MAT 127 Calculus C Spring 2003
Practice Midterm II Solutions

1. Determine whether each of the following sequences converges.

(a)

$$\left\{\frac{\sqrt{n}}{\sqrt{n}+1}\right\}_{n=1}^{\infty}$$

(b)

$$\left\{\frac{n^2+2}{n+1000}\right\}_{n=5}^{\infty}$$

(c)

$$\left\{e^{1+1/n}\right\}_{n=1}^{\infty}$$

(d)

$$\left\{\frac{\sin^2 n}{n}\right\}_{n=1}^{\infty}$$

(e)

$$\left\{\frac{\ln(n^2)}{n}\right\}_{n=1}^{\infty}$$

Solution

(a)

$$\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n}+1}=\lim_{n\to\infty}\frac{1}{1+1/\sqrt{n}}=1$$

— converges by the ratio law for limits.

(b)

$$\lim_{n\to\infty}\frac{n^2+2}{n+1000}=\lim_{n\to\infty}\frac{n+2/n}{1+1000/n}=\infty$$

— diverges by the ratio law for limits.

(c)

$$\lim_{n\to\infty}e^{1+1/n}=e$$

— converges since e^x is continuous (at $x=1$).

(d)

$$\lim_{n\to\infty}\frac{\sin^2 n}{n}=0$$

— converges by the squeeze theorem since $0<\sin^2 n<1$.

(e) Using l'Hospital rule,

$$\lim_{n\to\infty}\frac{\ln(n^2)}{n}=2\lim_{n\to\infty}\frac{\ln n}{n}=2\lim_{x\to\infty}\frac{\ln x}{x}=2\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=0$$

2. Determine whether each of the following series is convergent or divergent. Justify your answer and state which test (Integral, Comparison, p-Series, etc.) you are using. NOTE: if the series converges, you do not need to find its sum.

(a)
$$\sum_{n=1}^{\infty} \frac{10000}{1+2^n}$$

(b)
$$\sum_{n=1}^{\infty} \frac{3^{n+5}}{4^n}$$

(c)
$$\sum_{n=0}^{\infty} (-1)^n n$$

(d)
$$\sum_{n=10}^{\infty} \frac{5}{n^2-2n+1}$$

(e)
$$\sum_{n=1} (-1)^n n^{-3/2}$$

(f)
$$\sum_{n=1}^{\infty} \frac{\cos(2n^2)}{n^2}$$

(g)
$$\sum_{n=5}^{\infty} \frac{1}{n \ln n}$$

Solution

(a)
$$\sum_{n=1}^{\infty} \frac{10000}{1+2^n}$$

— converges by comparison with geometric series: $a_n < 1000/2^n$.

(b)
$$\sum_{n=1}^{\infty} \frac{3^{n+5}}{4^n}$$

— converges; it is geometric series with $r = 3/4$.

(c)

$$\sum_{n=0}^{\infty}(-1)^n n$$

— diverges, since $\lim_{n\to\infty} a_n$ does not exist.

(d)

$$\sum_{n=10}^{\infty}\frac{5}{n^2-2n+1}$$

— converges by the limit comparison test with the p-Series $b_n = 5/n^2$ with $p = 2$.

(e)

$$\sum_{n=1}(-1)^n n^{-3/2}$$

— converges by the test for the absolute convergence, since $|a_n| = 1/n^{3/2}$ — p-Series with $p = 3/2$.

(f)

$$\sum_{n=1}^{\infty}\frac{\cos(2n^2)}{n^2}$$

— converges by the test for the absolute convergence and comparison test. since $|a| < 1/n^2$ (because $|\cos(2n^2)| < 1$) — the p-Series with $p = 2$.

(g)

$$\sum_{n=5}^{\infty}\frac{1}{n\ln n}$$

— diverges by the integral test for $f(x) = 1/x\ln x$. The test is applicable since $f(x)$ is continuous, positive and decreasing on $[5,\infty)$ (because each of the functions x and $\ln x$ is increasing there). Using the substitution $u = \ln x$ we get:

$$\int_5^t \frac{1}{x\ln x}dx = \ln(\ln x)_5^t = \ln(\ln t) - \ln\ln 5$$

and it has no limit (goes to ∞) as $t\to\infty$.

4. Find the sum of the series

(a)

$$\sum_{n=1}^{\infty}\frac{3^n+2^n}{6^n}$$

(b)

$$\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$$

Solution

(a) It is a sum of two geometric series, one with $a = r = 1/2$ and another with $a = r = 1/3$. Therefore,

$$s = \frac{1}{2}\frac{1}{1-1/2} + \frac{1}{3}\, rac11 - 1/3 = 1 + \frac{1}{2} = \frac{3}{2}.$$

(b) Factoring $n^2+4n+3 = (n+1)(n+3)$ and using the partial fraction expansion

$$\frac{2}{(n+1)(n+3)} = \frac{1}{n+1} - \frac{1}{n+3}$$

we see that it is a telescopic series, convergent since $\lim_{n\to\infty} \frac{1}{n+3} = 0$. Writing down several terms and observing the cancellation we see that

$$s = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

.

5. Determine whether the following series is convergent or divergent.

$$\sum_{n=1} n^2 e^{-n^3}$$

Solution

Converges by the integral test for $f(x) = x^2 e^{-x^3}$. The test is applicable since $f(x)$ is continuous, positive and decreasing on $[1, \infty)$. The latter follows from the derivative test:

$$f'(x) = 2xe^{-x^3} - 3x^4 e^{-x^3} = (2-3x^3)xe^{-x^3} < 0$$

whenever $x \geq 1$. Using the substitution $u = x^3$ we get

$$\int_1^t x^2 e^{-x^3} dx = \frac{1}{3}\int_1^{t^3} e^{-u} du = \frac{1}{3}(e^{-1} - e^{-t^3}) \longrightarrow e^{-1}/3 \text{ as } t \to \infty.$$

6. It is given that

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Let $A = 1 + 1/2^2 + 1/3^2 + \cdots + 1/1000^2$. Estimate the difference $|\pi^2/6 - A|$. Is A greater than or less than $\pi^2/6$?

Solution

By the remainder estimate for the integral test,

$$0 < \pi^2/6 - A = \pi^2/6 - s_{1000} = R_{1000} \le \int_n^\infty \frac{dx}{x^2}.$$

We have

$$\int_n^\infty \frac{dx}{x^2} = \lim_{t\to\infty} \int_n^t \frac{dx}{x^2} = \lim_{t\to\infty} (-1/t + 1/n) = 1/n.$$

Therefore $|\pi^2/6 - A| = R_{1000} \le 1/1000 = 0.001$ and A is less than $\pi^2/6$.

y. Show that the series is convergent. How many terms of the series do we need to add in order to find the sum with indicated accuracy?

(a)

$$\sum_{n=1}^\infty \frac{(-1)^n}{n^2} \quad (|\text{error}| < 0.0001)$$

(b)

$$\sum_{n=1}^\infty \frac{(-1)^{n-1}}{\sqrt{n}} \quad (|\text{error} < 0.000005)$$

Solution

(a) It is alternating series which converges by the the alternating series test: $a_n = (-1)^n b_n$ with $b_n = 1/n^2$, which is clearly (i) positive; (ii) decreasing; (iii) goes to 0 as n goes to ∞. By the remainder estimate for the alternating series test, $|R_n| \le b_{n+1}$. Setting

$$|R_n| \le b_{n+1} = \frac{1}{(n+1)^2} < 0.0001 = 10^{-4}$$

and solving for n we get

$$(n+1)^2 > 10^4, \quad \text{or} \quad n+1 > 100,\ n > 99.$$

Thus we need to add 100 terms.

(b) It is alternating series which converges by the the alternating series test: $a_n = (-1)^n b_n$ with $b_n = 1/\sqrt{n}$, which is clearly (i) positive; (ii) decreasing; (iii) goes to 0 as n goes to ∞. By the remainder estimate for the alternating series test, $|R_n| \le b_{n+1}$. Setting

$$|R_n| \le b_{n+1} = \frac{1}{\sqrt{n+1}} < 0.000005 = 5 \cdot 10^{-6}$$

and solving for n we get

$$\sqrt{n+1} > \frac{10^6}{5} = 2\cdot 10^5, \quad \text{or} \quad n+1 > 4\cdot 10^{10},\ n > 4\cdot 10^{10} - 1.$$

Thus we need to add $4\cdot 10^{10}$ terms (which is a lot!).

8. Express the number 5.404040404040... as a ratio of integers.

Solution

$$5.404040\ldots = 5 + 40/100 + 40/100^2 + 40/100^3 + \cdots = 5 + 40\sum_{n=1}^{\infty}\frac{1}{100^n}$$

$$= 5 + \frac{40}{1-1/100} = 5 + 40/99 = 535/99.$$

9. A sequence is defined recursively as

$$a_1 = 1,\ a_{n+1} = \frac{n+1}{2n+3}a_n.$$

Determine whether the series

$$\sum_{n=1}^{\infty} a_n$$

converges.

Solution

Clearly all a_n are positive and

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{2n+3}a_n.$$

Applying the ratio test, we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{n+1}{2n+3} = \frac{1}{2} < 1$$

so that the series converges.

10. A hard rubber ball has the property that after being dropped from a height h onto a hard surface, it bounces back up to a height rh, where $r < 1$. If the ball is dropped from an initial height of H meters, and continues to bounce indefinitely, what will be the total distance that the ball travels?

Solution

The distance traveled by the ball is

$$H + rH + rH + r^2H + r^2H + \cdots = H + \sum_{n=1}^{\infty} 2Hr^n = H + \frac{2Hr}{1-r} = \frac{H+Hr}{1-r}.$$