Letters in Mathematical Physics **52:** 79–91, 2000.

Quantum Field Theories on an Algebraic Curve

Dedicated to the memory of Moshé Flato

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(Received: 30 November 1999)

Abstract. We formulate quantum field theories on an algebraic curve and outline a 'paradigm' interpreting Ward identities as reciprocity laws.

Mathematics Subject Classifications (2000). 14H05, 14H81, 81R10.

Key words. algebraic curves, Clifford and Heisenberg algebras, quantum field theories, Ward identities, reciprocity laws.

1. Introduction

1.1. OVERVIEW

Recent development in mathematical physics is characterized by applications of Quantum Field Theory (QFT) to various areas of mathematics. They include the following:*

(1) Application of Topological QFT to Geometry and Topology: $d = 3$ Jones–Witten theory [1], $d = 4$ Donaldson and Seiberg–Witten theories [2, 3].
(2) Application of CFT and vertex operator algebras to the 'monstrous moonshine' [4].
(3) Application of CFT to Complex and Algebraic Geometry: $d = 2$ quantum gravity and intersection theory on moduli spaces of stable curves [5–7], and Weil–Petersson geometry [8, 9].

In this talk**, we are interested in applying QFT to Arithmetics, i.e., in developing QFT methods for algebraic number fields and fields of algebraic functions. On several occasions, I discussed these topics with Moshé Flato, who had deep thoughts about possible relations between QFT and arithmetic. Thus, Moshé came up with

*This is not a complete list, of course; it rather reflects author's interests in topics somewhat related to the subject of the paper.
**The Letter is an expanded version of the talk, given at the Conférence Moshé Flato, September 5–8 1999, Dijon, France. A detailed presentation will be published elsewhere.

a beautiful idea to use factoring of polynomials in several variables for the special quantization of the Nambu bracket (called Zariski quantization), developed in our paper [10].

1.2. HISTORIC BACKGROUND

In the 1920s, E. Artin and H. Hasse introduced 'Calculus' to arithmetic, and in 1930–1940s, C. Chevalley and A. Weil introduced infinite-dimensional methods. Recent development can be succinctly described as follows.

- In 1968, J. Tate [11] defined residues of differentials on algebraic curves in terms of the traces of certain linear operators on infinite-dimensional spaces of adèles, and gave a new proof of the residue theorem.
- In 1987, E. Arbarello, C. de Concini and V. Kac [12] interpreted Tate's approach in terms of central extensions of infinite-dimensional Lie algebras and proved A. Weil reciprocity law using the infinite-wedge representation.
- In 1987–1988, D. Kazhdan [13] and E. Witten [14, 15] proposed an adèlic formulation of free fermions on an algebraic curve.
- In 1990, H. Garland and G. Zuckerman [16], constructed local Fock spaces for multiplicative bosons on a Riemann surface.
- In 1988–1991, A. Beilinson, B. Feigin, and B. Mazur [17] developed a representation theory approach for CFT on algebraic curves.
- In 1989–1991, A. Raina [18, 19], developed an algebro-geometric approach for free fermions on Riemann surfaces and gave a new proof of Fay's trisecant identity.
- In 1994, A. Beilinson and V. Drinfeld, in the course of their work on geometric Langlands correspondence, introduced 'chiral algebras' [20, 21] on algebraic curves as mathematical objects that imitate operator product expansions of quantum fields.

Results of papers [17, 20, 21] have been recently summarized by D. Gaitsgory [22].

Primarily based on [13–15], we interpret this development as emergence of a 'new paradigm', that introduces reciprocity laws and Class Field Theory for algebraic function fields in one variable (see [23] for a review) as Ward Identities – quantum Noether conservation laws in Conformal Field Theory (see [14, 24] for a review).

Conservation laws in physics include Noether symmetries, gauge symmetries, and discrete symmetries. Here are the simplest examples of conservation laws in arithmetic.

(a) Artin–Whaples product formula

$$\prod_{p\in S} |r|_p = 1, \quad r \in \mathbb{Q},$$

where S is the set of all places on $\mathbb{Q}$, consisting of all primes $p = 2, 3, 5, \ldots$, and an 'infinite prime' $p = \infty$, and $|\ |_p$ is the p-adic absolute value (it is the usual absolute value on $\mathbb{R}$ for $p = \infty$).

(b) Gauss quadratic reciprocity law

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}},$$

where p, q are distinct odd primes, and (p/q) is the Legendre symbol: $(p/q) = 1$ or $(p/q) = -1$ depending on whether p is quadratic residue or nonresidue modulo p.

(c) Cauchy residue theorem

$$\sum_{P \in X} \mathrm{Res}_P f \mathrm{d}g = 0,$$

where X is compact Riemann surface, and $f, g \in \mathbb{C}(X)$ are meromorphic functions on X.

(d) A. Weil reciprocity law

$$\prod_{Q \in (g)} f(Q)^{n_Q} = \prod_{P \in (f)} g(P)^{n_P}, f, g \in \mathbb{C}(X),$$

where

$$(f) = \sum_{P \in X} n_P \cdot P, (g) = \sum_{Q \in X} n_Q \cdot Q$$

are divisors of meromorphic functions f and g with the property $(f) \cap (g) = \emptyset$.

2. Mathematical Set-up

2.1. DEFINITIONS

Let X be a complete, irreducible, nonsingular, algebraic curve of genus g over an algebraically closed field k of characteristic zero, and let $K = k(X)$ be the field of rational functions on X – a finitely-generated algebraic extension of k of transcendence degree 1 (see, e.g., [25, 26]). When $k = \mathbb{C}$ – the field of complex numbers, X is a compact Riemann surface, and $K = \mathbb{C}(X)$ is the field of meromorphic functions on X. Points $P \in X$ correspond to regular discrete valuations v_P of K over k, i.e., surjective homomorphisms $v : K^* \to \mathbb{Z}$ of the multiplicative group of the field K into the additive group of rational integers $\mathbb{Z}$, that are trivial on the subgroup k^* in K^*, and satisfy the property

$$v(x + y) \geqslant \min\{v(x), v(y)\}, \quad \text{for all } x, y \in K^*.$$

Let $\mathcal{K}$ be the canonical line bundle (invertible sheaf) over X, and let $\mathcal{L}$ be a spin structure for X – a line bundle of degree $g - 1$ over X, such that $\mathcal{L}^2 = \mathcal{K}$. For any line bundle $\mathcal{D}$ of degree 0 over X set $\mathcal{L}^{\pm} = \mathcal{L} \otimes \mathcal{D}^{\pm 1}$. Denote by $\mathcal{K}_X$, $\mathcal{L}_X$, and $\mathcal{L}_X^{\pm}$ the corresponding infinite-dimensional k-vector spaces of rational sections over X of the line bundles $\mathcal{K}$, $\mathcal{L}$, and $\mathcal{L}^{\pm}$. These are the spaces of rational differentials on X, rational $\frac{1}{2}$ differentials on X and rational 'chiral' $\frac{1}{2}$ differentials on X, correspondingly. Set $\mathcal{L}(D)_X = \mathcal{L}_X^{+} \oplus \mathcal{L}_X^{-}$.

2.2. LOCAL DATA

The following local objects at every $P \in X$ are canonically associated with the trivial line bundle $\mathcal{O}_X$ over X, line bundle $\mathcal{L}$ and with the rank 2 vector bundle $\mathcal{L}(D) = \mathcal{L}^{+} \oplus \mathcal{L}^{-}$.

(i) Local field at $P \in X$ – a completion K_P of the field K with respect to the valuation v_P.
(ii) Local ring $\mathcal{O}_P = \{u \in K_P | v_P(u) \geqslant 0\}$.
(iii) Unique maximal ideal $\mathfrak{p} = \{u \in \mathcal{O}_P | v_P(u) > 0\}$ in $\mathcal{O}_P$.
(iv) Residue field $k = \mathcal{O}_P/\mathfrak{p}$.
(v) Group of units $\mathrm{U}_P = \{u \in K_P | v_P(u) = 0\}$ in $\mathcal{O}_P, P \in X$, with the property $\mathrm{U}_P \simeq k^* \times U_P^1$, where $\mathrm{U}_P^1 = 1 + \mathfrak{p}$.
(vi) k-vector spaces L_P and $L(D)_P$ – completions at $P \in X$ of $\mathcal{L}_X$ and $\mathcal{L}(D)_X$, correspondingly, with respect to the valuation v_P.
(vii) Subspaces $\mathcal{O}(L)_P$ and $\mathcal{O}(L(D))_P$ of regular elements in L_P and $L(D)_P$, correspondingly.

k-vector spaces K_P, L_P, and $L(D)_P$ are complete in the $\mathfrak{p}$-adic topology, the residue field k is discrete in the quotient topology in K_P, $P \in X$, and these vector spaces are Tate vector spaces – topological vector spaces with a base of neighborhoods of zero given by mutually commensurable subspaces.

k-vector spaces L_P, K_P and $L(D)_P$ and the group K_P^* come equipped with the following canonical structures.

- L_P has nondegenerate, continuous, symmetric, k-bilinear form $(\, ,)_P : L_P \otimes L_P \to k$,

$$(\phi_1, \phi_2)_P = \mathrm{Res}_P(\phi_1 \, \phi_2).$$

  The pair $L_P, (\, , \,)_P$ defines a Clifford algebra Cliff_P.
- $L(D)_P$ has nondegenerate, continuous, symmetric, k-bilinear form $(\, ,)_P : L(D)_P \otimes L(D)_P \to k$,

$$(\psi_1 + \bar{\psi}_1, \psi_2 + \bar{\psi}_2)_P = \mathrm{Res}_P(\psi_1 \bar{\psi}_2 + \bar{\psi}_1 \psi_2),$$

  where $\psi \in L_P^{+}, \bar{\psi} \in L_P^{-}$. The pair $L(D)_P, (\, , \,)_P$ defines a Clifford algebra $\mathrm{Cliff}(D)_P$.

- K_P, as Abelian Lie algebra $\mathfrak{g}_P = \mathfrak{gl}_1(K_P)$ – 'geometric current algebra', has continuous, skew-symmetric, k-bilinear form – canonical Lie algebra two-cocycle given by the residue symbol,

  $$c_P(u, \mathrm{v}) = -\mathrm{Res}_P(u\, \mathrm{dv}).$$

  The pair $\mathfrak{g}_P, c_P$ defines a central extension $\hat{\mathfrak{g}}_P$ – 'geometric affine algebra'.
- K_P^* as Abelian Lie group $G_P = \mathrm{GL}_1(K_P)$ – 'geometric loop group', has continuous, skew-symmetric, k-bilinear form – canonical group two-cocycle, given by Tate's tame symbol,

  $$(f, g)_P = (-1)^{v_P(f)v_P(g)} \frac{f^{v_P(g)}}{g^{v_P(f)}} \bmod \mathfrak{p} \in k^*.$$

  The pair $G_P, (\ ,\ )_P$ defines a central extension $\hat{G}_P$ – 'geometric affine group'.

2.3. GLOBAL DATA

Global versions of the local objects introduced in the previous section are the following.

- The ring of adèles for K, $\mathbb{A}_X = \coprod_{P \in X} K_P$ – restricted Cartesian product of spaces K_P over $P \in X$ with respect to subspaces $\mathcal{O}_P$. An element $a = \{a_P\}_{P \in X} \in \mathbb{A}_X$, if $a_P \in \mathcal{O}_P$ for all except finitely many $P \in X$.
- Diagonal embedding $K \ni f \mapsto \{f|_P\}_{P \in X} \in \mathbb{A}_X$ of the global field K into the ring of adèles $\mathbb{A}_X$.
- The group of idèles for K – the group of units in the ring of adèles $\mathbb{A}_X$, $\mathbb{J}_X = \coprod_{P \in X} K_P^*$ – restricted Cartesian product of multiplicative groups K_P^* with respect to subgroups U_P.
- Diagonal embedding $K^* \hookrightarrow \mathbb{J}_X$ of the multiplicative group of the global field into the group of idèles $\mathbb{J}_X$.
- Restricted Cartesian product of spaces L_P, $\mathbb{L}_X = \coprod_{P \in X} L_P$, with respect to subspaces $\mathcal{O}(L)_P$.
- Diagonal embedding $\mathcal{L}_X \hookrightarrow \mathbb{L}_X$ of the space of rational sections of $\mathcal{L}$ over X into the space of adèles $\mathbb{L}_X$.
- Restricted Cartesian product of the spaces $L(D)_P$, $\mathbb{L}(D)_X = \coprod_{P \in X} L(D)_P$, with respect to subspaces $\mathcal{O}(L(D))_P$.
- Diagonal embedding $\mathcal{L}(D)_X \hookrightarrow \mathbb{L}(D)_X$ of the space of rational sections of $\mathcal{L}(D)$ over X into the space of adèles $\mathbb{L}(D)_X$.

2.4. ARTIN–WHAPLES AND WEIL EXAMPLES

(1) For $x = \{x_P\}_{P \in X} \in \mathbb{J}_X$ define $||x|| = \prod_{P \in X} |x_P|_P$, where $|x_P|_P = c^{v_P(x_P)}$ for some fixed $0 < c < 1$ is the $\mathfrak{p}$-adic metric for the valuation v_P. The Artin–Whaples product formula [28] $||fx|| = ||x||$, where $x \in \mathbb{J}_X$ and $f \in K^* \hookrightarrow \mathbb{J}_X$, is an

example of 'conservation law' stating that 'the sum over X of residues of the meromorphic differential df/f is zero'.

(2) For $a = \{a_P\}_{P\in X}, b = \{b_P\}_{P\in X} \in \mathbb{J}_X$ define the global symbol $(a, b)_X = \prod_{P\in X}(a_P, b_P)_P \in k^*$. The A. Weil reciprocity law (see, e.g., [26]) $(f, g)_X = 1$, for all $f, g \in K^*$, is an example of 'conservation law'

$$\frac{(fa, gb)_X}{(f, b)_X(a, g)_X} = (a, b)_X,$$

for all $a, b \in \mathbb{A}_X$.

3. QFT Set-up

3.1. DEFINITIONS

In correspondence with different vector spaces of global rational sections $\mathcal{L}_X, \mathcal{L}(D)_X$, $K = k(X)$, and with the group K^*, we consider the following quantum field theories on algebraic curve X.

(a) Majorana–Weyl (one-component) free fermions $\longleftrightarrow \mathcal{L}_X$.
(b) Charged (two-component) free fermions $\longleftrightarrow \mathcal{L}(D)_X$.
(c) Additive $\mathfrak{gl}_1$-bosons $\longleftrightarrow K$.
(d) Multiplicative GL_1-bosons $\longleftrightarrow K^*$.

Specifically, for every $P \in X$, let $\mathcal{G}_P$ be either the Clifford algebra Cliff_P or $\mathrm{Cliff}(D)_P$, or the Lie algebra $\hat{\mathfrak{g}}_P$, or the Lie group $\hat{G}_P$. Also, let $\mathcal{G}_P^+$ be, correspondingly, either the Clifford subalgebra $\Lambda^*\mathcal{O}(L)_P$ or $\Lambda^*\mathcal{O}(L(D))_P$, or the Lie subalgebra $\mathcal{O}_P \oplus \{0\}$, or the Lie subgroup $\mathrm{U}_P^1 \times \{1\}$.

By a QFT with symmetry algebra or group $\mathcal{G}$ on an algebraic curve X, we understand the following local and global data.

3.2. LOCAL QFT

It consists of an irreducible highest weight $\mathcal{G}_P$-module $\mathcal{F}_P$ for $P \in X$ – fermionic or bosonic Fock space containing distinguished vector $\mathbf{1}_P$, which is either annihilated or is invariant with respect to the action of the corresponding algebra or group $\mathcal{G}_P^+$.

The correspondence between quantum observables and states in CFT asserts that $\mathcal{F}_P$ is the space of quantum observables at $P \in X$.

3.2.1. For Majorana–Weyl fermions, set

$$\mathcal{F}_P = \Lambda^* L_P^- \simeq \mathrm{Cliff}_P/\mathrm{Cliff}_P \cdot \mathcal{O}(L)_P,$$

with the left action of Cliff_P, where $L_P = \mathcal{O}(L)_P \oplus L_P^-$, is a decomposition of L_P into maximal isotropic subspaces with respect to the symmetric bilinear form $(\ ,\ )_P$, introduced in Section 2.1. Though subspace L_P^- – the complement of $\mathcal{O}(L)_P$ in L_P, depends on the choice of a uniformizer at P, the corresponding Cliff_P-module $\mathcal{F}_P$ is defined canonically.

In the case when the spin structure $\mathcal{L}$ has no regular global sections, i.e., $h^0(X, \mathcal{L}) = 0$, the subspace L_P^- can be canonically defined as follows. By the Riemann–Roch theorem, $h^0(X, \mathcal{L}(nP)) = n$ for all $n \in \mathbb{N}$, so that there exist global rational sections $s_P^{(n)}$ of $\mathcal{L}$ with the only pole at $P \in X$ of order n. The subspace L_P^- is the localization of the subspace $H^0(X, \mathcal{L}(*P))$ at P and has basis $\{v_P^{(n)} = s_P^{(n)}|_P\}_{n\in\mathbb{N}}$. The pairing $(\ ,\ )_P : \mathcal{O}(L)_P \otimes L_P^- \to k$ is nondegenerate and continuous with respect to the $\mathfrak{p}$-adic topology on L_P and discrete topology on k, so that the k-vector space $\mathcal{O}(L)_P$ is dual to L_P^-: $\mathcal{O}(L)_P = (L_P^-)^\vee$. The k-vector space $\mathcal{F}_P^\vee = \Lambda^*\mathcal{O}(L)_P$ is a dual Cliff_P-module to the Fock module $\mathcal{F}_P$.

3.2.2. For additive $\mathfrak{gl}_1$-bosons, set

$$\mathcal{F}_P = \mathrm{Sym}^* K_P^- \simeq W_P / W_P \cdot \mathcal{O}_P,$$

with the left action of $\hat{\mathfrak{g}}_P$, where $K_P = \mathcal{O}_P \oplus K_P^-$, is a decomposition of K_P into isotropic subspaces with respect to the skew-symmetric bilinear form c_P, introduced in Section 2.1. Here W_P is the Weyl algebra – a quotient of $\mathrm{U}(\hat{\mathfrak{g}}_P)$ by the ideal generated by the element $(0, 1) - \mathbf{1}$. Though the subspace K_P^- – the complement of $\mathcal{O}_P$ in K_P, depends on the choice of a uniformizer at P, the corresponding $\mathfrak{gl}_1$-module $\mathcal{F}_P$ is defined canonically.

The subspace K_P^- can be defined as follows. By the Riemann–Roch theorem, $H^0(X, \mathcal{K}(nP)) = g + n - 1$, so that there exist differentials of second kind $\theta_P^{(n)}$, with the only pole at P of order $n \geqslant 2$. Since $\mathrm{Res}_P \theta_P^{(n)} = 0$, there exist $v_P^{(n)} \in K_P$ such that $dv_P^{(n)} = \theta_P^{(n)}|_P$, and we define K_P^- as a k-vector space with basis $\{v_P^{(n)}\}_{n\in\mathbb{N}}$. The pairing $c_P : \mathfrak{p} \otimes K_P^- \to k$ is nondegenerate (note that k is the kernel of the bilinear form c_P), and continuous with respect to the $\mathfrak{p}$-adic topology on L_P and discrete topology on k, so that the k-vector space $\mathfrak{p}$ is dual to K_P^-: $\mathfrak{p} = (K_P^-)^\vee$. The k-vector space $\mathcal{F}_P^\vee = \mathrm{Sym}^*\mathfrak{p}$ is a dual $\hat{\mathfrak{g}}_P$-module to the Fock module $\mathcal{F}_P$.

Local Fock spaces for charged fermions and GL_1-bosons are constructed in a similar way, the bosonic Fock space construction being equivalent to the Heisenberg system representation of Garland–Zuckerman [16]. There is also an algebraic version of Fermi–Bose correspondence between Fock spaces for charged fermions and for $\mathfrak{gl}_1$-bosons (the latter should be extended to include 'zero modes').

3.2.3. *Applications*

Here we indicate applications of local QFT of $\mathfrak{gl}_1$-bosons to local CFT (Class Field Theory). We consider only the case when $\mathrm{char}\, k = 0$. Then one has the following (see, e.g., [27]).

(i) For every $P \in X$, $K_P \simeq k((t))$, formal Laurent series in t, the uniformizer at $P \in X$.
(ii) All finite algebraic extensions of $k((t))$ are Abelian with cyclic Galois group.
(iii) The algebraic closure K_P^{alg} of K_P is isomorphic to the union of the fields $K_P^{(n)} = k((t^{1/n}))$, $K_P^{\mathrm{alg}} = \bigcup_{n\in\mathbb{N}} K_P^{(n)}$.
(iv) The completion $\mathbb{K}_P$ of the field K_P^{alg} with respect to the valuation $\mathbf{v}_P$ extending the valuation v_P of K (the valuation $\mathbf{v}_P$ takes values in $\mathbb{Q}^*$) is isomorphic to the field $k((t^{1/\infty}))$, consisting of formal power series

$$f = \sum_{r_n\in\mathbb{Q}} c_n t^{r_n}, \text{ where } c_1 \neq 0 \text{ and } r_n \nearrow +\infty \text{ as } n \to \infty,$$

so that $\mathbf{v}_P(f) = r_1$.
(v) The Galois group of the extension K_P^{alg}/K_P – the absolute Galois group G, is isomorphic to $\hat{\mathbb{Z}}$ – a completion of the Abelian group $\mathbb{Z}$ with respect to the Krull topology. The Galois group G has generator F – a geometric Frobenius element, defined by $F(\sqrt[n]{t}) = \omega_n \sqrt[n]{t}$, where $\omega_n \in k$ are nth roots of unity in k, satisfying the property $\omega_n = \omega_{mn}^m$ for all $m, n \in \mathbb{N}$ (see [27]). The correspondence $\hat{\mathbb{Z}} \ni \nu \mapsto F^\nu \in G$ establishes an isomorphism $\hat{\mathbb{Z}} \simeq G$.

Let $\mathfrak{G}_P$ be the 'absolute' affine Lie algebra – a central extension of the Abelian Lie algebra $\mathbb{K}_P$ with respect to the two-cocycle c_P. It can be described (depending on the choice of the uniformizer) as the 'absolute' Heisenberg algebra with generators $\{\alpha_r\}_{r\in\mathbb{Q}}$ and C, and relations

$$[\alpha_r, \alpha_s] = r\delta_{r,-s}C, \quad [\alpha_r, C] = 0,$$

for all $r, s \in \mathbb{Q}$. The corresponding irreducible $\mathfrak{G}_P$-module is the 'absolute' bosonic Fock space $\mathbb{F}_P = k[x_r]_{r\in\mathbb{Q},r>0}$ of polynomials in infinitely many variables parameterized by a positive $r \in \mathbb{Q}$. The absolute Galois group G acts on the Lie algebra $\mathfrak{G}_P$ and on the Fock space $\mathbb{F}_P$, and direct sums of its invariant subspaces determine the Fock spaces $\mathcal{F}_P^{(n)}$ of additive $\mathfrak{gl}_1$-bosons for $K_P^{(n)}$.

This simple observation establishes a connection between local QFT and local CFT for the field $K_P = k((t))$ when $\operatorname{char} k = 0$. Of course, local CFT for this case is rather trivial. Similar relations exist for a really interesting case when $\operatorname{char} k = p$, i.e., $k = \overline{\mathbb{F}_p}$ – an algebraic closure of the finite field $\mathbb{F}_p$ of p elements.

3.3. GLOBAL QFT

Global QFT on algebraic curve X consists of the following data.

I Local QFT – a pair of algebra (group) $\mathcal{G}_P$ and local Fock space $\mathcal{F}_P$ with distinguished vector $\mathbf{1}_P$ for every $P \in X$.
II Global algebra (group) $\mathbb{G}_X$ – a restricted direct sum (direct product or graded tensor product) of local Lie algebras $\mathcal{G}_P$ (Lie groups or Clifford algebras) over

all $P \in X$, with the diagonal embedding of the subalgebra (subgroup) $\mathcal{G}_X$, generated by rational sections over X.

III Irreducible highest weight $\mathbb{G}_X$-module – global Fock space $\mathbb{F}_X$ of quantum observables with distinguished vector $\mathbf{1}_X$ – a restricted graded-symmetric tensor product of local Fock spaces $\mathcal{F}_P$ over all $P \in X$ such that $\mathbf{1}_X = \otimes_{P\in X}\mathbf{1}_P$.

IV The k-linear functional $\langle\ \rangle : \mathbb{F}_X \to k$ – the expectation value of quantum observables, that satisfies the following properties.

IV-1 $\langle \mathbf{1}_X \rangle = 1$

IV-2 $\langle \hat{f} \cdot v \rangle = 0$ – for additive QFT,
$\langle \hat{f} \cdot v \rangle = \langle v \rangle$ – for multiplicative QFT,
for all $f \in \mathcal{G}_X$ and $v \in \mathbb{F}_X$, where $\hat{f} \in \mathrm{End}_k \mathbb{F}_X$.

These are Ward identities, that express quantum symmetries of the theory.

3.3.1. *Applications*

The additive Ward identity for the Lie algebra case:

$$0 = \langle \widehat{[f,g]} \cdot v \rangle = c_X(f,g)\langle v \rangle + \langle [\hat{f}, \hat{g}] \cdot v \rangle,$$

for all $f, g \in \mathcal{G}_X$ and $v \in \mathbb{F}_X$, implies that

$$c_X(f,g) = \sum_{P \in X} c_P(f,g) = 0$$

– an additive ‘reciprocity law’. The additive Ward identity for the Clifford algebra case:

$$0 = \langle \widehat{fg + gf} \cdot v \rangle = (f,g)_X \langle v \rangle + \langle (\hat{f}\hat{g} + \hat{g}\hat{f}) \cdot v \rangle,$$

implies the additive conservation law

$$(f,g)_X = \sum_{P \in X} (f,g)_P = 0,$$

for all $f, g \in \mathcal{G}_X$.

Similarly, the multiplicative Ward identity:

$$\langle v \rangle = \langle \widehat{fg} \cdot v \rangle = c(f,g)_X \langle \hat{f}(\hat{g} \cdot v) \rangle,$$

for all $f, g \in \mathcal{G}_X$ and $v \in \mathbb{F}_X$, implies that

$$(f,g)_X = \prod_{P \in X} (f,g)_P = 1$$

– a multiplicative ‘reciprocity law’.

3.4. EXAMPLES

If the spin structure $\mathcal{L}$ for the algebraic curve X has no global regular sections, i.e., $h^0(X, \mathcal{L}) = 0$, or the degree 0 line bundle $\mathcal{D}$ over X is such that $h^0(X, \mathcal{L} \otimes \mathcal{D}) = 0$, then one can canonically construct a QFT of Majorana–Weyl and charged fermions on X.

Start with the case of Majorana–Weyl fermions and consider, for every $P \in X$, the k-vector spaces L_P^- and their dual spaces $(L_P^-)^\vee$, introduced in Section 3.2. For every $P \in X$ denote by $\{u_P^{(n)}\}_{n\in\mathbb{N}}$ the basis in $\mathcal{O}_P = (L_P^-)^\vee$, dual to the basis $\{v_P^{(n)}\}_{n\in\mathbb{N}}$ in L_P^- with respect to pairing $(\ ,\ )_P$. Let $\mathbb{F}_X^\vee = \bigotimes_{P\in X} \mathcal{F}_P^\vee$ be the dual space to the global Fock space $\mathbb{F}_X$ (here $\otimes$ is a $\mathbb{Z}/2\mathbb{Z}$-graded symmetric tensor product, which is unrestricted over X), and consider a vector $\Omega \in \mathbb{F}_X^\vee$ in the following form of the 'filled Dirac sea':

$$\Omega = \mathbf{1}_X^\vee + \sum_{n=1}^{\infty} \sum_{P_1\in X} \cdots \sum_{P_n\in X} \sum_{i_1=1}^{\infty} \cdots \sum_{i_n=1}^{\infty} c_{P_1\cdots P_n}^{i_1\cdots i_n} u_{P_1}^{(i_1)} \otimes \cdots \otimes u_{P_n}^{(i_n)}.$$

Consider the linear functional on $\mathbb{F}_X$, associated with $\Omega \in \mathbb{F}_X^\vee$,

$$\langle v \rangle = (\Omega, v),\ v \in \mathbb{F}_X.$$

Denoting by $\hat{f}^\vee$ the dual Cliff-action, we see that the Ward identity IV-2 for $\langle\ \rangle$ is equivalent to the following condition

$$\hat{f}^\vee \Omega = 0 \text{ for all } f \in \mathcal{L}_X.$$

By Riemann–Roch theorem, it follows from $h^0(X, \mathcal{L}) = 0$, that every $f \in \mathcal{L}_X$ admits a 'simple fraction decomposition', i.e., it is a linear combination of sections $s_P^{(n)}$ with $P \in (f)_\infty$ – the polar divisor of f. Therefore, it is sufficient to verify IV-2 only for sections $s_P^{(n)}$, and simple 'calculus of residues' gives the following result.

THEOREM 1. *Let a spin structure $\mathcal{L}$ for an algebraic curve X be such that $h^0(X, \mathcal{L}) = 0$. Then there exists a unique expectation value functional $\langle\ \rangle : \mathbb{F}_X \to k$ satisfying the additive Ward identities, and it has the form $\langle v \rangle = (\Omega, v)$, $v \in \mathbb{F}_X$, where*

$$\Omega = \exp\left\{ \sum_{m,n=1}^{\infty} \sum_{P,Q\in X} c_{PQ}^{mn} u_P^{(m)} \otimes u_Q^{(n)} \right\} \in \mathbb{F}_X^\vee,$$

and

$$c_{PQ}^{mn} = \mathrm{Res}_P(v_P^{(m)} v_Q^{(n)}).$$

The first statement of the theorem was established in [14]. The second statement gives the generating function for 'free fermion correlation functions' (cf. [18, 19]) in a closed form. In particular, setting $v_P = v_P^{(1)}, P \in X$, we get the following expression

$$\langle\phi(P_1)\cdots\phi(P_n)\rangle\rangle = (\Omega, v_{P_1}\otimes\cdots\otimes v_{P_n})$$

$$= \begin{cases} 0 & n \text{ is odd;} \\ \mathrm{Pf}(\langle\phi(P_i)\phi(P_j)\rangle\rangle) & n \text{ is even,} \end{cases}$$

for the n-point correlation function of fermion field operators $\phi_i = \phi(P_i)$ on X.

An analogous construction works for the case of charged fermions, when $h^0(X, \mathcal{L}\otimes\mathcal{D}) = 0$.

For additive $\mathfrak{gl}_1$-bosons, the expectation value functional is not unique and depends on the choice of differentials of the second kind on X. Still, similar arguments work for this case. Namely, for every $P\in X$ choose the basis $\{u_P^{(n)}\}_{n\in\mathbb{N}}$ in $\mathfrak{p}$, dual to the basis $\{v_P^{(n)}\}_{n\in\mathbb{N}}$ in K_P^-, and define the dual Fock space $\mathbb{F}_X^\vee$ as (unrestricted) symmetric tensor product of the dual Fock spaces $\mathcal{F}_P^\vee$ over all $P\in X$. Consider a vector $\Omega\in\mathbb{F}_X^\vee$ in the form of the "filled Dirac sea'. Contrary to the previous case, the simple fraction expansion is no longer valid for rational functions on X. However, one can consider a stronger condition than IV-2, by requiring the same equation to be valid for 'multi-valued' rational functions on X (cf. [29]). By definition, $\{f_P\}_{P\in X}\in\mathbb{A}_X$ is a multi-valued rational function on X, if there exists a rational differential θ on X (which is necessarily of the second kind) such that $\mathrm{d}f_P = \theta|_P$ for all $P\in X$. The multi-valued functions – 'Abelian integrals', act on the dual Fock space $\mathbb{F}_X^\vee$ and admit simple fraction expansion, so that we can use the previous arguments and determine a vector Ω. Thus, denoting by J the free boson field (current) on X, we get

$$\langle J(P)J(Q)\rangle = \mathrm{Res}_P(\theta_P\, d^{-1}\theta_Q).$$

Another way to construct additive $\mathfrak{gl}_1$-bosons is to apply the bosonization procedure to charged free fermions with $h^0(X, \mathcal{L}\otimes\mathcal{D}) = 0$. Thus the obtained QFT's are parameterized by $\mathrm{Pic}^{g-1}(X) - \Theta$, where Θ is the canonical theta-divisor (symmetric with respect to the involution $\mathcal{D}\mapsto\mathcal{K}\otimes\mathcal{D}^{-1}$).

The 'exponentiated' version of these constructions allows to define a QFT of multiplicative bosons and to prove A. Weil reciprocity law.

3.5. FURTHER DEVELOPMENT

Local QFT's discussed here can be also defined when $\mathrm{char}\,k = p > 0$. Thus, additive $\mathfrak{gl}_1$-bosons correspond to the Artin–Schreier extensions, whereas multiplicative GL_1-bosons correspond to the Kummer extensions of local fields, with global QFT's incorporating Artin reciprocity law for the norm residue symbol.

There is some evidence that these methods may also work for number fields, providing a QFT foundation for the Gauss quadratic reciprocity law and its generalizations. We plan to return to these issues elsewhere.

Acknowledgements

At different stages of this work I appreciated stimulating discussions with M. Flato, I. Frenkel and M. Semenov-Tian-Schansky. This research was supported by the NSF grant DMS-9802574, and was partially conducted for the Clay Mathematics Institute when the author was a CMI Prize Fellow.

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