MAT 127 Calculus C Spring 2003
Midterm I Solutions

Name:____

I.D.:____________________________ **Section number:** _________

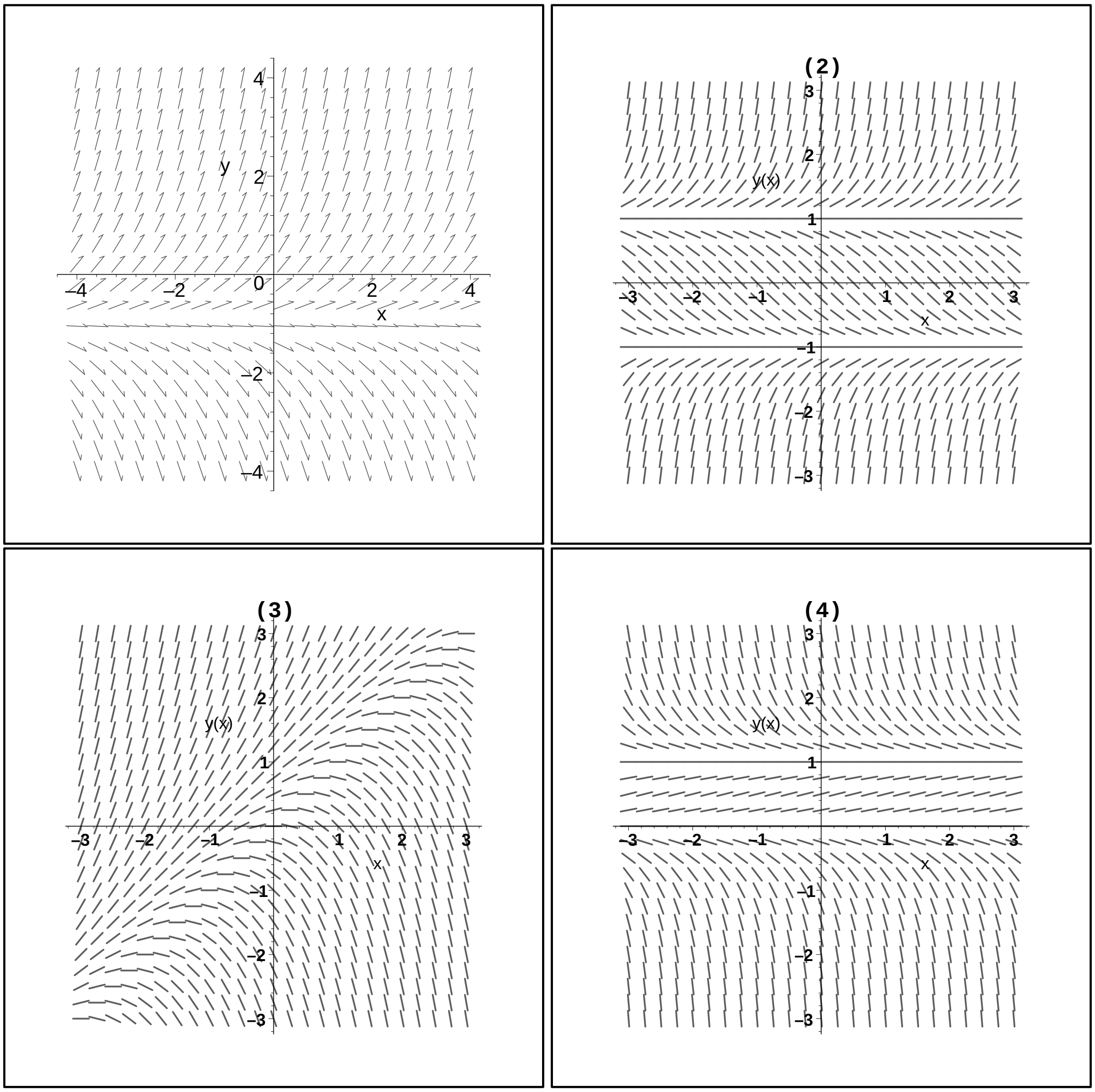
1	2	3	4	5	6	TOTAL
16 points	15 points	18 points	16 points	15 points	20 points	100 points

No books, notes, or calculators!

Please answer each question in the space provided and on the reverse side of the sheets. Show your work whenever possible. Unless otherwise indicated, **answers without justification will get little or no partial credit!** Cross out anything that grader should ignore and circle or box the final answer. Double-check your answers and remember to include units in word problems! Check that your exam has all 6 problems.

1. Shown below are four differential equations and four direction fields.

(a) $y' = y - x$ (b) $y' = y + 1$ (c) $y' = y(1 - y)$ (d) $y' = y^2 - 1$



(i) (8 points) Match the equations and the corresponding direction fields (no justification required).

(ii) (8 points) In each case, find all equilibrium solutions.

Solution (both parts, (i) and (ii))

First, look at the equilibrium solutions: equation (a) has none ($y = x$ is NOT an equilibrium solution!), (b) has $y = -1$, (c) has $y = 0$ and $y = 1$, (d) has $y = \pm 1$. Checking against direction fields we easily see that (a) goes with (3), (b) goes with (1), (c) — with (4), and (d) — with (2).

2. (i) (10 points) For what values of k does the function $y(x) = \sin kx$ satisfy the differential equation

$$y'' + 16y = 0$$

(ii) (5 points) Repeat part (i) for the function $y(x) = \cos kx$.

Solution Part (i) We successively compute

$$y' = k\cos kx,$$
$$y'' = -k^2 \sin kx.$$

This gives

$$y'' + 16y = -k^2 \sin kx + 16 \sin kx = (-k^2 + 16) \sin kx = 0$$

for all x. Therefore, either $k = 0$, or

$$k^2 - 16 = 0$$

and $\boxed{k = 0 \text{ and } \pm 4}$.

Part (ii) Similarly, $y' = -k \sin kx$ and $y'' = -k^2 \cos kx$ and we get

$$y'' + 16y = -k^2 \cos kx + 16 \cos kx = (-k^2 + 16) \cos kx = 0,$$

so that $k^2 - 16 = 0$, $\boxed{k = \pm 4}$.

3. Consider the differential equation

$$y' = xe^{-y}$$

(i) (12 points) Find all solutions $y(x)$.

(ii) (6 points) Find the solution to the initial value problem $y(1) = 0$.

Solution

Part (i). We have the equation for differentials

$$e^y dy = x dx$$

(note that $\frac{1}{e^{-y}} = e^y$!) so that

$$\int e^y dy = \int x dx,$$

which can be is easily integrated

$$e^y = \frac{x^2}{2} + C.$$

Taking the logarithms (Note that $\ln(a + b) \neq \ln a + \ln b$, NEVER use this kind of wrong formulas!), we get

$$\boxed{y = \ln\left(\frac{x^2}{2} + C\right)}$$

Part(ii). If $y(1) = 0$, then, using part (i), $y(1) = \ln(\frac{1}{2} + C) = 0$. Exponentiating (and using $e^0 = 1$) we get $\frac{1}{2} + C = 1$, so that $C = \frac{1}{2}$ and the solution to the initial value problem is

$$\boxed{y = \ln\left(\frac{x^2+1}{2}\right)}$$

4. (16 points) Find the orthogonal trajectories to the family

$$x^2 + y^2 = k$$

Solution

First, we obtain the differential equation for the family $x^2 + y^2 = k$. Taking the derivative of this equation (note that k is a constant) we get

$$2x + 2y\frac{dy}{dx} = 0,$$

or

$$\frac{dy}{dx} = -\frac{x}{y}.$$

The slope of an orthogonal family is negative reciprocal of the slope of the given family, so that

$$\frac{dy}{dx} = \frac{y}{x}$$

is the differential equation for the orthogonal family. Separating variables we get

$$\frac{dy}{y} = \frac{dx}{x}$$

and

$$\int \frac{dy}{y} = \int \frac{dx}{x}.$$

Integrating

$$\ln|y| = \ln|x| + C$$

and exponentiating we get

$$|y| = e^C|x| \quad \text{or} \quad y = Ax \quad \text{where} \quad A = \pm e^C \quad \text{or} \quad 0.$$

(Note that the latter case corresponds to the equilibrium solution $y = 0$ of the differential equation $y' = y/x$.) Thus the orthogonal trajectories form a family of straight lines through the origin $\boxed{y = Ax}$.

5. In the year 1800, £1,000 (British pounds) was deposited into the Bank of England, which was using continuous compounding of interest. In the year 1840, the income on this investment was 300%.

(i) (7 points) When did the investment double in value?
(ii) (5 points) Find the interest rate r (use $\ln 2 \sim 0.69$)
(iii) (3 points) When was the investment equal to £8,000?

Solution

Part (i) In the year 1840 the total value of investment was £4,000 (the value of the principal plus the income). To see that this is so, imagine that the value of the investment in the year 1840 was still £1,000. Then you would say that the income was 0% rather then 100% (and you would be rather disappointed). Let r be the interest rate, then

$$A(t) = 1,000\, e^{rt}$$

— the value of the investment in the year $1,800 + t$. It is given that

$$1,000\, e^{40r} = 4,000, \quad \text{or} \quad e^{40r} = 4.$$

For the doubling time we have $1,000\, e^{rT} = 2,000$, or $e^{rT} = 2$. We get from here that $T = 20$ years.

$$\boxed{\text{In the year 1820}}$$

Part (ii) We have from part (i) that $e^{40r} = 4$, or

$$40r = \ln 4 = 2\ln 2, \quad \text{so that} \quad r = \frac{\ln 2}{20} = 0.0345.$$

Thus the interest rate is

$$\boxed{r = 3.45\%}$$

Part (iii) Solving for t in $A(t) = 1,000\, e^{rt} = 8,000$, we get $e^{rt} = 8$. Comparing with $e^{40r} = 4$ and using that $8 = 2^3 = 4^{3/2}$ we get $t = 60$ years.

$$\boxed{\text{In the year 1860}}$$

6. A tank contains 300 L of pure water. Solution that contains 0.2 kg of salt per liter of water enters the tank at a rate of 10 L/min. From another source, pure water enters the tank at a rate 5 L/min. The solution is kept thoroughly mixed and drains from the tank at a rate 15 L/min.

(i) (10 points) Write a differential equation for $y(t)$, the amount of salt in the tank at time t (t is measured in minutes, $y(t)$ is measured in kilograms). What initial condition does $y(t)$ satisfy?

(ii) (10 points) Solve the initial value problem from part (i) and find the time when the concentration of salt in the tank is 0.1 kg/L.

Solution

Part (i) As usual,

$$\frac{dy}{dt} = \text{Rate in} - \text{Rate out}.$$

The rate in is 0.2 kg/L × 10 L/min = 2 kg/min (note that the second source brings pure water into the tank so that it does not contribute to the rate in). The rate out is

$$\frac{y(t)}{300}\ \text{kg/L} \times 15\ \text{L/min} = \frac{y(t)}{20}\ \text{kg/min}$$

and the differential equation for $y(t)$ is

$$\boxed{\frac{dy}{dt} = 2 - \frac{y}{20}}$$

with the initial condition

$$\boxed{y(0) = 0}$$

Part (ii) Writing the differential equation as

$$\frac{dy}{dt} = \frac{40 - y}{20},$$

separating the variables

$$\frac{dy}{40 - y} = \frac{dt}{20}$$

and integrating

$$\int \frac{dy}{40 - y} = \int \frac{dt}{20}$$

we get

$$-\ln|40 - y| = \frac{t}{20} + C, \quad \text{or} \quad |40 - y| = Ae^{-t/20},\ A = e^{C}.$$

Using $y(0) = 0$ we get $A = 40$ and $40 - y = 40e^{-t/20}$. (Note that $y(t)$ is always less then 40 since at $t = 0$ $y(0) = 0 < 40$.)

Solving for y,

$$\boxed{y(t) = 40(1 - e^{-t/20})}$$

Finally, when concentration of salt is 0.1 kg/L, the amount of salt is $300 \times 0.1 = 30$ kg, and the equation $y(t) = 30$ becomes

$$10 = 40e^{-t/20} \quad \text{or} \quad e^{-t/20} = \frac{1}{4}.$$

Thus $-t = 20 \ln(1/4) = -40 \ln 2$ and

$$\boxed{t = 40 \ln 2 = 27.6 \text{ min}}$$