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REAL PROJECTIVE CONNECTIONS, V. I. SMIRNOV'S APPROACH, AND BLACK-HOLE-TYPE SOLUTIONS OF THE LIOUVILLE EQUATION

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We consider real projective connections on Riemann surfaces and their corresponding solutions of the Liouville equation. We show that these solutions have singularities of a special type (*a black-hole type*) *on a finite number of simple analytic contours. We analyze the case of the Riemann sphere with four real punctures, considered in V. I. Smirnov's thesis* (*Petrograd,* 1918) *in detail.*

Keywords: uniformization, Riemann surface, projective connection, Fuchsian projective connection, monodromy group, Liouville equation, Liouville action, singular solution

Dedicated to my teacher Ludvig Dmitrievich Faddeev
on the occasion of his 80*th birthday*

1. Introduction

One of the central problems of mathematics in the second half of the 19th century and at the beginning of the 20th century was the problem of uniformization of Riemann surfaces. The classics, Klein [1] and Poincaré [2], associated it with studying second-order ordinary differential equations with regular singular points. Poincaré proposed another approach to the uniformization problem [3]. It consists in finding a complete conformal metric of constant negative curvature, and it reduces to the global solvability of the Liouville equation, a special nonlinear partial differential equation of elliptic type on a Riemann surface.

Here, we illustrate the relation between these two approaches and describe solutions of the Liouville equation corresponding to second-order ordinary differential equations with a real monodromy group. In the modern physics literature on the Liouville equation, it is rather commonly assumed that for the Fuchsian uniformization of a Riemann surface, it suffices to have a second-order ordinary differential equation with a real monodromy group. But the classics already knew that this is not the case, and they analyzed second-order ordinary differential equations with a real monodromy group on genus-0 Riemann surfaces with punctures in detail. Nonetheless, they did not consider the relation to the Liouville equation, and we partially fill this gap here.

Namely, in Sec. 2, following the lectures [4], we briefly describe the theory of projective connections on a Riemann surface—an invariant method for defining a corresponding second-order ordinary differential equation with regular singular points. Following [5], [6], we review the main results on the Fuchsian uniformization, the Liouville equation, and the complex geometry of the moduli space. In Sec. 3, following [7], we present the modern classification of projective connections with a real monodromy group and review the results of V. I. Smirnov's thesis [8] (Petrograd, 1918). This work, published in [9], [10], was the first

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where a complete classification of equations with a real monodromy group was given in the case of four real punctures. In Sec. 3.2, we give a modern interpretation of Smirnov's results. Finally, in Sec. 4, we describe solutions of the Liouville equation with black-hole-type singularities associated with real projective connections. To the best of our knowledge, these solutions have not been considered previously.

2. Projective connections, uniformization, and the Liouville equation

2.1. Projective connections. Let X_0 be a compact genus-g Riemann surface with marked points $x_1, \dots, x_n$, where $2g+n-2>0$, and let $\{U_\alpha, z_\alpha\}$ be a complex-analytic atlas with local coordinates z_α and transition functions $z_\alpha = g_{\alpha\beta}(z_\beta)$ on $U_\alpha \cap U_\beta$. Let $X = X_0 \setminus \{x_1, \dots, x_n\}$ denote a corresponding Riemann surface of type (g,n), a genus-g surface with n punctures. The collection $R = \{r_\alpha\}$, where r_α are holomorphic functions on $U_\alpha \cap X$, is called a (*holomorphic*) *projective connection* on X if on every intersection $U_\alpha \cap U_\beta \cap X$,

$$r_\beta = r_\alpha \circ g_{\alpha\beta} (g'_{\alpha\beta})^2 + \mathcal{S}(g_{\alpha\beta}),$$

where $\mathcal{S}(f)$ is the Schwarzian derivative of a holomorphic function f,

$$\mathcal{S}(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

In addition, we assume that if $x_i \in U_\alpha$ and $z_\alpha(x_i) = 0$, then

$$r_\alpha(z_\alpha) = \frac{1}{2z_\alpha^2} + O\left(\frac{1}{|z_\alpha|}\right), \quad z_\alpha \to 0. \tag{1}$$

Projective connections form an affine space $\mathcal{P}(X)$ over the vector space $\mathcal{Q}(X)$ of holomorphic quadratic differentials on X; elements of $\mathcal{Q}(X)$ are collections $Q = \{q_\alpha\}$ with the transformation law

$$q_\beta = q_\alpha \circ g_{\alpha\beta} (g'_{\alpha\beta})^2$$

and the additional condition that $q_\alpha(z_\alpha) = O(|z_\alpha|^{-1})$ as $z_\alpha \to 0$ if $x_i \in U_\alpha$ and $z_\alpha(x_i) = 0$. The vector space $\mathcal{Q}(X)$ has the complex dimension $3g-3+n$ (for more details on projective connections and quadratic differentials, see [4] and the references therein).

A projective connection R naturally determines a second-order linear differential equation on the Riemann surface X, the Fuchsian differential equation

$$\frac{d^2 u_\alpha}{dz_\alpha^2} + \frac{1}{2} r_\alpha u_\alpha = 0, \tag{2}$$

where $U = \{u_\alpha\}$ is understood as a multivalued differential of order $-1/2$ on X. Equation (2) determines the monodromy group, a representation of the fundamental group $\pi_1(X, x_0)$ of the Riemann surface X with the marked point x_0 in $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm I\}$. Condition (1) implies that the standard generators of $\pi_1(X, x_0)$, which correspond to the loops around the punctures x_i, are mapped to parabolic elements in $\mathrm{PSL}(2,\mathbb{C})$ under the monodromy representation.

2.2. Uniformization. According to the uniformization theorem

$$X \cong \Gamma \setminus \mathbb{H}, \tag{3}$$

where $\mathbb{H} = \{\tau \in \mathbb{C}\colon \operatorname{Im}\tau > 0\}$ is the Poincaré model of the Lobachevksy plane and $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is a type-(g,n) Fuchsian group acting on $\mathbb{H}$ by fractional linear transformations. In other words, there exists a complex-analytic covering $J\colon \mathbb{H} \to X$ whose automorphism group is Γ. The function inverse to J, a multivalued analytic function $J^{-1}\colon X \to \mathbb{H}$, is a locally univalent linear polymorphic function on X (this means that its branches are connected by fractional linear transformations in Γ). The Schwarzian derivatives of J^{-1} with respect to z_α are well defined on U_α and determine the *Fuchsian projective connection* $R_{\mathrm{F}} = \{\mathcal{S}_{z_\alpha}(J^{-1})\}$ on X, and the multivalued functions $1/\sqrt{(J^{-1})'}$ and $J^{-1}/\sqrt{(J^{-1})'}$ satisfy Fuchsian differential equation (2) with $R = R_{\mathrm{F}}$. The monodromy group of this equation, up to conjugation in $\mathrm{PSL}(2,\mathbb{R})$, is the Fuchsian group Γ.

Klein [1] and Poincaré [2] were solving the problem of uniformizing a Riemann surface X by choosing a projective connection in Fuchsian equation (2) such that its monodromy group is a Fuchsian group Γ with the property that (3) holds. But a direct proof of the existence of the Fuchsian projective connection R_{F} on X turned out to be a very difficult problem, which has not been completely solved to this day (see [11], [12]). In the case of type-$(0,n)$ Riemann surfaces, to which we further restrict ourself, this problem is formulated as follows.

Let X_0 be the Riemann sphere $\mathbb{P}^1$ and X be a genus-0 Riemann surface with n punctures $z_1,\dots,z_n$. Without loss of generality, we can assume that $z_{n-2}=0$, $z_{n-1}=1$, $z_n=\infty$, and $X = \mathbb{C}\backslash\{z_1,\dots,z_{n-3},0,1\}$. Equation (2) becomes

$$\frac{d^2u}{dz^2} + \frac{1}{2}\sum_{i=1}^{n-1}\left(\frac{1}{2(z-z_i)^2} + \frac{c_i}{z-z_i}\right)u = 0, \tag{4}$$

where z is a global complex coordinate on X. The complex parameters $c_1,\dots,c_{n-1}$ satisfy the two conditions

$$\sum_{i=1}^{n-1} c_i = 0, \qquad \sum_{i=1}^{n-1} z_i c_i = 1 - \frac{n}{2}, \tag{5}$$

which allow expressing c_{n-2} and c_{n-1} explicitly in terms of $z_1,\dots,z_{n-3}$ and the remaining $n-3$ parameters $c_1,\dots,c_{n-3}$.

In the classical approach of Klein and Poincaré to the uniformization problem, given the singular points $z_1,\dots,z_{n-3},0,1,\infty$, it was required to choose parameters $c_1,\dots,c_{n-3}$ such that the monodromy group of Eq. (4) is a Fuchsian group isomorphic to the fundamental group of the Riemann surface X. The ratio of two linear independent solutions of Eq. (4) up to a fractional linear transformation would then be the desired multivalued map $J^{-1}\colon X \to \mathbb{H}$ realizing the uniformization of the Riemann surface X. The corresponding Γ-automorphic function $J\colon \mathbb{H} \to \mathbb{C}$ is called Klein's Hauptmodul (Hauptfunktion). The complex numbers $c_1,\dots,c_{n-3}$, the *accessory parameters* of the Fuchsian uniformization of the surface X, are uniquely determined by the singular points $z_1,\dots,z_{n-3}$. Moreover,

$$\mathcal{S}(J^{-1})(z) = \sum_{i=1}^{n-1}\left(\frac{1}{2(z-z_i)^2} + \frac{c_i}{z-z_i}\right). \tag{6}$$

To prove the existence of the accessory parameters, Poincaré proposed the so-called continuity method in [2]. But a rigorous solution of the uniformization problem could not be obtained using this method, and being subjected to criticism, the method was soon abandoned. The ultimate solution of the uniformization problem was obtained by Koebe and Poincaré in 1907 using quite different methods, in particular, using potential theory (see, e.g., [13] for a modern exposition).

2.3. The Liouville equation. The projection on X of the Poincaré metric $(\operatorname{Im}\tau)^{-2}\,|d\tau|^2$ on $\mathbb{H}$ is a complete conformal metric on X of constant negative curvature -1. It has the form $e^{\varphi(z)}|dz|^2$, where

$$e^{\varphi(z)}=\frac{|(J^{-1})'(z)|^2}{(\operatorname{Im}J^{-1}(z))^2}. \tag{7}$$

The smooth function φ on X satisfies the Liouville equation

$$\varphi_{z\bar z}=\frac{1}{2}e^{\varphi} \tag{8}$$

and has the asymptotic behavior

$$\varphi(z)=\begin{cases}-2\log|z-z_i|-2\log|\log|z-z_i||+o(1), & z\to z_i,\quad i\neq n,\\ -2\log|z|-2\log\log|z|+o(1), & z\to\infty.\end{cases} \tag{9}$$

In [3], Poincaré proposed an approach to the uniformization problem based on the Liouville equation. Namely, he proved that Liouville equation (8) is uniquely solvable in the class of smooth real-valued functions on X with asymptotic behavior (9). It hence follows that $T_\varphi=\varphi_{zz}-\varphi_z^2/2$ is a rational function of form (6) and that the differential equation

$$\frac{d^2u}{dz^2}+\frac{1}{2}T_\varphi u=0$$

has a Fuchsian monodromy group that uniformizes the Riemann surface X.

The Liouville equation is the Euler–Lagrange equation for the functional

$$S(\psi)=\lim_{\varepsilon\to0}\left(\iint_{X_\varepsilon}(|\psi_z|^2+e^{\psi})\,d^2z+2\pi n\log\varepsilon+4\pi(n-2)\log|\log\varepsilon|\right),$$

where d^2z is the Lebesgue measure on $\mathbb{C}$,

$$X_\varepsilon=X\setminus\left(\bigcup_{i=1}^{n-1}\{|z-z_i|<\varepsilon\}\cup\left\{|z|>\frac{1}{\varepsilon}\right\}\right),$$

and ψ belongs to the class of smooth functions X with asymptotic behavior (9). The quantity $T_\psi=\psi_{zz}-\psi_z^2/2$ plays the role of the $(2,0)$-component of the stress–energy tensor in the classical Liouville theory, and

$$T_\varphi=\mathcal{S}(J^{-1}).$$

We let

$$\mathcal{M}_{0,n}=\{(z_1,\dots,z_{n-3})\in\mathbb{C}^{n-3}\colon z_i\neq z_j \text{ for } i\neq j \text{ and } z_i\neq 0,1\}$$

denote the moduli space of genus-0 Riemann surfaces with n ordered punctures (rational curves with n marked points). The critical values of the Liouville action functional $S(\psi)$ (the values on the extremums φ for all surfaces X) determine a smooth function $S\colon\mathcal{M}_{0,n}\to\mathbb{R}$, the *classical action for the Liouville equation.* As proved in [5], [6], the classical action for the Liouville equation plays a fundamental role in the complex geometry of the moduli space $\mathcal{M}_{0,n}$. Namely, S is a common antiderivative for the accessory parameters

$$c_i=-\frac{1}{2\pi}\frac{\partial S}{\partial z_i},\quad i=1,\dots,n-3,$$

and also a Kähler potential for the Weil–Petersson metric on $\mathcal{M}_{0,n}$,

$$-\frac{\partial^2 S}{\partial z_i \partial \bar{z}_j} = \left\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right\rangle_{WP}, \quad i, j = 1, \ldots, n-3.$$

The statement that the classical action for the Liouville equation is a common antiderivative for accessory parameters for genus-0 surfaces was conjectured by Polyakov[1] based on the semiclassical analysis of the conformal Ward identities of the quantum Liouville theory (see [14]).

3. Real projective connections and Smirnov's thesis

3.1. General case. Let X be a Riemann surface of type (g, n). A projective connection R on X is said to be *real* or *Fuchsian* if its monodromy group up to a conjugation in $\mathrm{PSL}(2, \mathbb{C})$ is respectively a subgroup in $\mathrm{PSL}(2, \mathbb{R})$ or a Fuchsian group. By the uniformization theorem, a Fuchsian projective connection R_{F} is uniquely characterized by the condition that its monodromy group is precisely the Fuchsian group Γ that uniformizes the Riemann surface X. It is rather natural to ask whether it is possible to characterize a projective connection R_{F} on X by simpler conditions like being real (see p. 224 in [15]) or Fuchsian. The answer to the question is negative.

Namely, for a compact genus-$g{>}1$ Riemann surface $X = \Gamma \setminus \mathbb{H}$, Goldman [7] showed that to every integral-measurable Thurston's lamination $\mu = \sum_i m_i \gamma_i$ (disjoint union), where γ_i are simple closed geodesics in the hyperbolic metric on X and the m_i are nonnegative integers, there corresponds a genus-g Riemann surface $Gr_\mu(X)$ with a projective connection $R(\mu)$ having the monodromy group Γ. Riemann surfaces $Gr_\mu(X)$ are obtained from X by the so-called grafting procedure that generalizes classic examples of Maskit–Hejhal and Sullivan–Thurston (see [7]). Moreover, the set of all Fuchsian projective connections on all genus-g Riemann surfaces is isomorphic to the direct product of the Teichmüller space T_g and the set of integral-measurable Thurston's laminations on a genus-g topological surface. It was proved in [16] that there are infinitely many Fuchsian projective connections on each genus-$g{>}1$ Riemann surface X.

Real projective connections on Riemann surfaces of type (g, n) were also studied by Faltings [15]. As shown for compact Riemann surfaces in [7], to each half-integral-measurable Thurston's lamination μ, there corresponds a genus-g Riemann surface $Gr_\mu(X)$ with a real projective connection.

3.2. Surfaces of type $(0, 4)$ and Smirnov's approach. The classics associated the uniformization problem of Riemann surfaces with differential equations. As a basic example, they considered the case of Riemann surfaces of type $(0, 4)$; the corresponding problem was to find an accessory parameter in Eq. (4) such that its monodromy group was a Fuchsian or Kleinian group (a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$). For the special case of real singular points, Klein [17] proposed an approach that uses Sturm's oscillation theorem, and Hilb proved [18] that Eq. (4) has a Fuchsian monodromy group for infinitely many values of the accessory parameter. Hilbert [19] reduced this problem to the study of a certain integral equation.

The problem of a real monodromy group of Eq. (4) with four real singular points was completely solved by Smirnov in his thesis [8], published in Petrograd in 1918 (its main content was presented in [9], [10]). Namely, we consider Eq. (4) with the singular points $z_1 = 0$, $z_2 = a$, $z_3 = 0$, and $z_4 = \infty$, where $0 < a < 1$. Writing the general solution of Eqs. (5) in the form

$$c_1 = 1 + \frac{1 + 2\lambda}{a}, \qquad c_2 = \frac{1 + 2\lambda}{a(a-1)}, \qquad c_3 = -\frac{a + 2\lambda}{a - 1},$$

where λ is the accessory parameter, and changing the dependent variable $y = \sqrt{z(z-a)(z-1)}\, u$, we obtain the equation

$$\frac{d}{dz}\left(p(z)\, \frac{dy}{dz}\right) + (z + \lambda) y = 0, \qquad p(z) = z(z-a)(z-1). \tag{10}$$

[1] Lecture at the Leningrad Branch of the Steklov Institute of Mathematics, 1982, unpublished.

Let $(y_i^{(1)}, y_i^{(2)})$ denote the standard basis in the solution space of (10), which in the neighborhood of a singular point z_i consists of normalized holomorphic solutions

$$y_i^{(1)}(z,\lambda) = 1 + \sum_{k=1}^{\infty} a_{ik}(z-z_i)^k, \quad i = 1,2,3,$$

$$y_4^{(1)}(z,\lambda) = \frac{1}{z} + \sum_{k=2}^{\infty} \frac{a_{4k}}{z^k}$$

and

$$y_i^{(2)}(z,\lambda) = y_i^{(1)}(z,\lambda)\log(z-z_i) + \tilde{y}_i(z,\lambda), \quad i = 1,2,3,$$

$$y_4^{(2)}(z,\lambda) = y_4^{(1)}(z,\lambda)\log\frac{1}{z} + \tilde{y}_4(z,\lambda),$$

where $\tilde{y}_i(z,\lambda)$ are holomorphic in the neighborhood of z_i. For real λ, the power series $y_i^{(1,2)}(z,\lambda)$ and $\tilde{y}_i(z,\lambda)$ have real coefficients.

To determine a real λ for which the monodromy group of Eq. (10) is real, the classics used the notion of a *real continuation*. Namely (see [8]), if we have

$$y(z) = c\log(a-z) + f(z)$$

in the neighborhood of a singular point a for real $z < a$, where $\log 1 = 0$ and the function $f(z)$ is holomorphic in a neighborhood of a, then the real continuation of y to the domain $z > a$ is defined as

$$y(z) = c\log(z-a) + f(z).$$

The following statement holds.

Theorem 1 (Klein, Hilbert)**.** *Equation* (10) *has a real monodromy group if λ is real and one of the following conditions holds*:

1. *The solution $y_0^{(1)}(z,\lambda)$ is holomorphic in a neighborhood of the singular point $z_2 = a$.*
2. *The solution $y_2^{(1)}(z,\lambda)$ is holomorphic in a neighborhood of the singular point $z_3 = 1$.*
3. *Under the real continuation through $z = a$, the solution $y_0^{(1)}(z,\lambda)$ is holomorphic in a neighborhood of the singular point $z_3 = 1$.*

Moreover, under conditions 1, 2, and 3, the respective ratios $\eta = \sqrt{-1}\, y_3^{(1)}/y_1^{(1)}$,$\eta = \sqrt{-1}\, y_2^{(1)}/y_4^{(1)}$, and $\eta = \sqrt{-1}\, y_1^{(1)}/y_2^{(1)}$ of linearly independent solutions of Eq. (10) are transformed by real fractional linear transformations when going around the singular points 0, a, and 1 (see [8] for details).

Conditions 1–3 determine three Sturm–Liouville type spectral problems for Eq. (10). Namely, we must determine the values of λ such that

1. there is a solution on the interval $[0, a]$ that is regular at 0 and a,
2. there is a solution on the interval $[a, 1]$ that is regular at a and 1, and
3. there is a solution regular at 0 such that it is regular at 1 under the real continuation through a.

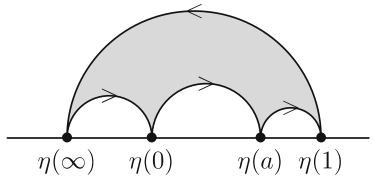

Fig. 1

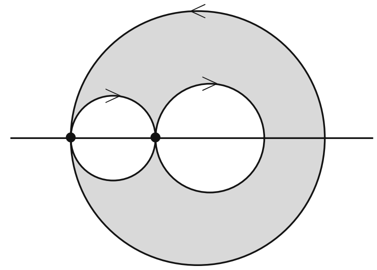

Fig. 2

Using the classical Sturm method, Smirnov proved the following result [8].

Theorem 2 (Smirnov)**.** *Each Sturm–Liouville problem* 1–3 *has a simple unbounded discrete spectrum. Namely, the following statements hold*:

1. *Spectral problem* 1 *has infinitely many eigenvalues* μ_k, $k \in \mathbb{N}$, *accumulating at* ∞ *and satisfying the inequalities*
$$-a < \mu_1 < \mu_2 < \dots.$$

2. *Spectral problem* 2 *has infinitely many eigenvalues* μ_{-k}, $k \in \mathbb{N}$, *accumulating at* $-\infty$ *and satisfying the inequalities*
$$-a > \mu_{-1} > \mu_{-2} > \dots.$$

3. *Spectral problem* 3 *has infinitely many eigenvalues* λ_k, $k \in \mathbb{Z}$, *accumulating at* $\pm\infty$ *and satisfying the inequalities*
$$\dots < \mu_{-2} < \lambda_{-1} < \mu_{-1} < \lambda_0 < \mu_1 < \lambda_1 < \mu_2 < \dots.$$

The case $\lambda = \lambda_0$ corresponds to the Fuchsian uniformization of the Riemann surface $X = \mathbb{C} \setminus \{0, a, 1\}$ and the ratio $\eta = \sqrt{-1}\, y_1^{(1)}/y_2^{(1)}$ bijectively maps the upper half-plane of z to the interior of a circular rectangle with zero angles and sides orthogonal to $\mathbb{R} \cup \{\infty\}$. Normalizing η by a real fractional linear transformation such that the images of all singular points 0, a, 1, and ∞ are finite, we obtain the rectangle in Fig. 1 (cf. Fig. 9 in [17]). Analytically continuing $\eta(z)$ to the lower half-plane of z, we obtain a multi-valued linearly polymorphic function $\eta\colon X \to \mathbb{H}$ with a Fuchsian group Γ such that $J = \eta^{-1}$ determines isomorphism (3).

The corresponding ratio η is also a bijective function on the upper half-plane of z in the cases $\lambda = \mu_{\pm 1}$. Hence, if $\lambda = \mu_1$, then we have $\eta(0) = \eta(a) = \infty$ and $\eta(1) = \eta(\infty) = 0$. Normalizing $\eta = \sqrt{-1}\, y_3^{(1)}/y_1^{(1)}$ such that the images of the singular points are finite, we obtain a bijective map η of the upper half-plane of z onto the interior of the degenerate circular rectangle in Fig. 2. The corresponding monodromy groups are Schottky groups.

For all other values of λ_k and μ_k, the corresponding map η is no longer a bijective map on the upper half-plane of z. Hence, if $\lambda = \lambda_1$, then the upper half-plane is mapped onto the interior of the annulus in Fig. 3 (cf. Fig. 10 in [17]). Here, the function η takes the values twice from the marked darker domain, which corresponds to the rectangle in Fig. 1. If $\lambda = \lambda_k$, then this rectangle is wrapped over itself $2|k|$ times.

Similarly, if $\lambda = \mu_2$, then the upper half-plane of z maps onto the interior of the annulus in Fig. 4. Here, the function η takes the values twice from the marked darker domain, which corresponds to the degenerate rectangle in Fig. 2. If $\lambda = \mu_k$, then this rectangle is wrapped over itself $|k|$ times.

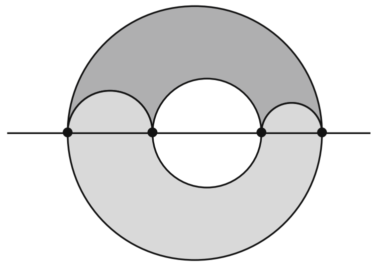

Fig. 3

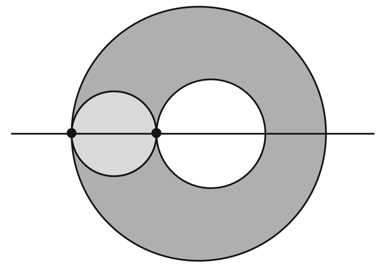

Fig. 4

It is instructive to compare these results of Smirnov with Goldman's classification of Fuchsian and real projective connections on Riemann surfaces generalized to surfaces of type (g, n). The Fuchsian series $\lambda = \lambda_k$ corresponds to integral laminations in [7], while the series $\lambda = \mu_k$ corresponds to half-integral laminations.

4. Black-hole-type solutions of the Liouville equation

The Fuchsian uniformization of a Riemann surface X determines a solution of the Liouville equation: a smooth function φ on X satisfying Eq. (8) and having asymptotic behavior (9) (see Sec. 2.3). The function φ is obtained from the ratio J^{-1} of linearly independent solutions of Eq. (4) by formula (7). This formula is well defined because of the condition that the monodromy group Γ of Eq. (4) is real; the smoothness of φ is ensured by the condition that Γ uniformizes the Riemann suface X and its image under the multivalued map J^{-1} is the upper half-plane $\mathbb{H}$.

Similarly, with each Eq. (4) with a real monodromy group, we associate a solution of the Liouville equation. We set

$$e^{\varphi(z)} = \frac{|\eta'(z)|^2}{(\operatorname{Im}\eta(z))^2}, \tag{11}$$

where η is the ratio of linearly independent solutions of Eq. (4), which transforms by fractional linear transformations when going around the singular points ($\eta = J^{-1}$ in the Fuchsian case). The function φ is well defined because of the realness of the monodromy group and has asymptotic behavior (9). The latter follows from the theory of Fuchsian equations with equal exponents. But solution (11) is no longer smooth: the image of X under the multivalued map η has a nontrivial intersection with the real axis, and the function φ is singular on $\eta^{-1}(\mathbb{R})$.

Namely, it follows from results in [15] (see Sec. 6 there) that the inverse image $\eta^{-1}(\mathbb{R})$ is a disjoint union of finitely many simple closed analytic curves on X. Let C be one such curve. There is a branch of the multivalued function η that maps C bijectively onto the circle, and hence $C = \{z = \eta^{-1}(t),\ t \in [\alpha, \beta]\}$. It is convenient to introduce the *Schwarz function* S of the analytic contour C by the formula

$$S = \bar{\eta}^{-1} \circ \eta,$$

where $\bar{\eta}^{-1}(z) = \overline{\eta^{-1}(\bar{z})}$. The Schwarz function is defined in some neighborhood of the contour C and determines it by the equation $\bar{z} = S(z)$ (see [20]). It is easy to show that in terms of the Schwarz function, the solution φ has the same singularities on C as the function

$$-\frac{4\overline{S'(z)}}{(z - \overline{S(z)})^2}. \tag{12}$$

Namely, as $z \to z_0 \in C$ along any direction not tangent to C,

$$e^{\varphi(z)} = -\frac{4S'(z_0)}{(\bar{z} - \bar{z}_0 - S'(z_0)(z - z_0))^2}(1 + O(|z - z_0|)). \tag{13}$$

We note that because of the condition $\overline{S}(S(z)) = z$, the function in the right-hand side of (13) is real and positive. The singularities of type (12), (13) on a contour C are similar to the singularity on $\mathbb{R}$ of the Poincaré metric on $\mathbb{C} \setminus \mathbb{R}$, which corresponds to the Schwarz function $S(z) = z$.

We can therefore state the following problem: on the Riemann surface $X = \mathbb{C} \setminus \{z_1, \ldots, z_{n-3}, 0, 1\}$, find simple analytic contours $C_1, \ldots, C_k$ and a function φ such that on $X \setminus \bigcup_{j=1}^{k} C_j$, the function φ satisfies Liouville equation (8), has asymptotic behavior (9) at the punctures z_i, and has singularities of type (12), (13) on the contours C_j. On each connected component of X, $e^{\varphi(z)}\,|dz|^2$ determines a complete metric of constant negative curvature -1. The boundary C_j can be interpreted as the horizon of a black hole, and we therefore call corresponding solutions of the Liouville equation solutions of the *black-hole type*. It follows from Goldman's classification of real projective connections [7] that there exists a family of such solutions parameterized by the "integral lattice" of integral and measurable Thurston's laminations, implicitly defined by the grafting procedure.

From the results in Smirnov's thesis, we obtain a rather explicit description of black-hole-type solutions in the case of four real singular points. Namely, we obtain the following result from Theorem 2.

Theorem 3. *All black-hole-type solutions of the Liouville equation with four real punctures* 0, a, 1, *and* ∞ *are described as follows*:

1. *Solutions of the Fuchsian type, which correspond to the values of the accessory parameter* $\lambda = \lambda_k$ *with integer* k *and have* $2|k|$ *contours* C_j: *these contours go over the points* 0 *and* a *if* $k > 0$ *and over the points* a *and* 1 *if* $k < 0$.

2. *Solutions of the Schottky type, which correspond to the values of the accessory parameter* $\lambda = \mu_k$ *with integer* $k \neq 0$ *and have* $2|k| - 1$ *contours* C_j: *these contours go over the points* 0 *and* a *if* $k > 0$ *and over the points* a *and* 1 *if* $k < 0$.

In the general case, it is convenient to substitute $\chi(z) = e^{-\varphi(z)/2}$, which transforms Liouville equation (8) into

$$-\chi\,\chi_{z\bar{z}} + |\chi_z|^2 = \frac{1}{4} \tag{14}$$

and asymptotic behavior (9) into

$$\chi(z) = \begin{cases} |z - z_i|\,|\log|z - z_i||(1 + o(1)), & z \to z_i, \quad i \neq n, \\ |z|\log|z|(1 + o(1)), & z \to \infty. \end{cases} \tag{15}$$

Singularities (12) transform into the vanishing condition on the contour C,

$$\chi(z) \sim \frac{z - \overline{S(z)}}{2\sqrt{-\overline{S'(z)}}}, \tag{16}$$

and the real-valued function $\chi(z)$ hence changes sign under the Schwarz reflection $z^* = S(z)$ through C. Elliptic partial differential equation (14) with asymptotic behavior (15) and vanishing conditions (16) on the contours C_j is a boundary value problem with a free boundary. It would be interesting to use the method of continuation with respect to a parameter together with the a priori estimates to solve it, as was done in [3] for Liouville equation (8) with asymptotic behavior (9).

In conclusion, we note that the function χ plays an important role in the theory of the Liouville equation. Namely, it is a bilinear form in solutions of Eq. (4) and their complex conjugates and satisfies Eq. (4)

$$\chi_{zz} + \frac{1}{2}T_{\varphi}\chi = 0 \tag{17}$$

and the complex conjugate equation. In the quantum Liouville theory, the field $\chi = e^{-\varphi(z)/2}$ describes a vector degenerate at the level 2 in a Verma module for the Virasoro algebra. For black-hole-type solutions, the function χ still satisfies Eq. (17). It would be interesting to elucidate what role it plays in the quantum Liouville theory.

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