}$, which are analogues of the Jost solutions in the theory of one-dimensional Schrödinger operators. In terms of the Jost solutions, (3.5) takes the form

$$R_0(x-y;\lambda) = \frac{2\omega}{C(f_-,f_+)(k)}\left(\frac{f_-(x,k)f_+(y,k)}{1-e^{\frac{\pi i}{\omega'}(x-y)}} + \frac{f_-(y,k)f_+(x,k)}{1-e^{-\frac{\pi i}{\omega'}(x-y)}}\right), \tag{3.7}$$

[2]Here and in what follows we denote different constants by C.

where

$$C(f,g)(x,k)=f(x+2\omega',k)g(x,k)-f(x,k)g(x+2\omega',k)$$

is the so-called *Casorati determinant* (a difference analogue of the Wronskian) of the solutions of the functional-difference equation (3.6). It is a $2\omega'$-periodic function of x. For the Jost solutions, $C(f_-,f_+)(x,k)=2\sinh\bigl(\frac{\pi ik}{\omega}\bigr)$.

Remark 3.1. Using formula (3.7), one can check directly that the integral operator (3.2) is inverse to $H-\lambda I$ for $\lambda\in\mathbb{C}\setminus[2,\infty)$. Indeed, for smooth compactly supported functions $g(x)$ we easily see that

$$h(x)=\int_{-\infty}^{+\infty}R_0(x-y;\lambda)g(y)\,dy\in D(H_0)$$

and $(H_0-\lambda I)h=g$. The last assertion reduces to verifying that

$$R_0(x+2\omega'-y-i0;\lambda)+R_0(x-2\omega'-y+i0;\lambda)-\lambda R_0(x-y;\lambda)=\delta(x-y) \quad (3.8)$$

in the sense of distributions. Since the functions $f_\pm(x,k)$ satisfy (3.6), the distribution on the left-hand side of (3.8) is supported only at $x=y$, and its singular part coincides with the singular part of the distribution

$$-\frac{2\omega\omega'}{\pi iC(f_-,f_+)(k)}\biggl(\frac{f_-(x+2\omega',k)f_+(y,k)-f_-(y,k)f_+(x+2\omega',k)}{x-y-i0} + \frac{f_-(x-2\omega',k)f_+(y,k)-f_-(y,k)f_+(x-2\omega',k)}{x-y+i0}\biggr)$$

in the neighbourhood of $x=y$. This singular part is equal to

$$-\frac{2\omega\omega'}{\pi i}\biggl(\frac{1}{x-y-i0}-\frac{1}{x-y+i0}\biggr)=\delta(x-y),$$

where we have used the definition of the Casorati determinant, the normalization (2.3) and the Sokhotski–Plemelj formula.

Remark 3.2. It is instructive to compare the formula (3.7) for the resolvent of H_0 with that for the one-dimensional Schrödinger operator. The latter formula (see, for example, [2], Ch. 1, § 1 and [9], Ch. 3) is

$$G(x,y;\lambda)=\frac{1}{W(k)}\bigl(f_-(x,k)f_+(y,k)\theta(y-x)+f_-(y,k)f_+(x,k)\theta(x-y)\bigr), \quad (3.9)$$

where $\lambda=k^2$, $f_-(x,k)$ and $f_+(x,k)$ are the Jost solutions normalized at $-\infty$ and $+\infty$ respectively, and $W(k)$ is their Wronskian. A key role in the process of verification of the analogue of (3.8) is played by the formula $\theta'(x)=\delta(x)$, where $\theta(x)$ is the Heaviside function, $\theta(x)=1$ when $x>0$ and $\theta(x)=0$ when $x<0$. The formula (3.7) has a remarkable similarity to (3.9) with $\theta(x)$ replaced by the smoothed analogue

$$\theta_{\omega'}(x)=\frac{1}{1-e^{-\frac{\pi ix}{\omega'}}}$$

of the Heaviside function. The analogue of the formula $\theta'(x)=\delta(x)$ is

$$\theta_{\omega'}(x+2\omega'-i0)-\theta_{\omega'}(x+2\omega'+i0)=2\omega'\delta(x),$$

which is equivalent to the Sokhotski–Plemelj formula.

Remark 3.3. Polyakov noted that if we identify x with the energy ϵ, and $\frac{\pi i}{\omega'}=\frac{2\pi}{b}$ with the inverse temperature $\frac{1}{kT}$, then the function $\theta_{\omega'}(x)$ coincides with the one-particle partition function $\mathscr{Z}=(1-e^{-\frac{\epsilon}{kT}})^{-1}$ in Bose–Einstein statistics.

§ 4. The operator *H*

The operator $H=H_0+V$ is given by the formal functional-difference expression

$$(H\psi)(x)=\psi(x+2\omega')+\psi(x-2\omega')+e^{\frac{\pi ix}{\omega}}\psi(x)$$

on $D(H_0)\cap D(V)$. In particular, H is defined and symmetric on the domain $\mathscr{D}\subset D(H_0)\cap D(V)$ consisting of linear combinations of the functions $p(x)e^{-x^2+cx}$, where $p(x)$ is a polynomial and $c\in\mathbb{C}$. The domain $\mathscr{D}$ is dense in $L^2(\mathbb{R})$ and invariant under H. We shall prove that H is essentially self-adjoint on $\mathscr{D}$, and its unique self-adjoint extension, still denoted by H, has a simple absolutely continuous spectrum filling $[2,\infty)$. As in the case of the free operator H_0, we use the parametrization (3.4) and pose the following problem for generalized eigenfunctions of H:

$$\psi(x+2\omega',k)+\psi(x-2\omega',k)+e^{\frac{\pi ix}{\omega}}\psi(x,k)=2\cosh\biggl(\frac{\pi ik}{\omega}\biggr)\psi(x,k). \tag{4.1}$$

4.1. The momentum representation and the Kashaev wave function. In the momentum representation, the eigenfunction problem for H is the first-order functional-difference equation

$$\widehat{\psi}(p+2\omega',k)=2\biggl(\cosh\biggl(\frac{\pi ik}{\omega}\biggr)-\cosh\biggl(\frac{\pi ip}{\omega}\biggr)\biggr)\widehat{\psi}(p,k),\qquad p\in\mathbb{R}, \tag{4.2}$$

where $\widehat{\psi}=\mathscr{F}(\psi)$. Remarkably, the general solution of (4.2) (up to multiplication by a quasi-constant) is explicitly expressible in terms of the modular quantum dilogarithm:

$$\widehat{\psi}(p,k)=c(k)e^{-\pi i(p-\omega'')^2}\gamma(p+k-\omega'')\gamma(p-k-\omega''),\qquad 0\leqslant\operatorname{Im}k\leqslant|\omega|, \tag{4.3}$$

where $c(k)$ is a constant to be chosen later. For real k, the product of γ-functions is singular at $p=\pm k$ and is understood as the distribution $\gamma(p+k-\omega''+i0)\times\gamma(p-k-\omega''+i0)$.

The fundamental role of the generalized solution (4.3) of (4.2) was revealed in [6]. We call this solution the *Kashaev wave function*. The distribution $\widehat{\psi}(p,k)$ has the following asymptotics:

$$\widehat{\psi}(p,k)=\begin{cases}c(k)e^{-\pi i(p-\omega'')^2}(1+o(1)) & \text{as}\quad p\to+\infty,\\ c(k)e^{\pi i(p-\omega'')^2+2i\beta+2\pi ik^2}(1+o(1)) & \text{as}\quad p\to-\infty,\end{cases} \tag{4.4}$$

and decays exponentially for large p :

$$|\widehat{\psi}(p,k)| = |c(k)| \exp\{-2\pi|p|\,|\omega''|\}(1+o(1)) \quad \text{as} \quad |p| \to \infty. \tag{4.5}$$

Putting

$$c(k) = e^{-i\beta - \pi i k^2} \tag{4.6}$$

in (4.3) and denoting the corresponding solution by $\widehat{\varphi}(x,k)$, we obtain an important property,

$$\overline{\widehat{\varphi}(p,k)} = \widehat{\varphi}(-p,-\bar{k}). \tag{4.7}$$

Moreover, the definition of $\widehat{\varphi}(p,k)$ shows that $\widehat{\varphi}(p,-k) = \widehat{\varphi}(p,k)$ for real k.

4.2. The scattering solution. For real x and k we put

$$\varphi(x,k) = \int_{-\infty}^{+\infty} \widehat{\varphi}(p,k) e^{2\pi i p x}\, dp, \tag{4.8}$$

where $\widehat{\varphi}(p,k)$ is given by (4.3) and (4.6), and the contour of integration passes above the singularities at $p = \pm k$. Comparing (4.8) with formula (6.3) in the appendix, we see that $\varphi(x,k)$ plays the role of a q-deformed modified Bessel function $K_{2\pi i k}(e^x)$. It follows from the asymptotic formulae (4.4) that the Kashaev wave function decays exponentially as $|\operatorname{Re} p| \to \infty$ along the lines $\operatorname{Im} p = \sigma < |\omega''|$. Therefore,

$$\varphi(x,k) = \int_{-\infty+i\sigma}^{+\infty+i\sigma} \widehat{\varphi}(p,k) e^{2\pi i p x}\, dp. \tag{4.9}$$

The formula (4.9) determines the function $\varphi(x,k)$ for real x and k in the physical strip $0 < \operatorname{Im} k \leqslant |\omega|$, where $|\omega| < \sigma < |\omega''|$.

The analytic properties of the function $\varphi(x,k)$ are described in the following lemma.

Lemma 4.1. (i) *The function $\varphi(x,k)$ has the following asymptotic behaviour for real x and k:*

$$\varphi(x,k) = M(k) e^{2\pi i k x} + M(-k) e^{-2\pi i k x} + o(1) \quad \textit{as} \quad x \to -\infty,$$

where

$$M(k) = e^{i(\beta + \frac{\pi}{4}) - 2\pi i k (k - \omega'')} \gamma(2k - \omega'').$$

We have $\overline{M(k)} = M(-k)$ and

$$\frac{1}{|M(k)|^2} = 4 \sinh\left(\frac{\pi i k}{\omega}\right) \sinh\left(\frac{\pi i k}{\omega'}\right).$$

(ii) *For real x, the function $\varphi(x,k)$ admits analytic continuation to the physical strip $0 < \operatorname{Im} k \leqslant |\omega|$ and satisfies the reality condition*

$$\overline{\varphi(x,k)} = \varphi(x,-\bar{k}).$$

For real x and k, $\varphi(x,k)$ is an even real-valued function of k.

(iii) *For every fixed k in the physical strip, the function $\varphi(x,k)$ extends to an entire function of the complex variable x and satisfies the equation*

$$\varphi(x+2\omega',k)+\varphi(x-2\omega',k)+e^{\frac{\pi i x}{\omega}}\varphi(x,k)=2\cosh\left(\frac{\pi i k}{\omega}\right)\varphi(x,k).$$

(iv) *The following estimates hold*:

$$|\varphi(x,k)|\leqslant Ce^{-2\pi\kappa x}$$

uniformly in $-\infty<x\leqslant a$, where $0\leqslant\kappa=\operatorname{Im}k\leqslant|\omega|$, and

$$|\varphi(x,k)|\leqslant Ce^{-2\pi(|\omega|+|\omega'|)x},\qquad |\varphi(x\pm 2\omega',k)|\leqslant Ce^{2\pi(|\omega'|-|\omega|)x}$$

uniformly in $a\leqslant x<\infty$.

Proof. Shifting the contour of integration in (4.8) to the lower half-plane for negative x and passing through the poles of the integrand at $p=-k$ and $p=k$, we get the first formula in (i). The formulae for the coefficient $M(k)$ are obtained from parts 1)–4) of Proposition 2.1. Part (ii) follows directly from Proposition 2.1 and property (4.7). In particular, $\overline{M(k)}=M(-\bar{k})$.

To prove (iii), we deform the contour of integration in (4.9) to a contour L by replacing the semi-infinite intervals $-\infty<\operatorname{Re}p\leqslant-|\operatorname{Re}k|-1$ and $|\operatorname{Re}k|+1\leqslant\operatorname{Re}p<\infty$ on the line $\operatorname{Im}p=\sigma$ by the rays $p=-|\operatorname{Re}k|-1+i\sigma+e^{\frac{\pi i}{4}}t$, $-\infty<t\leqslant 0$, and $p=|\operatorname{Re}k|+1+i\sigma+e^{-\frac{\pi i}{4}}t$, $0\leqslant t<\infty$ (Fig. 1 shows the contour L for real k).

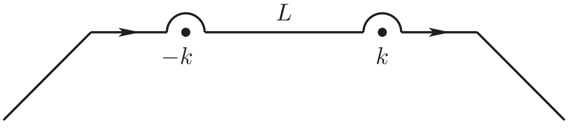

Figure 1

Thus,

$$\varphi(x,k)=\int_L\widehat{\varphi}(p,k)e^{2\pi i p x}\,dp, \tag{4.10}$$

and it follows from Proposition 2.1 that the integrand in (4.10) decays along L at the rate of $e^{-\pi t^2}$ as $t\to\pm\infty$. Hence the formula (4.10) determines $\varphi(x,k)$ as an entire function of x. The difference equation for $\varphi(x,k)$ is obtained from (4.2) using the Fourier transform.

Finally, (iv) follows in the standard way from the integral representation (4.10) using the asymptotic behaviour of $\gamma(z)$ in Proposition 2.1 and the method of steepest descent. □

Remark 4.1. The function $\varphi(x,k)$ is invariant under the interchange of ω and ω' and satisfies the dual equation $\check{H}\varphi=\check{\lambda}\varphi$, where $\check{H}=\check{U}+\check{U}^{-1}+\check{V}$ (see Remark 2.1) and $\check{\lambda}=2\cosh\left(\frac{\pi i k}{\omega'}\right)$.

4.3. Jost solutions. Since equation (4.1) takes the free form (3.6) as $x \to -\infty$, it is natural to assume that it has Jost solutions, that is, solutions $f_\pm(x,k)$ with the asymptotic behaviour

$$f_\pm(x,k) = e^{\pm 2\pi i k x} + o(1) \quad \text{as} \quad x \to -\infty. \tag{4.11}$$

Here we prove the existence of such solutions. We first compare equation (4.1) with equation (1.2). Comparison of the formulae (6.3) and (6.4) in the appendix with formula (4.10) suggests considering the solution of (4.2) obtained by multiplying $\widehat{\varphi}(p,k)$ by the quasi-constant $\sinh\bigl(\frac{\pi i p}{\omega'}\bigr) + \sinh\bigl(\frac{\pi i k}{\omega'}\bigr)$. Thus, for real x and k we put

$$f(x,k) = \frac{1}{2\sinh\bigl(\frac{\pi i k}{\omega'}\bigr)M(k)} \int_L \widehat{\varphi}(p,k)\biggl(\sinh\biggl(\frac{\pi i p}{\omega'}\biggr) + \sinh\biggl(\frac{\pi i k}{\omega'}\biggr)\biggr)e^{2\pi i p x}\,dp. \tag{4.12}$$

The following lemma shows that the functions $f_+(x,k) = f(x,k)$ and $f_-(x,k) = f(x,-k)$ indeed play the role of Jost solutions of (4.1).

Lemma 4.2. (i) *For real x and k, the functions $f_\pm(x,k)$ have the following asymptotics as $x \to -\infty$:*

$$f_\pm(x,k) = e^{\pm 2\pi i k x} + o(1).$$

(ii) *For real x, the functions $f_\pm(x,k)$ admit analytic continuation to the physical strip $0 < \operatorname{Im} k \leqslant |\omega|$ and satisfy*

$$\overline{f_\pm(x,k)} = f_\pm(x,-\bar{k}).$$

(iii) *For every fixed k in the physical strip, the functions $f_\pm(x,k)$ are entire functions of x and satisfy* (4.1). *The asymptotic formulae in part* (i) *remain valid in the strip $0 \leqslant \operatorname{Im} x \leqslant 2|\omega'|$.*

(iv) *We have*

$$\varphi(x,k) = M(k)f_+(x,k) + M(-k)f_-(x,k).$$

(v) *The following estimates hold*:

$$|f_\pm(x,k)| \leqslant Ce^{\mp 2\pi\kappa x}$$

uniformly in $-\infty < x \leqslant a$, where $0 \leqslant \kappa = \operatorname{Im} k \leqslant |\omega|$, and

$$|f_\pm(x,k)| \leqslant Ce^{2\pi(|\omega|-|\omega'|)x}, \qquad |f_\pm(x+2\omega',k)| \leqslant Ce^{2\pi(|\omega|+|\omega'|)x}$$

uniformly in $a \leqslant x < \infty$.

Proof. To prove (i), it suffices to shift the contour of integration in (4.12) to the lower half-plane for negative x and use the regularity of the integrand at $p = -k$ (because of multiplication by the quasi-constant $\sinh\bigl(\frac{\pi i p}{\omega'}\bigr) + \sinh\bigl(\frac{\pi i k}{\omega'}\bigr)$). Parts (ii)–(iv) follow immediately from the representation (4.12) written in the form

$$f(x,k) = \frac{1}{4\sinh\bigl(\frac{\pi i k}{\omega'}\bigr)M(k)}\biggl(\varphi(x-2\omega,k) - \varphi(x+2\omega,k) + 2\sinh\biggl(\frac{\pi i k}{\omega'}\biggr)\varphi(x,k)\biggr) \tag{4.13}$$

and similar properties of the function $\varphi(x,k)$ in Lemma 4.1. Since the potential $e^{\frac{\pi i x}{\omega}}$ in (4.1) has period 2ω, the functions $\varphi(x \pm 2\omega, k)$ also satisfy (4.1). The proof of (v) is standard and uses the integral representation (4.12). □

Remark 4.2. The formula (4.13) is a difference analogue of (6.7). The function $f(x,k)$ is not invariant under the interchange ω and ω' and, therefore, does not satisfy the dual eigenvalue equation (compare with Remark 4.1).

Remark 4.3. In the case when $\operatorname{Im}\tau>0$, the Jost solutions $f_\pm(x,k)$ can be defined using power series in $e^{\frac{\pi i x}{\omega}}$. These series converge absolutely for all $x\in\mathbb{R}$. In our case when $\tau=b^2>0$, we encounter the problem of small denominators and the corresponding series are no longer absolutely convergent for all x. This is why we are using the integral representation (4.12).

4.4. The Casorati determinant. As in § 3.2, a direct verification shows that the Casorati determinant

$$C(f,g)(x,k)=f(x+2\omega',k)g(x,k)-f(x,k)g(x+2\omega',k)$$

of two solutions of (4.1) is a $2\omega'$-periodic function of x. The Casorati determinant need not in general be a constant, unlike its continuous analogue (the Wronskian). Nevertheless, the following assertion holds.

Lemma 4.3. *We have*

$$C(f_-,f_+)(x,k)=2\sinh\biggl(\frac{\pi i k}{\omega}\biggr).$$

Proof. Put $F(x)=C(f_-,f_+)(x,k)$. It follows from parts (i) and (iii) of Lemma 4.2 that the following asymptotics as $x\to-\infty$ holds in the strip $0\leqslant\operatorname{Im}x\leqslant 2|\omega'|$:

$$F(x)=2\sinh\biggl(\frac{\pi i k}{\omega}\biggr)+o(1).$$

When $x\to\infty$, using the formula

$$F(x)=2\sinh\biggl(\frac{\pi i k}{\omega'}\biggr)C(f_-,\varphi)(x,k)$$

and the bounds in Lemma 4.1(iv) and Lemma 4.2(v), we obtain that the function $F(x)$ is bounded on the lines $\operatorname{Im}x=0$ and $\operatorname{Im}x=2|\omega'|$. Furthermore, it follows from the integral representation (4.12) that $F(x)$ has at most exponential growth as $x\to\infty$. Using the Phragmén–Lindelöf theorem, we conclude that the $2\omega'$-periodic function $F(x)$ is bounded in the strip $0\leqslant\operatorname{Im}x\leqslant 2|\omega'|$. Therefore, $F(x)=2\sinh\bigl(\frac{\pi i k}{\omega}\bigr)$. □

§ 5. The eigenfunction expansion theorem

5.1. The resolvent of *H*. Consider an integral operator $R(\lambda)$ on $L^2(\mathbb{R})$ with integral kernel

$$R(x,y;\lambda)=\frac{\omega}{\sinh\bigl(\frac{\pi i k}{\omega}\bigr)M(k)}\biggl(\frac{f_-(x,k)\varphi(y,k)}{1-e^{\frac{\pi i}{\omega'}(x-y)}}+\frac{f_-(y,k)\varphi(x,k)}{1-e^{-\frac{\pi i}{\omega'}(x-y)}}\biggr), \tag{5.1}$$

so that

$$R(y,x;\lambda)=R(x,y;\lambda),\qquad \overline{R(x,y;\lambda)}=R(x,y;\bar\lambda).$$

Proposition 5.1. *The operator* $R(\lambda)$, $\lambda \in \mathbb{C} \setminus [2, \infty)$, *is the resolvent of* H. *In other words,* $R(\lambda) = (H - \lambda I)^{-1}$.

Proof. As in the case of H_0 (see Remark 3.1), we claim that for every smooth compactly supported function $g(x)$ we have

$$h(x) = \int_{-\infty}^{+\infty} R(x, y; \lambda) g(y)\, dy \in D(H)$$

and $(H - \lambda I)h = g$. Indeed, it suffices to verify that

$$R(x + 2\omega' - i0, y; \lambda) + R(x - 2\omega' + i0, y; \lambda) + (e^{\frac{\pi i x}{\omega}} - \lambda) R(x, y; \lambda) = \delta(x - y) \quad (5.2)$$

in the sense of distributions. As in Remark 3.1, since $\varphi(x, k)$ and $f_-(x, k)$ satisfy equation (4.1), we see that the distribution on the left-hand side of (5.2) is supported only at $x = y$, and its singular part coincides with the singular part of the function

$$-\frac{2\omega\omega'}{2\pi \sinh\left(\frac{\pi i k}{\omega}\right) M(k)} \left(\frac{f_-(x + 2\omega', k)\varphi(y, k) - f_-(y, k)\varphi(x + 2\omega', k)}{x - y - i0} \right.$$
$$\left. + \frac{f_-(x - 2\omega', k)\varphi(y, k) - f_-(y, k)\varphi(x - 2\omega', k)}{x - y + i0} \right)$$

in a neighbourhood of $x = y$. Arguing as in Remark 3.1, we deduce (5.2) from the formula

$$C(f_-, \varphi)(x, k) = 2 \sinh\left(\frac{\pi i k}{\omega}\right) M(k),$$

which in turn follows from Lemma 4.3 and the Sokhotski–Plemelj formula.

It remains to show that the kernel (5.1) determines a bounded operator on $L^2(\mathbb{R})$ for $\lambda \in \mathbb{C} \setminus [2, \infty)$. This follows immediately from the bound

$$|R(x, y; \lambda)| \leqslant C e^{-2\pi\kappa|x-y|}, \qquad \kappa = \operatorname{Im} k,$$

which is a consequence of the bounds in Lemmas 4.1, 4.2. Indeed, since $R(x, y; \lambda) = R(y, x; \lambda)$, we may assume that $y \leqslant x$. Rewrite (5.1) in the form

$$R(x, y; \lambda) = \frac{\omega\left(f_-(x, k)\varphi(y, k) e^{2\pi i \omega(x-y)} - f_-(y, k)\varphi(x, k) e^{-2\pi i \omega(x-y)}\right)}{2 \sinh(2\pi i \omega(x - y)) \sinh\left(\frac{\pi i k}{\omega}\right) M(k)}$$

and consider first the case when $0 \leqslant y \leqslant x$. Using Lemma 4.1(iv) and Lemma 4.2(v), we get

$$|R(x, y; \lambda)| \leqslant C e^{-2\pi|\omega|(x-y)} \big(e^{2\pi(|\omega| - |\omega'|)x} e^{-2\pi(|\omega| + |\omega'|)y} e^{-2\pi|\omega|(x-y)}$$
$$+ e^{-2\pi(|\omega| + |\omega'|)x} e^{-2\pi\kappa y} e^{2\pi|\omega|(x-y)}\big) \leqslant 2C e^{-2\pi|\omega|(x-y)}.$$

In the case $y < 0 \leqslant x$ we have

$$|R(x, y; \lambda)| \leqslant C e^{-2\pi|\omega|(x-y)} \big(e^{2\pi(|\omega| - |\omega'|)x} e^{-2\pi\kappa y} e^{-2\pi|\omega|(x-y)}$$
$$+ e^{-2\pi(|\omega| + |\omega'|)x} e^{2\pi\kappa y} e^{2\pi|\omega|(x-y)}\big)$$
$$\leqslant C\big(e^{2\pi(|\omega| - \kappa)y} e^{-2\pi|\omega|(x-y)} + e^{-2\pi|\omega|x} e^{2\pi\kappa y}\big) \leqslant 2C e^{-2\pi\kappa(x-y)}.$$

Finally, in the remaining case $y \leqslant x < 0$ we have

$$|R(x,y;\lambda)| \leqslant Ce^{-2\pi|\omega|(x-y)}\big(e^{2\pi\kappa x}e^{-2\pi\kappa y}e^{-2\pi|\omega|(x-y)} + e^{-2\pi\kappa x}e^{2\pi\kappa y}e^{2\pi|\omega|(x-y)}\big) \leqslant 2Ce^{-2\pi\kappa(x-y)}. \qquad \square$$

Remark 5.1. Formula (5.1) can also be used as the definition of the operator H.

5.2. The eigenfunction expansion. The explicit formula (5.1) for the resolvent $R(\lambda)$ leads immediately to the eigenfunction expansion theorem for H. Namely, let $E(\Delta)$ be the resolution of the identity for the self-adjoint operator H, where Δ ranges over all Borel subsets of $\mathbb{R}$ (see [21], [22]). When there is no point spectrum, we have the formula

$$E(\Delta) = \lim_{\varepsilon\to 0^+}\frac{1}{2\pi i}\int_\Delta (R(\lambda+i\varepsilon) - R(\lambda-i\varepsilon))\,d\lambda$$

(see [22], Ch. XII), which is sometimes referred to as *Stone's formula.* In particular, putting $\Delta = [2,\infty)$, we get the formula

$$I = \lim_{\varepsilon\to 0^+}\frac{1}{2\pi i}\int_2^{+\infty} (R(\lambda+i\varepsilon) - R(\lambda-i\varepsilon))\,d\lambda \tag{5.3}$$

for the operator H. It is this formula that serves as a basis for the derivation of the eigenfunction expansion theorem.

Theorem 5.1. (i) *Define an operator* $\mathscr{U}$ *by the formula*

$$(\mathscr{U}\psi)(k) = \int_{-\infty}^{+\infty}\psi(x)\varphi(x,k)\,dx, \qquad \psi(x)\in L^2(\mathbb{R}).$$

Then $\mathscr{U}$ *maps* $L^2(\mathbb{R})$ *isometrically onto the Hilbert space* $\mathscr{H}_0 = L^2([0,\infty),\rho(k)dk)$ *with the spectral function*

$$\rho(k) = \frac{1}{|M(k)|^2} = 4\sinh\Big(\frac{\pi i k}{\omega}\Big)\sinh\Big(\frac{\pi i k}{\omega'}\Big).$$

In other words, $\mathscr{U}\colon L^2(\mathbb{R})\to\mathscr{H}_0$ *and we have* $\mathscr{U}^*\mathscr{U} = I$, $\mathscr{U}\mathscr{U}^* = I_0$, *where* I_0 *is the identity operator on* $\mathscr{H}_0$.

(ii) *The operator* $\mathscr{U}H\mathscr{U}^{-1}$ *is the operator of multiplication by the function* $2\cosh\big(\frac{\pi i k}{\omega}\big)$ *on* $\mathscr{H}_0$. *Hence* H *has a simple absolutely continuous spectrum filling* $[2,\infty)$.

Proof. We claim that the following identity holds for all functions $\psi(x)\in\mathscr{D}$:

$$\psi(x) = \int_0^{+\infty}\Big(\int_{-\infty}^{+\infty}\psi(y)\varphi(y,k)\,dy\Big)\varphi(x,k)\rho(k)\,dk. \tag{5.4}$$

Indeed, using the equation (4.1) for $\varphi(x,k)$, we see that $(\mathscr{U}\psi)(k)$ decays faster than any power of $e^{-\frac{\pi i k}{\omega}}$ as $k\to\infty$. Hence all the integrals are absolutely convergent. One can prove (5.4) either by the method of complex integration as in [1], § 2 (see also [9], Ch. 3) or by using (5.3), which is what we do here. Namely, we apply (5.3) to the function $\psi(x)\in\mathscr{D}$. Explicitly computing the jump of the resolvent kernel $R(x,y;\lambda)$ across the absolutely continuous spectrum and using Lemma 4.2(iv), we get

$$R(x,y;\lambda+i0)-R(x,y;\lambda-i0)=\frac{\omega}{\sinh\left(\frac{\pi ik}{\omega}\right)}\frac{\varphi(x,k)\varphi(y,k)}{|M(k)|^2}$$
$$=\frac{\omega}{\sinh\left(\frac{\pi ik}{\omega}\right)}\varphi(x,k)\varphi(y,k)\rho(k).$$

Here we take into account that the case $\lambda+i0$ corresponds to the variable $k>0$, and the case $\lambda-i0$ corresponds to the variable $-k$. Using $d\lambda=\frac{2\pi i}{\omega}\sinh\left(\frac{\pi ik}{\omega}\right)dk$, we arrive at (5.4). Multiplying (5.4) by $\overline{\psi(x)}$ and integrating, we obtain

$$\|\psi\|^2_{L^2(\mathbb{R})}=\|\mathscr{U}\psi\|^2_{\mathscr{H}_0}$$

(the change of order of integration is legitimate by Fubini's theorem). Hence the operator $\mathscr{U}$, which is defined on the dense vector subspace $\mathscr{D}\subset L^2(\mathbb{R})$, maps $\mathscr{D}$ to the Hilbert space $\mathscr{H}_0$ and is an isometry. Therefore $\mathscr{U}$ admits an isometric extension to the whole of $L^2(\mathbb{R})$. This proves the *completeness relation*

$$\mathscr{U}^*\mathscr{U}=I.$$

The *orthogonality relation*

$$\mathscr{U}\mathscr{U}^*=I_0$$

is equivalent to saying that the image of $\mathscr{U}$ in $\mathscr{H}_0$ (the closed subspace $\operatorname{Im}\mathscr{U}$) coincides with $\mathscr{H}_0$. This is proved in the standard way (see, for example, [9], Ch. 3). Namely, we have $\mathscr{U}(H-\lambda I)=(\widehat{H}-\lambda I)\mathscr{U}$ on the domain $\mathscr{D}$, where $\widehat{H}$ is the operator of multiplication by $2\cosh\left(\frac{\pi ik}{\omega}\right)$ on $\mathscr{H}_0$. Hence we get

$$\mathscr{U}R(\lambda)=\widehat{R}(\lambda)\mathscr{U},$$

where $\widehat{R}(\lambda)$ is the resolvent of $\widehat{H}$. Thus $\operatorname{Im}\mathscr{U}$ is an invariant subspace for $\widehat{R}(\lambda)$ for all $\lambda\in\mathbb{C}\setminus[2,\infty)$. It follows that $\widehat{R}(\lambda)$ commutes with the orthogonal projector P onto the subspace $\operatorname{Im}\mathscr{U}$. Hence P is a function of $\widehat{H}$. Since $\widehat{H}$ is in turn a function of the operator of multiplication by k on $\mathscr{H}_0$, we obtain that P is the operator of multiplication by the characteristic function χ_Δ of some Borel subset Δ in $[0,\infty)$. On the other hand, if for some $k>0$ we have

$$\int_{-\infty}^{+\infty}\psi(x)\varphi(x,k)\,dx=0$$

for all $\psi(x)\in C_0(\mathbb{R})$, then $\varphi(x,k)=0$ for all x, whence necessarily $\Delta=[0,\infty)$. This completes the proof of (i). Part (ii) follows from the arguments above. □

Remark 5.2. The resolvent kernel $\check{R}(\check{\lambda})$ for the dual operator $\check{H}$ (see Remark 4.1) is obtained from (5.1) by interchanging ω and ω'. Therefore, repeating the proof of Theorem 5.1, we arrive at the same operator $\mathscr{U}$. Hence $\mathscr{U}$ also diagonalizes $\check{H}$, and $\mathscr{U}\check{H}\mathscr{U}^{-1}$ is the operator of multiplication by the function $\check{\lambda}=2\cosh\left(\frac{\pi ik}{\omega'}\right)$ on $\mathscr{H}_0$. As a result, we obtain that H and $\check{H}$ commute as self-adjoint operators on $L^2(\mathbb{R})$.

Remark 5.3. In the physics literature the completeness and orthogonality relations, understood in the sense of distributions, are written as follows:

$$\int_0^{+\infty} \varphi(x,k)\varphi(y,k)\rho(k)\,dk = \delta(x-y),$$
$$\int_{-\infty}^{+\infty} \varphi(x,k)\varphi(x,l)\,dx = \frac{1}{\rho(k)}\delta(k-l), \qquad k,l>0.$$

As in the case of one-dimensional Schrödinger operators (see, for example, [9], Ch. 3), the last relation can be proved directly using the Casorati determinant. Namely, we put $\Phi(x) = C(\varphi(x,k),\varphi(x,l))$ and integrate this function over the contour D shown in Fig. 2.

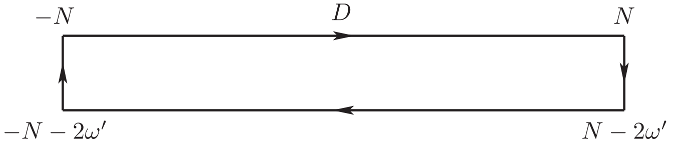

Figure 2

By Cauchy's theorem,

$$\int_D \Phi(x)\,dx = 0.$$

On the other hand, using the formula

$$\Phi(x) - \Phi(x-2\omega') = (\lambda-\mu)\varphi(x,k)\varphi(x,l),$$

where $\lambda = 2\cosh\bigl(\frac{\pi i k}{\omega}\bigr)$ and $\mu = 2\cosh\bigl(\frac{\pi i l}{\omega}\bigr)$, we get

$$\int_{-N}^{N} \varphi(x,k)\varphi(x,l)\,dx = \frac{1}{\lambda-\mu}\biggl(\int_{N-2\omega'}^{N} \Phi(x)\,dx - \int_{-N-2\omega'}^{-N} \Phi(x)\,dx\biggr).$$

By Lemma 4.1(iv), the first integral decays exponentially as $N\to\infty$. Using parts (i) and (iv) of Lemma 4.2 along with the well-known formula

$$\lim_{N\to\infty} \frac{\sin(2\pi(k-l)N)}{k-l} = \pi\delta(k-l)$$

and the Riemann–Lebesgue lemma, we obtain

$$\lim_{N\to\infty} \frac{1}{\mu-\lambda}\int_{-N-2\omega'}^{-N} \Phi(x)\,dx = \frac{1}{\rho(k)}\delta(k-l).$$

Remark 5.4. Since $W(I_\nu, K_\nu) = -1$ (see formula (6.6) in the appendix), the resolvent kernel $\widetilde{R}(\lambda)$ of the operator $\widetilde{H} = -\frac{d^2}{dx^2} + e^{2x}$ is of the form

$$\widetilde{R}(x,y;\lambda) = \frac{1}{2ik\widetilde{M}(k)}\bigl(\widetilde{f}_-(x,k)\widetilde{\varphi}(y,k)\theta(y-x) + \widetilde{f}_-(y,k)\widetilde{\varphi}(x,k)\theta(x-y)\bigr),$$

where $\operatorname{Im} k > 0$ (see Remark 3.2). As in Theorem 5.1, using (6.8) and defining an operator $\widetilde{\mathscr{U}}$ by the formula

$$(\widetilde{\mathscr{U}}\psi)(k) = \int_{-\infty}^{+\infty} \psi(x)\widetilde{\varphi}(x,k)\,dx,$$

we see that $\widetilde{\mathscr{U}}$ maps $L^2(\mathbb{R})$ isometrically onto $\widetilde{\mathscr{H}_0} = L^2([0,\infty), \widetilde{\rho}(k)\,dk)$, where

$$\widetilde{\rho}(k) = \frac{1}{2\pi|\widetilde{M}(k)|^2} = \frac{2k\sinh(\pi k)}{\pi^2}$$

by (6.9). The operator $\widetilde{\mathscr{U}}\widetilde{H}\widetilde{\mathscr{U}}^{-1}$ is the operator of multiplication by k^2 on $\widetilde{\mathscr{H}_0}$. After the change of variables $x = \ln y$, the formulae

$$\widetilde{\psi}(k) = \int_{-\infty}^{+\infty} \psi(x)K_{ik}(e^x)\,dx, \qquad \psi(x) = \frac{2}{\pi^2}\int_0^{+\infty} \widetilde{\psi}(k)K_{ik}(e^x)k\sinh(\pi k)\,dk$$

are known in the theory of special functions as the Kontorovich–Lebedev transform and its inverse (see [23], Ch. XII), and the equality

$$\int_{-\infty}^{+\infty} |\psi(x)|^2\,dx = \frac{2}{\pi^2}\int_0^{+\infty} |\widetilde{\psi}(k)|^2 k\sinh(\pi k)\,dk$$

is known as Parseval's theorem. The eigenfunction expansion for $\widetilde{H}$ gives a spectral interpretation of the Kontorovich–Lebedev transform. Hence Theorem 5.1 may be regarded as a q-analogue of this transform.

5.3. The scattering theory. Here we briefly outline the scattering theory for the operator H. Put

$$\varphi^{(+)}(x,k) = \frac{1}{M(k)}\varphi(x,k).$$

We have

$$\varphi^{(+)}(x,k) = e^{2\pi ikx} + S(k)e^{-2\pi ikx} + o(1)$$

as $x \to -\infty$, where

$$S(k) = \frac{M(-k)}{M(k)} = e^{-4\pi i\omega'' k}\frac{\gamma(-2k-\omega'')}{\gamma(2k-\omega'')}.$$

According to the stationary scattering theory (see [1], [2]), the operator of multiplication by $S(k)$ plays the role of the scattering operator on $\mathscr{H}_0$ as well as on $L^2([0,\infty))$. Defining an operator $\mathscr{U}^{(+)}$ by the formula

$$(\mathscr{U}^{(+)}\psi)(k) = \int_{-\infty}^{+\infty} \psi(x)\varphi^{(+)}(x,k)\,dx,$$

we see that $\mathscr{U}^{(+)}$ maps $L^2(\mathbb{R})$ isometrically onto $L^2([0,\infty))$. As in [8], [24], it is convenient to interpret the latter space as the subspace of $L^2(\mathbb{R})$ consisting of the functions $\chi(k)$ such that

$$\chi(k) = S(k)\chi(-k).$$

We similarly define an operator $\mathscr{U}^{(-)}$ using the solution $\varphi^{(-)}(x,k) = \overline{\varphi^{(+)}(x,k)}$. The operators $\mathscr{U}^{(\pm)}$ are used in the non-stationary approach to the scattering theory (see [1], [2]), but we shall not dwell on this.

Remark 5.5. One can similarly construct the scattering theory for $\widetilde{H}$. By (6.9), the scattering operator $\widetilde{S}$ satisfies

$$\widetilde{S}(k) = \frac{\widetilde{M}(-k)}{\widetilde{M}(k)} = -2^{2ik}\frac{\Gamma(1+ik)}{\Gamma(1-ik)}$$

(compare with the formula (5.19) in [10]).

§ 6. Appendix

Here we briefly present known properties of the solutions of (1.2). In the momentum representation, equation (1.2) takes the form

$$\widehat{\widetilde{\psi}}\left(p+\frac{i}{\pi},k\right) = 4\pi^2(k^2-p^2)\widehat{\widetilde{\psi}}(p,k), \tag{6.1}$$

where we put $\lambda = (2\pi k)^2$. A solution of (6.1) is given by the product of Euler gamma-functions

$$\widehat{\widetilde{\psi}}(p,k) = 2^{-2\pi ip-2}\Gamma(\pi i(p+k))\Gamma(\pi i(p-k)), \tag{6.2}$$

and the general solution is obtained by multiplying by a quasi-constant (a periodic function of p with period i/π). Performing the inverse Fourier transform and putting $s=-2\pi ip$, we get the Mellin–Barnes representation for the modified Bessel function of the second kind:

$$K_\nu(e^x) = \frac{1}{8\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{e^x}{2}\right)^{-s}\Gamma\left(\frac{s-\nu}{2}\right)\Gamma\left(\frac{s+\nu}{2}\right)ds, \tag{6.3}$$

where $\nu = 2\pi ik$ and $\sigma = \operatorname{Re}s > |\operatorname{Re}\nu|$ (see [20], Ch. 7, formula (27) for the Mellin transform of $K_\nu(z)$). When $\operatorname{Re}\nu = 0$, the integration is performed over the imaginary axis $\sigma = 0$ bypassing the poles at $s = \pm\nu$ in the half-plane $\operatorname{Re}s > 0$. The function $K_\nu(e^x)$ is an entire function of x.

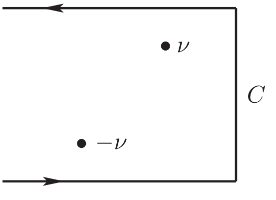

Figure 3

The modified Bessel functions of the first kind $I_\nu(e^x)$ and $I_{-\nu}(e^x)$ are also solutions of (1.2), where $\lambda = -\nu^2$. We have

$$W(I_{-\nu},I_\nu)(x) = I_{-\nu}(x)I'_\nu(x) - I'_{-\nu}(x)I_\nu(x) = \frac{2\sin(\pi\nu)}{\pi}.$$

The function $I_\nu(e^x)$ is obtained by multiplying the solution (6.2) of equation (6.1) by the quasi-constant $\frac{e^{-\pi i\nu}-e^{-\pi is}}{\pi i}$, where $s=-2\pi ip$, and replacing the contour of integration in (6.3) by the contour C shown in Fig. 3. As a result, we get the integral representation

$$I_\nu(e^x) = -\frac{1}{8\pi^2}\int_C \left(\frac{e^x}{2}\right)^{-s}\Gamma\left(\frac{s-\nu}{2}\right)\Gamma\left(\frac{s+\nu}{2}\right)(e^{-\pi i\nu} - e^{-\pi i s})\,ds. \tag{6.4}$$

Shifting the contour of integration C to the left to $-\infty$, we obtain the standard representation of $I_\nu(e^x)$ as a power series in e^x. The factor $e^{-\pi i\nu} - e^{-\pi i s}$ guarantees that there are no poles at $s = \nu + 2 - 2n$, $n \in \mathbb{N}$. The function $I_\nu(e^x)$ is an entire function of x, and $I_\nu(e^{x+\pi i}) = e^{\pi i\nu} I_\nu(e^x)$. By (6.3) and (6.4),

$$K_\nu(e^x) = \frac{\pi}{2\sin(\pi\nu)}\bigl(I_{-\nu}(e^x) - I_\nu(e^x)\bigr), \tag{6.5}$$

whence

$$W(I_\nu, K_\nu) = -1. \tag{6.6}$$

We also have

$$K_\nu(e^{x+\pi i}) = \frac{\pi}{2\sin(\pi\nu)}\bigl(e^{-\pi i\nu} I_{-\nu}(e^x) - e^{\pi i\nu} I_\nu(e^x)\bigr),$$

whence

$$I_\nu(e^x) = \frac{1}{\pi i}\bigl(e^{-\pi i\nu} K_\nu(e^x) - K_\nu(e^{x+\pi i})\bigr). \tag{6.7}$$

The function $I_\nu(e^x)$ has the following asymptotics as $x \to -\infty$:

$$I_{ik}(e^x) = \frac{2^{-ik}}{\Gamma(1+ik)}(e^{ikx} + o(1)).$$

Moreover, $I_\nu(e^x)$ grows like a double exponent as $x \to \infty$, whereas $K_\nu(e^x) = O(e^{-e^x})$ as $x \to \infty$.

The role of the scattering solution of (1.2) is played by the function $\widetilde{\varphi}(x,k) = K_{ik}(e^x)$. The corresponding Jost solutions are the functions $\widetilde{f}_+(x,k) = 2^{ik}\Gamma(1+ik)\times I_{ik}(e^x)$ and $\widetilde{f}_-(x,k) = \widetilde{f}(x,-k)$ with asymptotic behaviour $\widetilde{f}_\pm(x,k) = e^{\pm ikx} + o(1)$ as $x \to -\infty$. We have

$$\widetilde{\varphi}(x,k) = \widetilde{M}(k)\widetilde{f}_+(x,k) + \widetilde{M}(-k)\widetilde{f}_-(x,-k), \tag{6.8}$$

where

$$\widetilde{M}(k) = -\frac{2^{-1-ik}\pi}{\sin(\pi ik)\Gamma(1+ik)} = 2^{-1-ik}\Gamma(-ik), \qquad |\widetilde{M}(k)|^2 = \frac{\pi}{4k\sinh(\pi k)}. \tag{6.9}$$

This concludes our description of the properties of the solutions of (1.2).

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