MAT 127 Calculus C Spring 2003
Midterm II Solutions

1. (a) (10 points) Express the number $1.\overline{75} = 1.757575\ldots$ as a ratio of integers.

Solution

$$\begin{aligned}1.757575\ldots &= 1 + \frac{75}{100} + \frac{75}{100^2} + \cdots = 1 + \sum_{n=1}^{\infty} \frac{75}{100^n} \\ &= 1 + \frac{75}{100} \cdot \frac{1}{1 - 1/100} = 1 + \frac{75}{99} = \frac{174}{99},\end{aligned}$$

where we have used that the sum is geometric series with $a = 75/100$ and $r = 1/100$. The answer is represented as the ratio of integers (which is a fraction).

(b) (5 points) Find value of q if

$$\sum_{n=0}^{\infty} q^n = 2$$

Solution

This is geometric series with $a = 1$ and $r = q$. Assuming that the series converges, we find that its sum is $1/(1-q)$. Thus we have an equation

$$\frac{1}{1-q} = 2, \quad \text{that is} \quad 1 = 2 - 2q, \quad \text{or} \quad 2q = 1,$$

so that $q = 1/2$ (and the series indeed converges).

2. Determine whether each of the following sequences converges or diverges. If it converges, find the limit. In either case, justify your answer.

Solutions

(a) (3 points)

$$a_n = \frac{7n^3 + 5n}{13n^3 + n^2}$$

It converges and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{7 + 5/n^2}{13 + 1/n} = \frac{7}{13}.$$

(b) (3 points)

$$a_n = \sin\left(\frac{\pi}{2} + \pi n\right)$$

It diverges since actually (known properties of the values of the sin function) $a_n = (-1)^n$, i.e., it is 1 for even n and -1 for odd n.

(c) (3 points)

$$a_n = \ln(n+2) - \ln n$$

It converges and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{n+2}{n}\right) = \lim_{n\to\infty} \ln\left(1 + \frac{2}{n}\right) = \ln 1 = 0.$$

(d) (3 points)

$$a_n = \frac{\cos^2 n}{2^n}$$

It converges by the squeeze theorem: $0 < \cos^2 n < 1$ and $\lim_{n\to\infty} 1/2^n = 0$, so that the sequence converges to 0.

(e) (3 points)

$$a_n = \frac{\ln n}{n}$$

It converges by the L'Hospital rule, since $x' = 1$, $(\ln x)' = 1/x$ and

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0.$$

3. (40 points) Determine whether each of the following series is convergent or divergent. Justify your answer and state which test (Integral, Comparison, p-Series, etc.) you are using. NOTE: if the series converges, you do not need to find its sum.

Solutions

(a) (5 points)

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

It is convergent by the comparison test with convergent p-Series $b_n = 1/n^2$ with $p = 2$, because

$$\frac{1}{n^2+1} < \frac{1}{n^2}$$

for all n.

(b) (5 points)

$$\sum_{n=1}^{\infty} \frac{6}{n^3 - 7n + 15}$$

It is convergent by the limit comparison test with convergent p-Series $b_n = 1/n^3$ with $p = 3$, because

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{6n^3}{n^3-7n+15} = \lim_{n\to\infty}\frac{6}{1-7/n^2+10/n^3} = 6 > 0$$

(c) (4 points)

$$\sum_{n=1}^{\infty}\tan^{-1} n$$

It is divergent by the divergence test since $\lim_{n\to\infty} a_n = \pi/2 \neq 0$.

(d) (4 points)

$$\sum_{n=1}^{\infty}\frac{\sin^2 n}{3^n}$$

It is convergent by comparison test with convergent geometric series $b_n = 1/3^n$ with $r = 1/3$, because

$$0 < \frac{\sin^2 n}{3^n} < \frac{1}{3^n}$$

for all n.

(e) (7 points)

$$\sum_{n=1}^{\infty} ne^{-n^2}$$

It is convergent by the integral test with $f(x) = xe^{-x^2}$. Indeed, $f(x)$ is continuous, positive and decreasing on $[1,\infty)$ since

$$f'(x) = e^{-x^2} - 2x^2e^{-x^2} = (1-2x^2)e^{-x^2} < 0$$

whenever $x \geq 1$. Using the substitution $u = x^2$, we get

$$\int_1^{\infty} f(x)dx = \lim_{t\to\infty}\int_1^t xe^{-x^2}dx = \lim_{t\to\infty}\int_1^{t^2} e^{-u}du$$
$$= \lim_{t\to\infty}\frac{1}{2}\left(e^{-1}-e^{-t^2}\right) = e^{-1}/2$$

— the improper integral is convergent.

(f) (6 points)

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{4^n}$$

It is convergent by the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2 4^n}{n^2 4^{n+1}} = \frac{1}{4}\frac{(n+1)^2}{n^2},$$

so that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{4}\lim_{n\to\infty}\frac{(n+1)^2}{n^2} = \frac{1}{4} < 1.$$

(g) (4 points)

$$\sum_{n=1}^{\infty}\frac{\ln^2 n}{n}$$

It is divergent by the comparison test with harmonic series $b_n = 1/n$ (harmonic series is divergent), because $\ln^2 > 1$ for $n \geq 3$ (since $\ln e = 1$ and $e = 2.71...$) and

$$\frac{\ln n}{n} > \frac{1}{n}$$

whenever $n \geq 3$.
NOTE One can also use the integral test applied to the function $f(x) = \ln^2 x/x$ and use the sustitution $u = \ln x$, though it would require more work.

(h) (5 points)

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$$

It is convergent by the alternating series test: $a_n = (-1)^n b_n$ with $b_n = 1/\sqrt{n}$. It clearly satisfies (i) $b_n > 0$; (ii) b_n is decreasing; (iii) $b_n \to 0$ as $n \to \infty$, so that the test is applicable.

4. (10 points) Determine whether the series

$$\sum_{n=2}^{\infty}\frac{1}{n(n-1)}$$

is convergent or divergent. If it is convergent, find its sum.
Solution
It is convergent telescopic series. Indeed, by partial fractions,

$$\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$$

so that

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n}\right)$$
$$= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots = 1$$

since $s_n = 1 - 1/n$ converges to 1.

5. (a) (10 points) Show that the series is convergent. How many terms of the series do we need to add in order to find the sum to the indicated accuracy?

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt[3]{n}} \quad (|\text{error}| < 0.0001).$$

Solution

It is alternating series with $a_n = (-1)^{n-1}b_n$ and $b_n = 1/\sqrt[3]{n}$. Clearly, (i) $b_n > 0$; (ii) b_n is decreasing; (iii) $b_n \to 0$ as $n \to \infty$, so that the alternating series test is applicable. By the remainder estimate for the alternating series test,

$$|R_n| \le b_{n+1}.$$

Thus the desired accuracy will be achieved if

$$b_{n+1} = \frac{1}{\sqrt[3]{n+1}} = (n+1)^{-1/3} < 0.0001, \quad \text{or} \quad (n+1)^{1/3} > 10000 = 10^4.$$

Solving for n we get $n+1 > (10^4)^3 = 10^{12}, \quad n > 10^{12} - 1$. Thus 10^{12} terms will be sufficient.

(b) (10 points) It is given that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Give an estimate for the error $|s_{999} - \pi^2/6|$. Is s_{999} greater or less than $\pi^2/6$?

Solution

By the remainder estimate for the integral test,

$$R_n \le \int_n^{\infty} f(x)dx, \text{ where } f(x) = \frac{1}{x^2}$$

and $|\pi^2/6 - s_{999}| \le R_{999}$. We have, doing this integral,

$$\int_n^t \frac{dx}{x^2} = \frac{1}{n} - \frac{1}{t} \longrightarrow \frac{1}{n}$$

as $t \to \infty$. Thus $R_{999} \leq 1/999$ and

$$0 < \pi^2/6 - s_{999} \leq 1/999,$$

since obviuosly s_{999} is less than $s = \pi^2/6$ — the sum of the series (since to get the sum of the series we need to add more terms, which are all positive).