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Potentials and Chern forms for Weil–Petersson and Takhtajan–Zograf metrics on moduli spaces

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A B S T R A C T

For the TZ metric on the moduli space $\mathscr{M}_{0,n}$ of n-pointed rational curves, we construct a Kähler potential in terms of the Fourier coefficients of the Klein's Hauptmodul. We define the space $\mathfrak{S}_{g,n}$ as holomorphic fibration $\mathfrak{S}_{g,n} \to \mathfrak{S}_g$ over the Schottky space $\mathfrak{S}_g$ of compact Riemann surfaces of genus g, where the fibers are configuration spaces of n points. For the tautological line bundles $\mathscr{L}_i$ over $\mathfrak{S}_{g,n}$, we define Hermitian metrics h_i in terms of Fourier coefficients of a covering map J of the Schottky domain. We define the regularized classical Liouville action S and show that $\exp\{S/\pi\}$ is a Hermitian metric in the line bundle $\mathscr{L} = \otimes_{i=1}^n \mathscr{L}_i$ over $\mathfrak{S}_{g,n}$. We explicitly compute the Chern forms of these Hermitian line bundles

$$c_1(\mathscr{L}_i, h_i) = \frac{4}{3}\omega_{\mathrm{TZ},i}, \quad c_1(\mathscr{L}, \exp\{S/\pi\}) = \frac{1}{\pi^2}\omega_{\mathrm{WP}}.$$

We prove that a smooth real-valued function $-\mathscr{S} = -S + \pi\sum_{i=1}^n \log h_i$ on $\mathfrak{S}_{g,n}$, a potential for this special difference of WP and TZ metrics, coincides with the renormalized hyper-

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bolic volume of a corresponding Schottky 3-manifold. We extend these results to the quasi-Fuchsian groups of type (g, n).

Contents

1. Introduction

Weil introduced the Weil–Petersson (WP) metric on the moduli spaces of Riemann surfaces by using the Petersson inner product on the holomorphic cotangent spaces, the complex vector spaces of cusp forms of weight 4. Ahlfors proved that the WP metric is Kähler and its Ricci, holomorphic sectional and scalar curvatures are all negative [1,2], and Wolpert found a closed formula for the Riemann tensor of the WP metric and obtained explicit bounds for its curvatures [17].

In [19,20] it was shown that for the moduli space $\mathscr{M}_{0,n}$ of marked Riemann surfaces of type $(0, n)$, $n > 3$ (n-pointed rational curves) and for the Schottky space $\mathfrak{S}_g$ of compact Riemann surfaces of genus $g > 1$ the WP metric has global Kähler potential, the so-called *classical Liouville action* (for precise definitions, see Sects. 2 and 3). In [12,13] a new Kähler metric was introduced on the moduli space $\mathfrak{M}_{g,n}$ of Riemann surfaces of genus g with $n > 0$ punctures, $3g - 3 + n > 0$. In [8,10,11,16,18] it was called Takhtajan–Zograf (TZ) metric (for its precise definition, see Sect. 2.1.2). Unlike the WP metric, the curvature properties of the TZ metric are not known.

Here we present explicit formula for a Kähler potential h_i of the i-th TZ metric on the moduli space $\mathscr{M}_{0,n}$, $i = 1, \dots, n$. Specifically, in Proposition 1 we prove that h_i is expressed in terms of the first Fourier coefficients of Fourier expansions of the Klein's Hauptmodul J at the cusps, introduced in (2.5)–(2.6). The functions h_i on $\mathscr{M}_{0,n}$ provide explicit expressions for trivializations of the Hermitian metrics in the (holomorphically trivial) tautological line bundles $\mathscr{L}_i$ on $\mathscr{M}_{0,n}$, introduced in [16,18]. Proposition 1 is the statement that the first Chern form of the Hermitian line bundle $\mathscr{L}_i$ is $\frac{4}{3}\omega_{\mathrm{TZ},i}$, the symplectic form of the i-th TZ metric on $\mathscr{M}_{0,n}$, $i = 1, \dots, n$.

The function $H = h_1 \dots h_{n-1}/h_n$ on $\mathscr{M}_{0,n}$ determines a Hermitian metric in the line bundle $\lambda_{0,n}$ over the moduli space $\mathfrak{M}_{0,n}$ of type $(0, n)$ Riemann surfaces, introduced by Zograf [21] (see Lemma 1 and Sect. 2.2 for details). We show (see Corollary 2) that on $\mathfrak{M}_{0,n}$

$$c_1(\lambda_{0,n}, H) = \frac{4}{3}\omega_{\mathrm{TZ}},$$

where $\omega_{\mathrm{TZ}} = \omega_{\mathrm{TZ},i} + \cdots + \omega_{\mathrm{TZ},n}$ is the symplectic form of the TZ metric on $\mathfrak{M}_{0,n}$. Comparison with the known result (see [19,21])

$$c_1(\lambda_{0,n}, \exp\{S/\pi\}) = \frac{1}{\pi^2}\omega_{\mathrm{WP}},$$

where S is the classical Liouville action and ω_{WP} is the symplectic form of the WP metric, shows that a real-valued function $\mathscr{S} = S - \pi H$ on $\mathfrak{M}_{0,n}$ is a global Kähler potential for a special linear combination $\omega_{\mathrm{WP}} - \frac{4\pi^2}{3}\omega_{\mathrm{TZ}}$ of the WP and TZ metrics.

We also study WP and TZ metrics on the deformation spaces of punctured Riemann surfaces of genus $g > 1$. Namely, we introduce the Schottky space $\mathfrak{S}_{g,n}$ of type (g, n) Riemann surfaces as a holomorphic fibration $\mathfrak{S}_{g,n} \to \mathfrak{S}_g$ whose fibers are configuration spaces of n points (for details, see Sect. 2.3). Denote by J the corresponding covering map of the Schottky domain Ω and put $h_i = |a_i(1)|^2$, where $a_i(1)$ are the first Fourier coefficients of J at the punctures z_i, $i = 1, \dots, n$, given by (2.23). In Lemma 2 we prove that h_i determine Hermitian metrics on the tautological line bundles $\mathscr{L}_i$ — holomorphic line bundles dual to the vertical tangent bundle on $\mathfrak{S}_{g,n}$ along the fibers of the projection $p_i : \mathfrak{S}_{g,n} \to \mathfrak{S}_{g,n-1}$ which 'forgets' the marked point w_i, $i = 1, \dots, n$.

In Sect. 3.2 we define regularized classical Liouville action S and prove that $\exp\{S/\pi\}$ determines a Hermitian metric in the holomorphic line bundle $\mathscr{L} = \mathscr{L}_1 \otimes \cdots \otimes \mathscr{L}_n$ over $\mathfrak{S}_{g,n}$ (see Lemma 3). Sect. 4 contains the main results of the paper. Thus in Sect. 4.1 we present explicit potentials for the TZ metrics on $\mathscr{M}_{0,n}$, and in Theorems 1 and 2 we explicitly compute canonical connections and Chern forms of the Hermitian line bundles $(\mathscr{L}_i, h_i)$ and $(\mathscr{L}, \exp\{S/\pi\})$. Namely, we show that

$$c_1(\mathscr{L}_i, h_i) = \frac{4}{3}\omega_{\mathrm{TZ},i}, \quad i = 1, \dots, n, \tag{1.1}$$

$$c_1(\mathscr{L}, \exp\{S/\pi\}) = \frac{1}{\pi^2}\omega_{\mathrm{WP}}. \tag{1.2}$$

Here ω_{WP} and ω_{TZ} are, respectively, symplectic forms of the WP and TZ metrics on $\mathfrak{S}_{g,n}$.

The statement that the first Chern class of the line bundles $\mathscr{L}_i$ is $\frac{4}{3}\omega_{\mathrm{TZ},i}$ was proved in [13] at the level of cohomology classes and in [16,18] at the level of Chern forms. Hermitian metrics h_i in the line bundles $\mathscr{L}_i$ on $\mathfrak{S}_{g,n}$ provide explicit expressions for the pullbacks of the Hermitian metrics in tautological line bundles over the moduli space $\mathscr{M}_{g,n}$ of n-pointed curves of genus $g > 1$, introduced in [16,18].

The quantity

$$\mathscr{S} = S - \pi \sum_{i=1}^{n} \log h_i$$

is a smooth real-valued function on the Schottky space $\mathfrak{S}_{g,n}$. It follows from (1.1) and (1.2) that $-\mathscr{S}$ is a Kähler potential for a special linear combination of the WP and TZ metrics,

$$\bar{\partial}\partial\mathscr{S} = -2\sqrt{-1}\left(\omega_{\mathrm{WP}} - \frac{4\pi^2}{3}\omega_{\mathrm{TZ}}\right), \tag{1.3}$$

where ∂ and $\bar{\partial}$ are $(1,0)$ and $(0,1)$ components of the de Rham differential on $\mathfrak{S}_{g,n}$. This linear combination, with the overall factor $1/12\pi$, is precisely the one that appears in the local index theorem for families on punctured Riemann surfaces for $k = 0, 1$ in [13, Theorem 1].

In Sect. 5 we extend the approach in [14] to quasi-Fuchsian groups of type (g, n). Namely, we define the classical Liouville action and in Theorem 4 prove that it is a Kähler potential of the WP metric on the quasi-Fuchsian deformation space. In Sect. 6 we study renormalized volumes of the corresponding Schottky and quasi-Fuchsian 3-manifolds. In Theorem 5 we prove that the renormalized hyperbolic volume of the corresponding Schottky 3-manifold is related to the above-mentioned function $\mathscr{S}$ and in Theorem 6 we prove that for quasi-Fuchsian 3-manifolds it is related to the regularized Liouville action. These extend the results obtained in [14] to punctured Riemann surfaces.

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2. Basic facts

Here we recall the necessary basic facts from the complex-analytic theory of Teichmüller spaces (see the classic book [3] and [1,2], and the modern exposition in [5,9]) and the results from [19,20].

2.1. Teichmüller space $T(\Gamma)$ of a Fuchsian group

Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a Fuchsian group of type (g, n) acting on the Lobachevsky plane $\mathbb{H} = \{z = x + \sqrt{-1}\,y \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. The group Γ is generated by $2g$ hyperbolic transformations $A_1, B_1, \ldots, A_g, B_g$ and n parabolic transformations $S_1, \ldots, S_n$, where $3g - 3 + n > 0$, satisfying the single relation

$$A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1} S_1 \cdots S_n = 1.$$

The group Γ with a given, up to a conjugation in $\mathrm{PSL}(2,\mathbb{R})$, set of generators $A_1, B_1, \ldots,$ $A_g, B_g, S_1, \ldots, S_n$ is called a *marked* Fuchsian group.

Let $\mathcal{A}^{-1,1}(\mathbb{H}, \Gamma)$ be the space of Beltrami differentials for Γ — a complex Banach space of $\mu \in L^\infty(\mathbb{H})$ satisfying

$$\mu(\gamma z)\frac{\overline{\gamma'(z)}}{\gamma'(z)} = \mu(z) \quad \forall \gamma \in \Gamma.$$

For every $\mu \in \mathcal{A}^{-1,1}(\mathbb{H}, \Gamma)$ with

$$\|\mu\|_\infty = \sup_{z\in\mathbb{H}} |\mu(z)| < 1$$

there exists unique quasi-conformal (q.c.) homeomorphism $f^\mu : \mathbb{H} \to \mathbb{H}$ satisfying the Beltrami equation

$$f^\mu_{\bar z} = \mu f^\mu_z, \quad z \in \mathbb{H},$$

and fixing the points $0, 1, \infty$. Then $\Gamma^\mu = f^\mu \circ \Gamma \circ (f^\mu)^{-1}$ is a Fuchsian group of type (g, n) and the Teichmüller space $T(\Gamma)$ is defined by

$$T(\Gamma) = \{\mu \in \mathcal{A}^{-1,1}(\mathbb{H}, \Gamma) \mid \|\mu\|_\infty < 1\}/\sim.$$

Here $\mu \sim \nu$ if and only if $f^\mu \circ \gamma \circ (f^\mu)^{-1} = f^\nu \circ \gamma \circ (f^\nu)^{-1}$ for all $\gamma \in \Gamma$ (or equivalently, $f^\mu = f^\nu$ on $\mathbb{R}$). The group Γ corresponds to $\mu = 0$ and is the origin (the base point) of $T(\Gamma)$.

2.1.1. The complex structure

The Teichmüller space $T(\Gamma)$ admits a natural structure of a complex manifold, which is uniquely determined by the condition that canonical projection which sends $\mu \in \mathcal{A}^{-1,1}(\mathbb{H}, \Gamma)$ with $\|\mu\|_\infty < 1$ to its equivalence class $[\mu] \in T(\Gamma)$ is a holomorphic map. For the Fuchsian group Γ of type (g, n) the complex dimension of $T(\Gamma)$ is $d = 3g - 3 + n$.

Explicitly this complex structure is described as follows. Denote by $\mathscr{H}^{-1,1}(\mathbb{H}, \Gamma)$ the finite-dimensional subspace of harmonic Beltrami differentials for Γ with respect to the hyperbolic metric on $\mathbb{H}$. It consists of $\mu \in \mathcal{A}^{-1,1}(\mathbb{H}, \Gamma)$ satisfying $\partial_z(\rho\mu) = 0$, where

$\rho(z) = y^{-2}$ and has complex dimension $d = 3g - 3 + n$. The complex vector space $\mathscr{H}^{-1,1}(\mathbb{H}, \Gamma)$ is identified with the holomorphic tangent space $T_0T(\Gamma)$ to $T(\Gamma)$ at the origin $\mu = 0$. Every $\mu \in \mathscr{H}^{-1,1}(\Gamma)$ has the form $\mu(z) = y^2\overline{q(z)}$, where $q \in \mathscr{H}^{2,0}(\mathbb{H}, \Gamma)$ is a cusp form of weight 4 for Γ — a holomorphic function on $\mathbb{H}$ that vanishes at the cusps of Γ and satisfies

$$q(\gamma z)\gamma'(z)^2 = q(z) \quad \forall \gamma \in \Gamma.$$

Correspondingly, the holomorphic cotangent space $T_0^*T(\Gamma)$ to $T(\Gamma)$ at the origin is naturally identified with the complex vector space $\mathscr{H}^{2,0}(\mathbb{H}, \Gamma)$, and the pairing between $T_0^*T(\Gamma)$ and $T_0T(\Gamma)$ is given by

$$(q, \mu) = \iint_{\Gamma\backslash\mathbb{H}} q(z)\mu(z)d^2z, \quad \text{where} \quad d^2z = dxdy.$$

Choose a basis $\mu_1, \ldots, \mu_d$ for $\mathscr{H}^{-1,1}(\mathbb{H}, \Gamma)$, put $\mu = \varepsilon_1\mu_1 + \cdots + \varepsilon_d\mu_d$ and for $\|\mu\|_\infty < 1$ let f^μ be the normalized solution of the Beltrami equation. Then the correspondence $(\varepsilon_1, \ldots, \varepsilon_d) \mapsto \Gamma^\mu = f^\mu \circ \Gamma \circ (f^\mu)^{-1}$ defines the complex coordinates in a neighborhood of the origin in $T(\Gamma)$, called *Bers coordinates*.

There is a natural isomorphism between the Teichmüller spaces $T(\Gamma)$ and $T(\Gamma^\mu)$, which maps $\Gamma^\nu \in T(\Gamma)$ to $(\Gamma^\mu)^\lambda \in T(\Gamma^\mu)$, where, in accordance with $f^\nu = f^\lambda \circ f^\mu$,

$$\lambda = \left(\frac{\nu - \mu}{1 - \nu\bar{\mu}} \frac{f_z^\mu}{\bar{f}_{\bar{z}}^\mu}\right) \circ (f^\mu)^{-1}.$$

This isomorphism allows to identify the holomorphic tangent space $T_{[\mu]}T(\Gamma)$ at $[\mu] \in T(\Gamma)$ with the complex vector space $\mathscr{H}^{-1,1}(\mathbb{H}, \Gamma^\mu)$, and the holomorphic cotangent space $T_{[\mu]}^*T(\Gamma)$ — with the complex vector space $\mathscr{H}^{2,0}(\mathbb{H}, \Gamma^\mu)$. It also allows to introduce the Bers coordinates in the neighborhood of Γ^μ in $T(\Gamma)$, and to prove that these coordinates transform complex-analytically.

Remark 1. A *marked* Riemann surface of type (g, n) is a Riemann surface with a set of standard generators of its fundamental group, defined up to an inner automorphism. Whence the Teichmüller space $T(\Gamma)$ can be interpreted as a Teichmüller space of marked Riemann surfaces of type (g, n) by assigning to each $[\mu] \in T(\Gamma)$ a marked surface $X^\mu \cong \Gamma^\mu\backslash\mathbb{H}$, with the surface $X \cong \Gamma\backslash\mathbb{H}$ playing the role of a base point. According to the isomorphism $T(\Gamma) \simeq T(\Gamma^\mu)$, the choice of a base point is inessential and we will often use the notation $T_{g,n}$ for $T(\Gamma)$.

Variation formulas of the hyperbolic metric $\rho(z)|dz|^2$ on $\mathbb{H}$ play an important role in the complex-analytic theory of Teichmüller spaces. Put

$$(f^\mu)^*(\rho) = \frac{|f_z^\mu|^2}{(\operatorname{Im} f^\mu)^2}.$$

The first formula is the classic result of Ahlfors [1] (the so-called Ahlfors lemma) that for $\mu \in \mathscr{H}^{-1,1}(\mathbb{H}, \Gamma)$

$$\left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} (f^{\varepsilon\mu})^*(\rho) = 0. \tag{2.1}$$

The formula for the second variation

$$\left.\frac{\partial^2}{\partial \varepsilon_1 \partial \bar{\varepsilon}_2}\right|_{\varepsilon=0} (f^{\varepsilon_1\mu+\varepsilon_2\nu})^*(\rho) = \frac{1}{2}\rho f_{\mu\bar{\nu}}, \tag{2.2}$$

where $\mu, \nu \in \mathscr{H}^{-1,1}(\mathbb{H}, \Gamma)$ and Γ-automorphic function $f_{\mu\bar{\nu}}$ is uniquely determined by

$$-y^2\frac{\partial^2 f_{\mu\bar{\nu}}}{\partial z \partial \bar{z}} + \frac{1}{2} f_{\mu\bar{\nu}} = \mu\bar{\nu} \quad \text{and} \quad \iint\limits_{\Gamma\backslash\mathbb{H}} |f_{\mu\bar{\nu}}(z)|^2 \rho(z) d^2 z < \infty, \tag{2.3}$$

was proved by S. Wolpert [17, Theorem 3.3].

Remark 2. It is shown in [15, Proposition 6.3], that formulas (2.2)–(2.3) can be obtained from Ahlfors' earlier result in [2].

2.1.2. Kähler metrics on $T(\Gamma)$

The cotangent spaces $T^*_{[\mu]}T(\Gamma) = \mathscr{H}^{2,0}(\mathbb{H}, \Gamma^\mu)$ carry a natural inner product — the Petersson's inner product on the space of cusp forms of weight 4. It determines the Weil–Petersson metric on the Teichmüller space $T(\Gamma)$ by the formula

$$\langle \mu_1, \mu_2 \rangle_{\mathrm{WP}} = \iint\limits_{\Gamma^\mu\backslash\mathbb{H}} \mu_1(z)\overline{\mu_2(z)}\rho(z) d^2 z, \quad \mu_1, \mu_2 \in T_{[\mu]}T(\Gamma) = \mathscr{H}^{-1,1}(\mathbb{H}, \Gamma^\mu).$$

The Weil–Petersson metric is real-analytic and Kähler and is invariant with respect to the Teichmüller modular group $\mathrm{Mod}(\Gamma)$.

In case when Γ is a Fuchsian group of type (g, n) and $n > 0$, a new Kähler metric on $T(\Gamma)$ was introduced in [12,13]. Namely, let by $z_1, \ldots, z_n \in \mathbb{R} \cup \{\infty\}$ be the set of non-equivalent cusps for Γ — the fixed points of the parabolic generators $S_1, \ldots, S_n$. For each $i = 1, \ldots, n$ denote by Γ_i the cyclic subgroup $\langle S_i \rangle$ and let $\sigma_i \in \mathrm{PSL}(2, \mathbb{R})$ be such that $\sigma_i\infty = z_i$ and $\sigma_i^{-1}S_i\sigma_i = \left(\begin{smallmatrix} 1 & \pm 1 \\ 0 & 1 \end{smallmatrix}\right)$. Let $E_i(z, s)$ be the Eisenstein–Maass series associated with the cusp z_i, which for $\mathrm{Re}\, s > 1$ is defined by the following absolutely convergent series

$$E_i(z, s) = \sum_{\gamma\in\Gamma_i\backslash\Gamma} \mathrm{Im}(\sigma_i^{-1}\gamma z)^s.$$

The inner product

$$\langle\mu_1,\mu_2\rangle_i=\iint\limits_{\Gamma\backslash\mathbb{H}}\mu_1(z)\overline{\mu_2(z)}E_i(z,2)\rho(z)d^2z,\quad i=1,\dots,n,$$

in $\mathscr{H}^{-1,1}(\mathbb{H},\Gamma)$, and the corresponding inner products in all $\mathscr{H}^{-1,1}(\mathbb{H},\Gamma^\mu)$ determine another Hermitian metric on $T(\Gamma)$. It was proved in [12,13] that this metric is Kähler for each $i=1,\dots,n$. In [10,16,18] it was called TZ metric and we will denote it by $\langle\ ,\ \rangle_{\mathrm{TZ},i}$. The metric $\langle\ ,\ \rangle_{\mathrm{TZ}}=\langle\ ,\ \rangle_{\mathrm{TZ},1}+\dots+\langle\ ,\ \rangle_{\mathrm{TZ},n}$ is invariant with respect to the Teichmüller modular group $\mathrm{Mod}(\Gamma)$. Denote by $\omega_{\mathrm{TZ},i}$ the symplectic form of i-th TZ metric,

$$\omega_{\mathrm{TZ},i}=\frac{\sqrt{-1}}{2}\sum_{j,k=1}^{d}\langle\mu_j,\mu_k\rangle_{\mathrm{TZ},i}d\varepsilon_j\wedge d\bar{\varepsilon}_k,$$

and put $\omega_{\mathrm{TZ}}=\omega_{\mathrm{TZ},1}+\dots+\omega_{\mathrm{TZ},n}$.

The TZ metric is intrinsically related to the second variation of the hyperbolic metric on $\mathbb{H}$ (see Sect. 2.1.1). Namely, the following result was proved in [13]

$$\lim_{y\to\infty}\mathrm{Im}(\sigma_i z)f_{\mu\bar{\nu}}(\sigma_i z)=\frac{4}{3}\langle\mu,\nu\rangle_{\mathrm{TZ},i},\quad i=1,\dots,n,\tag{2.4}$$

where $\mu,\nu\in\mathscr{H}^{-1,1}(\mathbb{H},\Gamma)$ and $f_{\mu\bar{\nu}}$ is defined in (2.3).

2.2. The moduli space $\mathscr{M}_{0,n}$

Here we consider the moduli space[1] $\mathscr{M}_{0,n}$ of Riemann surfaces of type $(0,n)$ with labeled punctures (n-pointed rational curves). Each such surface is uniquely realized as $\bar{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$ with n labeled punctures such that the last three of them are, respectively, $0,1$ and ∞. Let $\mathscr{F}_n(\mathbb{C})$ be the configuration space of n ordered distinct points in $\mathbb{C}$ with the $\mathrm{PSL}(2,\mathbb{C})$-action. The moduli space is defined by $\mathscr{M}_{0,n}=\mathscr{F}_n(\mathbb{C})/\mathrm{PSL}(2,\mathbb{C})$ and is realized as the following domain in $\mathbb{C}^{n-3}$,

$$\mathscr{M}_{0,n}=\{(w_1,\dots,w_{n-3})\in\mathbb{C}^{n-3}\,|\,w_i\neq 0,1\ \text{ and }\ w_i\neq w_k\ \text{ for }\ i\neq k\}.$$

Let $X=\mathbb{C}\setminus\{w_1,\dots,w_{n-3},0,1\}$ be a Riemann surface of type $(0,n)$. By the uniformization theorem, $X\cong\Gamma\backslash\mathbb{H}$, where type $(0,n)$ Fuchsian group Γ is normalized such that the fixed points of S_{n-2}, S_{n-1}, S_n are, respectively, $z_{n-2}=0$, $z_{n-1}=1$, $z_n=\infty$. Denote by $\mathbb{H}^*$ the union of $\mathbb{H}$ and all cusps for Γ. There is unique covering map $J:\mathbb{H}\to X$ with the group of deck transformations Γ, which extends to a holomorphic isomorphism $J:\Gamma\backslash\mathbb{H}^*\xrightarrow{\sim}\bar{\mathbb{C}}$ that fixes $0,1,\infty$ and has the property that $w_i=J(z_i)$, $i=1,\dots,n-3$.

[1] In [19,21] this moduli space was denoted by W_n.

In the classical terminology J is called *Klein's Hauptmodul.* It is a unique Γ-automorphic function on $\mathbb{H}$ that fixes 0 and 1 and has a simple pole at ∞. The function J is univalent in any fundamental domain for Γ and has the following Fourier expansions at the cusps,

$$J(\sigma_i z) = w_i + \sum_{k=1}^{\infty} a_i(k) q^k, \quad i = 1, \dots, n-1, \tag{2.5}$$

$$J(\sigma_n z) = \sum_{k=-1}^{\infty} a_n(k) q^k, \quad i = n, \tag{2.6}$$

where $q = e^{2\pi\sqrt{-1}z}$. The first Fourier coefficients of J determine the following smooth positive functions on $\mathscr{M}_{0,n}$: $h_i = |a_i(1)|^2$, $i = 1, \dots, n-1$, and $h_n = |a_n(-1)|^2$.

The symmetric group $\mathrm{Symm}(n)$ acts on $\mathscr{M}_{0,n}$ (see [21, §1]) and let $\mathfrak{M}_{0,n} = \mathscr{M}_{0,n}/\mathrm{Symm}(n)$ be the moduli of Riemann surfaces of type $(0, n)$. As in [21], let $\{f_\sigma\}_{\sigma \in \mathrm{Symm}(n)}$ be a 1-cocycle for $\mathrm{Symm}(n)$ on $\mathscr{M}_{0,n}$ defined by

$$f_{\sigma_{kn}}(w_1, \dots, w_{n-3}) = \begin{cases} \displaystyle\prod_{\substack{i=1 \\ i \neq k}}^{n-3} \frac{(w_i - w_k)^2}{w_k(w_k - 1)}, & k = 1, \dots, n-3, \\ \displaystyle\prod_{i=1}^{n-3} w_i^2, & k = n-2, \\ \displaystyle\prod_{i=1}^{n-3} (w_i - 1)^2, & k = n-1 \end{cases}$$

where σ_{kn} is the transposition interchanging the points with indices k and n, and extended to the full group by $f_{\sigma_1 \sigma_2} = (f_{\sigma_1} \circ \sigma_2) f_{\sigma_2}$. Let $\lambda_{0,n}$ be the holomorphic line bundle over $\mathfrak{M}_{0,n}$ determined by the 1-cocycle f, the quotient of the trivial line bundle $\mathscr{M}_{0,n} \times \mathbb{C} \to \mathscr{M}_{0,n}$ by the symmetric group action

$$(\boldsymbol{w}, z) \mapsto (\sigma \cdot \boldsymbol{w}, f_\sigma(\boldsymbol{w}) z), \quad \boldsymbol{w} \in \mathscr{M}_{0,n},\ z \in \mathbb{C},\ \sigma \in \mathrm{Symm}(n).$$

Lemma 1. *A positive function $H = h_1 \cdots h_{n-1}/h_n$ on $\mathscr{M}_{0,n}$ determine a Hermitian metric in the line bundle $\lambda_{0,n}$ over $\mathfrak{M}_{0,n}$.*

Proof. It readily follows from the description of the symmetric group action on $\mathscr{M}_{0,n}$ in [21, §1] that $H(\sigma \cdot \boldsymbol{w}) |f_\sigma(\boldsymbol{w})|^2 = H(\boldsymbol{w})$ for all $\boldsymbol{w} \in \mathscr{M}_{0,n}$ and $\sigma \in \mathrm{Symm}(n)$. □

The hyperbolic metric $e^{\varphi(w)}|dw|^2$ on X is a push-forward by the map J of the hyperbolic metric $\rho(z)|dz|^2$ on $\mathbb{H}$,

$$e^{\varphi(w)} = \frac{\left|(J^{-1})'(w)\right|^2}{\left(\operatorname{Im} J^{-1}(w)\right)^2}, \tag{2.7}$$

and satisfies the Liouville equation

$$\varphi_{w\bar{w}} = \frac{1}{2}e^{\varphi}, \quad w \in X. \tag{2.8}$$

From the Fourier expansions (2.5)–(2.6) one gets the following asymptotic behavior of $\varphi(w)$ as $w \to w_i$ (see [19, Lemma 2])

$$\varphi(w) = -2\log|w - w_i| - 2\log\left|\log\left|\frac{w - w_i}{a_i(1)}\right|\right| + O(|w - w_i|), \quad i \neq n, \tag{2.9}$$

$$\varphi(w) = -2\log|w| - 2\log\log\left|\frac{w}{a_n(-1)}\right| + O(|w|^{-1}), \quad i = n. \tag{2.10}$$

Denote by $\mathcal{S}(f)$ the Schwarzian derivative,

$$\mathcal{S}(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

We have

$$\mathcal{S}(J^{-1})(w) = \varphi_{ww}(w) - \frac{1}{2}\varphi_w(w)^2 = \sum_{i=1}^{n-1}\left(\frac{1}{2(w - w_i)^2} + \frac{c_i}{w - w_i}\right), \tag{2.11}$$

and

$$\mathcal{S}(J^{-1})(w) = \frac{1}{2w^2} + O(|w|^{-3}) \quad \text{as} \quad w \to \infty$$

where $c_i = -a_i(2)/(a_i(1))^2$, $i = 1, \ldots, n-1$, (see [19, Lemma 1]) are *accessory parameters* of the Fuchsian uniformization of the surface X.

Consider the Riemann surface $X = \mathbb{C} \setminus \{w_1, \ldots, w_{n-3}, 0, 1\} \cong \Gamma\backslash\mathbb{H}$ as a base point in $T_{0,n}$. For each $[\mu] \in T_{0,n}$ the Fuchsian group $\Gamma^\mu = f^\mu \circ \Gamma \circ (f^\mu)^{-1}$ is normalized and we realize the Riemann surface $X^\mu \cong \Gamma^\mu\backslash\mathbb{H}$ as $X^\mu = \bar{\mathbb{C}} \setminus \{w_1^\mu, \ldots, w_{n-3}^\mu, 0, 1, \infty\}$. Denote by J_μ the corresponding normalized covering map $J_\mu : \mathbb{H} \to X^\mu$ and consider the map $p : T_{0,n} \to \mathscr{M}_{0,n}$, defined by

$$T_{0,n} \ni [\mu] \mapsto p([\mu]) = (w_1^\mu, \ldots, w_{n-3}^\mu) \in \mathscr{M}_{0,n}, \quad \text{where} \quad w_i^\mu = (J_\mu \circ f^\mu)(z_i).$$

According to [19, Lemma 3], the map p is a complex-analytic covering. Consider the commutative diagram

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{f^\mu} & \mathbb{H} \\ \big\downarrow J & & \big\downarrow J_\mu \\ X & \xrightarrow{F^\mu} & X^\mu \end{array} \tag{2.12}$$

It follows from (2.12) that

$$F^{\mu}_{\bar{w}} = M F^{\mu}_{w}, \quad \text{where} \quad M = (\mu \circ J^{-1}) \frac{\overline{(J^{-1})'}}{(J^{-1})'}. \tag{2.13}$$

The function $f^{\varepsilon\mu}(z)$ is real-analytic in ε for all $z \in \mathbb{C}$. Put $\dot{f}^{\mu}(z) = \partial f^{\varepsilon\mu}/\partial\varepsilon|_{\varepsilon=0}$. It satisfies $\dot{f}^{\mu}_{\bar{z}} = \mu$ and is given by

$$\dot{f}^{\mu}(z) = -\frac{1}{\pi} \iint\limits_{\mathbb{H}} \mu(\zeta) R(\zeta, z) d^2\zeta, \quad \text{where} \quad R(\zeta, z) = \frac{z(z-1)}{(\zeta - z)\zeta(\zeta - 1)}.$$

Correspondingly, the function $F^{\varepsilon\mu}$ is holomorphic in ε and

$$\dot{F}^{\mu}(w) = -\frac{1}{\pi} \iint\limits_{\mathbb{C}} M(v) R(v, w) d^2 v, \tag{2.14}$$

where $\dot{F}^{\mu} = (\partial F^{\varepsilon\mu}/\partial\varepsilon)|_{\varepsilon=0}$.

Denote by r_i, $i = 1, \ldots, n-3$, the basis in $\mathscr{H}^{2,0}(\mathbb{H}, \Gamma) = T_0^* T_{0,n}$ defined as

$$r_i(z) = R_i(J(z)) J'(z)^2, \quad \text{where} \quad R_i(w) = -\frac{1}{\pi} R(w, w_i),$$

and let $q_i(z)$, $i = 1, \ldots, n-3$, be the basis in $\mathscr{H}^{2,0}(\mathbb{H}, \Gamma)$, dual to $r_i(z)$ with respect to the Petersson inner product. Finally, let $\mu_i(z) = y^2 \overline{q_i(z)}$ be the corresponding basis in $\mathscr{H}^{-1,1}(\mathbb{H}, \Gamma) = T_0 T_{0,n}$ (see [19, Sects. 2.4–2.6]). The bases $r_i^{\mu}(z)$ in $\mathscr{H}^{2,0}(\mathbb{H}, \Gamma^{\mu})$ and $\mu_i^{\mu}(z)$ in $\mathscr{H}^{-1,1}(\mathbb{H}, \Gamma^{\mu})$ for $[\mu] \in T_{0,n}$ are defined similarly. Then for the covering map $p : T_{0,n} \to \mathscr{M}_{0,n}$ we have (see [19, Lemma 3])

$$dp_{[\mu]}(\mu_i^{\mu}) = \frac{\partial}{\partial w_i} \quad \text{and} \quad p^*_{[\mu]}(dw_i) = r_i^{\mu}, \quad i = 1, \ldots, n-3. \tag{2.15}$$

We also have

$$\mathcal{S}(J^{-1})(w) = \sum_{i=1}^{n} \mathcal{E}_i(w) - \pi \sum_{i=1}^{n-3} c_i R_i(w), \tag{2.16}$$

where

$$\mathcal{E}_i(w) = \frac{1}{2(w - w_i)^2} - \frac{1}{2w(w-1)}, \quad i \neq n, \quad \mathcal{E}_n(w) = \frac{1}{2w(w-1)}. \tag{2.17}$$

The corresponding functions $e_i(z) = \mathcal{E}_i(J(z)) J'(z)^2$ on $\mathbb{H}$ are automorphic forms of weight 4 for Γ with non-zero constant term at the cusp z_i, $i = 1, \ldots, n$.

Put $\dot{F}^i = \dot{F}^{\mu_i}$, $i = 1, \dots, n-3$. The functions $\dot{F}^i(w)$ are given by (2.14) where

$$M_i(v) = e^{-\varphi(v)}\overline{Q_i(v)}, \quad Q_i(v) = q_i(J^{-1}(v))(J^{-1})'(v)^2. \tag{2.18}$$

It follows from (2.15) that

$$\dot{F}^i(w_j) = \delta_{ij}, \quad i, j = 1, \dots, n-3$$

and

$$\dot{F}^i(0) = \dot{F}^i(1) = 0, \quad \dot{F}^i(w) = o(|w|^2) \quad \text{as} \quad w \to \infty.$$

Also we have

$$\dot{F}^i(w) = \delta_{ij} + (w - w_j)\dot{F}^i_w(w) + o\left(\left|\frac{w - w_j}{\log|w - w_j|}\right|\right) \tag{2.19}$$

as $w \to w_j$, $j \neq n$, and

$$\dot{F}^i(w) = w\dot{F}^i_w(w) + o\left(\frac{|w|}{\log|w|}\right) \quad \text{as} \;\; w \to \infty. \tag{2.20}$$

Remark 3. One can easily prove (2.19)–(2.20) (with better error terms) using integral representation (2.14) and asymptotic behavior (2.9)–(2.10). Here is the sketch of the proof of (2.20). We have

$$\dot{F}^i(w) - w\dot{F}^i_w(w) = -\frac{1}{\pi}\iint\limits_{\mathbb{C}} M_i(v)\left\{\frac{v - 2w}{(v - w)^2} - \frac{1}{v}\right\} d^2v,$$

where the integral is understood in the principal value sense as in [3]. Putting $v = uw$ we have

$$\dot{F}^i(w) - w\dot{F}^i_w(w) = -\frac{\bar{w}}{\pi}\iint\limits_{\mathbb{C}} M_i(uw)\left\{\frac{u - 2}{(u - 1)^2} - \frac{1}{u}\right\} d^2u.$$

It follows from (2.18) that

$$M_i(w) = O\left(\frac{\log^2|w|}{|w|}\right) \quad \text{as} \quad w \to \infty,$$

whence the integral over $|u| \geq 2$ is estimated by $O(\log^2|w|)$. Now choose $\alpha^3 = \log^2|w|/|w|$. Estimating the integral over $|u| \leq \alpha$ by the area, and the integral over $\alpha \leq |u| \leq 2$ — by the estimate of $M_i(\alpha|w|)$, we obtain that both of these integrals are estimated by $|w|\alpha^2 = |w|^{1/3}\log^{4/3}|w|$.

Let $e^{\varphi^\mu(w)}|dw|^2$ be the hyperbolic metric on the Riemann surface X^μ. It follows from commutative diagram (2.12) that $F^\mu \circ J = J_\mu \circ f^\mu$, and we have

$$(F^\mu)^*(e^{\varphi^\mu}) = (J^{-1})^*(f^\mu)^*(\rho).$$

Whence the first and second variations of the family of hyperbolic metrics $e^{\varphi^{\varepsilon\mu}(w)}|dw|^2$ on the Riemann surfaces $X^{\varepsilon\mu} = F^{\varepsilon\mu}(X)$ are given by the same formulas (2.1)–(2.3), where ρ is replaced by e^φ and $f_{\mu\bar\nu}$ — by $(J^{-1})^*(f_{\mu\bar\nu}) = f_{\mu\bar\nu} \circ J^{-1}$. Moreover, since for any $\alpha \in \mathbb{R}$ we have

$$(F^\mu)^*(e^{\alpha\varphi^\mu}) = \Big((F^\mu)^*(e^{\varphi^\mu})\Big)^\alpha,$$

we get from (2.1) and (2.2)

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} (F^{\varepsilon\mu})^*(e^{\alpha\varphi^{\varepsilon\mu}}) = 0 \tag{2.21}$$

and

$$\frac{\partial^2}{\partial\varepsilon_1\partial\bar\varepsilon_2}\bigg|_{\varepsilon=0} (F^{\varepsilon_1\mu+\varepsilon_2\nu})^*(e^{\alpha\varphi^{\varepsilon\mu}}) = \frac{\alpha}{2} e^{\alpha\varphi} f_{\mu\bar\nu} \circ J^{-1}. \tag{2.22}$$

Finally, each TZ metric $\langle\ ,\ \rangle_{\mathrm{TZ},i}$ on $T_{0,n}$ is invariant with respect to the automorphism group of the covering $p : T_{0,n} \to \mathscr{M}_{0,n}$ and determines a Kähler metric on $\mathscr{M}_{0,n}$, which we continue to denote by $\langle\ ,\ \rangle_{\mathrm{TZ},i}$, $i = 1, \dots, n$.

2.3. The Schottky space $\mathfrak{S}_{g,n}$

A Schottky group Σ is a free finitely generated strictly loxodromic Kleinian group. Its limit set Λ is a Cantor set and the region of discontinuity $\Omega = \bar{\mathbb{C}} \setminus \Lambda$ is connected. Let Σ be a Schottky group of rank $g > 1$, considered as a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$. The group Σ acts on Ω freely, and the quotient space $\Sigma\backslash\Omega$ is compact Riemann surface of genus g. A Schottky group Σ of rank g with a relation-free system of generators $L_1, \dots, L_g$ is called *marked*. For each such system of free generators there is a fundamental domain D for Σ in Ω which is a region in $\bar{\mathbb{C}}$ bounded by $2g$ disjoint Jordan curves $C_1, \dots, C_g, C'_1, \dots, C'_g$ with $C'_i = -L_i(C_i)$, $i = 1, \dots, g$. Here C_i and C'_i are oriented as components of the boundary of D, and the minus sign means the reverse orientation. Each element L_i can be represented in the normal form

$$\frac{L_i w - a_i}{L_i w - b_i} = \lambda_i \frac{w - a_i}{w - b_i}, \quad w \in \bar{\mathbb{C}},$$

where a_i and b_i are the respective attracting and repelling fixed points of the transformation L_i and $0 < |\lambda_i| < 1$. In what follows we always assume that a marked Schottky is *normalized*, that is $a_1 = 0$, $b_1 = \infty$ and $a_2 = 1$. In particular, this implies that $\infty \notin D$. The mapping

$$(\Sigma; L_1, \dots, L_g) \mapsto (a_3, \dots, a_g, b_2, \dots, b_g, \lambda_1, \dots, \lambda_g) \in \mathbb{C}^{3g-3}$$

establishes an one-to-one correspondence between the set of normalized marked Schottky groups and a region $\mathfrak{S}_g$ in $\mathbb{C}^{3g-3}$, called the *Schottky space*.

Equivalently, the Schottky space is defined as follows. Let $\mathcal{A}^{-1,1}(\Omega, \Sigma)$ be the complex Banach space of $L^\infty(\Omega)$ of Beltrami differentials for Σ (cf. Sect. 2.1). Let $\mathfrak{D}(\Sigma)$ be a deformation space of the Schottky group Σ,

$$\mathfrak{D}(\Sigma) = \{\mu \in \mathcal{A}^{-1,1}(\Omega, \Sigma) \,|\, \|\mu\|_\infty < 1\}/\sim,$$

where $\mu \sim \nu$ if and only if $F^\mu \circ \sigma \circ (F^\mu)^{-1} = F^\nu \circ \sigma \circ (F^\nu)^{-1}$ for all $\sigma \in \Sigma$ (or equivalently, $F^\mu = F^\nu$ on Λ).[2] The group Σ corresponds to $\mu = 0$ and is the origin (the base point) of $\mathfrak{D}(\Sigma)$. The deformation space $\mathfrak{D}(\Sigma)$ is complex-analytically isomorphic to the Schottky space $\mathfrak{S}_g$ with the choice of a base point.

The Schottky space $\mathfrak{S}_{g,n}$ of type (g, n) Riemann surfaces is defined by a holomorphic fibration $\jmath : \mathfrak{S}_{g,n} \to \mathfrak{S}_g$ whose fibers over the points $[\mu] \in \mathfrak{S}_g$ are configuration spaces $\mathscr{F}_n(\Sigma^\mu \backslash \Omega^\mu)$, where $\Sigma^\mu = F^\mu \circ \Sigma \circ (F^\mu)^{-1}$ and $\Omega^\mu = F^\mu(\Omega)$. Equivalently it is defined as follows. Consider the deformation space of a Schottky group Σ together with a point $(w_1, \dots, w_n) \in \mathscr{F}_n(D)$,

$$\mathfrak{D}(\Sigma; w_1, \dots, w_n) = \{(\mu; w_1^\mu, \dots, w_n^\mu) \in \mathcal{A}^{-1,1}(\Omega, \Sigma) \times \mathscr{F}_n(D^\mu) \,|\, \|\mu\|_\infty < 1\}/\sim .$$

Here $w_i^\mu = F^\mu(w_i)$, $D^\mu = F^\mu(D)$ and $\mu \sim \nu$ if and only if $F^\mu \circ \sigma \circ (F^\mu)^{-1} = F^\nu \circ \sigma \circ (F^\nu)^{-1}$ for all $\sigma \in \Sigma$ and $w_i^\mu = w_i^\nu$, $i = 1, \dots, n$. The deformation space $\mathfrak{D}(\Sigma; w_1, \dots, w_n)$ is complex-analytically isomorphic to the Schottky space $\mathfrak{S}_{g,n}$ with the choice of a base point.

Let $X = \Sigma\backslash\Omega$ be compact Riemann surface of genus g with n marked points $x_1, \dots, x_n$, and let Γ be a Fuchsian group of type (g, n) such that $X_0 = X \setminus \{x_1, \dots, x_n\} \cong \Gamma\backslash\mathbb{H}$. One can choose generators $A_1, B_1, \dots, A_g, B_g$ and $S_1, \dots, S_n$ of Γ such that Σ is isomorphic to the quotient group Γ/N, where N is the smallest normal subgroup of Γ which contains $A_1, \dots, A_g$ and $S_1, \dots, S_n$. As in Sect. 2.2, let $\mathbb{H}^*$ be the union of $\mathbb{H}$ and all cusps for Γ. The complex-analytic covering $\pi_\Gamma : \mathbb{H} \to \Gamma\backslash\mathbb{H} \cong X_0$ extends to the map $\pi_\Gamma^* : \mathbb{H}^* \to X$ such that $\pi_\Gamma^*(z_i) = x_i$, where z_i are fixed points of S_i, $i = 1, \dots, n$.

The Schottky uniformization of a compact Riemann surface X with n marked points $x_1, \dots, x_n$ is related to the Fuchsian uniformization of a punctured surface $X_0 = X \setminus \{x_1, \dots, x_n\}$ by the commutative diagram

[2] Here and in what follows F^μ is a normalized solution of the Beltrami equation on $\mathbb{C}$ with Beltrami coefficient μ.

$$\begin{tikzcd} \mathbb{H}^* \arrow[r, "J"] \arrow[dr, "\pi_\Gamma^*"'] & \Omega \arrow[d, "\pi_\Sigma"] \\ & X \end{tikzcd}$$

where π_Σ is unramified while J and π_Γ^* are branched covering maps. The map J is considered as a meromorphic function on $\mathbb{H}$ which is automorphic with respect to N and satisfies $J \circ B_i = L_i \circ J$, $j = 1, \dots, g$. It has the following Fourier series expansions at the cusps of Γ,

$$J(\sigma_i z) = w_i + \sum_{k=1}^{\infty} a_i(k) q^k, \quad i = 1, \dots, n, \tag{2.23}$$

where $w_i = J(z_i)$. (Cf. (2.5) and note that since Σ is normalized, $\infty \notin \Omega$.)

Let $\mathscr{L}_i$ be i-th the *relative dualizing sheaf* on $\mathfrak{S}_{g,n}$ — a holomorphic line bundle dual to the vertical tangent bundle on $\mathfrak{S}_{g,n}$ along the fibers of the projection $p_i : \mathfrak{S}_{g,n} \to \mathfrak{S}_{g,n-1}$ which 'forget' the marked point w_i, $i = 1, \dots, n$. The bundles $\mathscr{L}_i$, also called *tautological line bundles*, are characterized by the property that the fiber of $\mathscr{L}_i$ over a point $([\Sigma]; w_1, \dots, w_n) \in \mathfrak{S}_{g,n}$ is the cotangent line $T^*_{w_i}(\Sigma\backslash\Omega)$.

Since $J \circ B_k = L_k \circ J$, the points $w_1, \dots, L_k w_i, \dots, w_n$ correspond to the cusps $z_1, \dots, B_k z_i, \dots, z_n$, and the first Fourier coefficient of $J(z)$ at the equivalent cusp $B_k z_i$ is $L_k'(w_i) a_i(1)$. Correspondingly, $h_i = |a_i(1)|^2$ gets replaced by $h_i |L_k'(w_i)|^2$, and using the above interpretation of the line bundles $\mathscr{L}_i$ we arrive at the following statement.

Lemma 2. *The quantities h_i determine Hermitian metrics in the holomorphic line bundles $\mathscr{L}_i$, $i = 1, \dots, n$.*

Let $e^{\varphi(w)}|dw|^2$ be the push-forward of the hyperbolic metric on $\mathbb{H}$ by the map J. It is given by the same formula (2.7), where $\varphi(w)$ is smooth on $\Omega_0 = \Omega \setminus \Sigma \cdot \{w_1, \dots, w_n\}$, a complement in Ω of the Σ-orbit of $\{w_1, \dots, w_n\}$. The function $\varphi(w)$ satisfies

$$\varphi(\sigma w) = \varphi(w) - \log |\sigma'(w)|^2, \quad \forall \sigma \in \Sigma,\ w \in \Omega_0, \tag{2.24}$$

and has the same asymptotics (2.9) as $w \to w_i$, $i = 1, \dots, n$.

Remark 4. From asymptotics (2.9) it follows that

$$\log h_i = \lim_{w \to w_i} \left(\log |w - w_i|^2 + \frac{2e^{-\varphi(w)/2}}{|w - w_i|} \right), \quad i = 1, \dots, n.$$

To each marked Fuchsian group Γ of type (g, n) there is a unique marked normalized Schottky group $\Sigma \simeq \Gamma/N$ with the domain of discontinuity Ω such that $\Gamma\backslash\mathbb{H}^* \cong \Sigma\backslash\Omega$. This determines a map

$$\pi : T_{g,n} \to \mathfrak{S}_{g,n}$$

by putting $w_i = J(z_i)$, $i = 1, \dots, n$. As in $n = 0$ case (see [20, Sect. 2.4]), the map π is a complex-analytic covering. It plays the same role as the corresponding covering map p in Sect. 2.2.

Specifically, the push-forward by the map J of the vector space $\mathscr{H}^{2,0}(\mathbb{H}, \Gamma)$ is a vector space $\mathscr{H}^{2,0}(\Omega_0, \Sigma)$ of holomorphic functions on Ω_0, defined as

$$Q(w) = q(J^{-1}(w))(J^{-1})'(w)^2, \quad q(z) \in \mathscr{H}^{2,0}(\mathbb{H}, \Gamma).$$

They are automorphic forms of weight 4 for group Σ which admit a meromorphic extension to Ω with at most simple poles at $\Sigma \cdot \{w_1, \dots, w_n\}$. The space $\mathscr{H}^{2,0}(\Omega_0, \Sigma)$ is naturally identified with the holomorphic cotangent space $T_0^*\mathfrak{S}_{g,n}$ to $\mathfrak{S}_{g,n}$ at the origin. Correspondingly, the holomorphic tangent space $T_0\mathfrak{S}_{g,n}$ is the complex vector space $\mathscr{H}^{-1,1}(\Omega_0, \Sigma)$ of Beltrami differentials, harmonic with respect to the hyperbolic metric on Ω_0. Namely, each $M \in \mathscr{H}^{-1,1}(\Omega_0, \Sigma)$ has the form

$$M(w) = e^{-\varphi(w)}\overline{Q(w)}, \quad Q \in \mathscr{H}^{2,0}(\Omega_0, \Sigma).$$

The tangent and cotangent spaces to $\mathfrak{S}_{g,n}$ at each point $(\Sigma^\mu, w_1^\mu, \dots, w_n^\mu)$ are identified, respectively, with $\mathscr{H}^{-1,1}(\Omega_0^\mu, \Sigma^\mu)$ and $\mathscr{H}^{2,0}(\Omega_0^\mu, \Sigma^\mu)$, where $\Omega_0^\mu = F^\mu(\Omega_0)$. We have the following analog of commutative diagram (2.12),

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{f^\mu} & \mathbb{H} \\ \Big\downarrow J & & \Big\downarrow J_\mu \\ \Omega_0 & \xrightarrow{F^\mu} & \Omega_0^\mu \end{array} \tag{2.25}$$

Here $F^{\varepsilon\mu}$ satisfies Beltrami equation (2.13), is complex-analytic in ε and $\dot{F}^\mu$ is given by (2.14).

From the fibration $\jmath : \mathfrak{S}_{g,n} \to \mathfrak{S}_g$ it follows that $T_0^*\mathfrak{S}_{g,n}$ has a subspace $\jmath^*(T_0^*\mathfrak{S}_g) \cong \mathscr{H}^{2,0}(\Omega, \Sigma)$ with a natural basis $P_1(w), \dots, P_{3g-3}(w)$ given by holomorphic automorphic forms of weight 4 for Σ which represent the cotangent vectors $d\lambda_1, \dots, d\lambda_g, da_3, \dots, da_g, db_2, \dots, db_g$ as in [21, formulas (2.2)]. The complementary subspace to $\jmath^*(T_0^*\mathfrak{S}_g)$ in $T_0^*\mathfrak{S}_{g,n}$ is isomorphic to the subspace $T_0^*\mathscr{F}_n(X)$, the cotangent space to the configuration space at the base point $(w_1, \dots, w_n)$. Its natural basis, as it follows from (2.14), is given by the following meromorphic automorphic forms of weight 4,

$$P_{3g-3+i}(w) = -\frac{1}{\pi}\sum_{\sigma\in\Sigma} R(\sigma w, w_i)\sigma'(w)^2, \quad w \in \Omega, \tag{2.26}$$

which represent dw_i, $i = 1, \dots, n$.

Denote by $M_1(w), \dots, M_d(w)$ the basis in $\mathscr{H}^{-1,1}(\Omega_0, \Sigma)$, dual to the basis $P_1(w), \dots, P_d(w)$ in $\mathscr{H}^{2,0}(\Omega_0, \Sigma)$ with respect to the pairing

$$(Q, M) = \iint\limits_{D} Q(w)M(w)d^2w. \tag{2.27}$$

Here $M_{3g-3+1}, \dots, M_{3g-3+n}$ represent the tangent vectors $\partial/\partial w_1, \dots, \partial/\partial w_n$ in $T_0\mathfrak{S}_{g,n}$. The corresponding bases in tangent and cotangent spaces to $\mathfrak{S}_{g,n}$ at arbitrary point $(\Sigma^\mu; w_1^\mu, \dots, w_n^\mu)$ are defined similarly.

As in Sect. 2.2, we have $\mathcal{S}(J^{-1})(w) = \varphi_{ww}(w) - \frac{1}{2}\varphi_w(w)^2$. It follows from the asymptotic behavior of $\varphi(w)$ as $w \to w_i$, that

$$\mathcal{S}(J^{-1})(w) = \sum_{i=1}^{n} \mathscr{E}_i(w) - \pi \sum_{l=1}^{3g-3+n} c_l P_l(w), \tag{2.28}$$

where (cf. (2.16) and (2.17))

$$\mathscr{E}_i(w) = \frac{1}{2} \sum_{\sigma \in \Sigma} \left(\frac{1}{(\sigma w - w_i)^2} - \frac{1}{\sigma w(\sigma w - 1)} \right) \sigma'(w)^2, \quad i = 1, \dots, n, \tag{2.29}$$

are meromorphic automorphic forms of weight 4 for Σ with the second order poles at $\Sigma \cdot w_i$, and $c_1, \dots, c_d$ are the analogs of accessory parameters.[3]

For the first and second variations of the family of hyperbolic metrics on the Schottky domains Ω^μ we have the same formulas (2.21)–(2.22). Finally, each TZ metric $\langle\ ,\ \rangle_{\mathrm{TZ},i}$ on $T_{g,n}$ is invariant with respect to the automorphism group of the covering $\pi : T_{g,n} \to \mathfrak{S}_{g,n}$ and determines a Kähler metric on $\mathfrak{S}_{g,n}$, which we continue to denote by $\langle\ ,\ \rangle_{\mathrm{TZ},i}$, $i = 1, \dots, n$.

3. Liouville action

3.1. Punctured spheres

Let $X = \mathbb{C} \setminus \{w_1, \dots, w_{n-3}, 0, 1\}$ be a marked Riemann surface of type $(0, n)$. The regularized classical Liouville action is defined by the following formula (see [19]),

$$\begin{aligned} &S(w_1, \dots, w_{n-3}) \\ &= \lim_{\delta \to 0^+} \left(\iint\limits_{X_\delta} (|\varphi_w|^2 + e^\varphi) d^2w + 2\pi n \log \delta + 4\pi(n-2) \log |\log \delta| \right), \end{aligned} \tag{3.1}$$

where $X_\delta = \mathbb{C} \setminus \cup_{i=1}^{n-1} \{|w - w_i| < \delta\} \cup \{|w| > 1/\delta\}$. It is a critical value of the *Liouville action*, the Euler–Lagrange functional for the Liouville equation (2.8) with the asymptotic behavior (2.9)–(2.10) on the Riemann surface X, and defines the smooth function

[3] Note that for $i = 1, \dots, 3g - 3$ parameters c_i introduced here are $-1/\pi$ times accessory parameters in [20].

$S:\mathscr{M}_{0,n}\to\mathbb{R}$. Denote by ∂ and $\bar\partial$ the $(1,0)$ and $(0,1)$ components of de Rham differential on $\mathscr{M}_{0,n}$. It is proved in [19, Theorem 1],

$$\partial S=-2\pi\sum_{i=1}^{n-3}c_iR_i,$$

so that the regularized Liouville action is a generating function for the accessory parameters,

$$c_i=-\frac{1}{2\pi}\frac{\partial S}{\partial w_i},\quad i=1,\dots,n-3.$$

Also, according to [19, Theorem 2], the function $-S$ is a Kähler potential for the Weil–Petersson metric on $\mathscr{M}_{0,n}$,

$$\bar\partial\partial S=-2\sqrt{-1}\omega_{\mathrm{WP}}.$$

Let $\mathfrak{M}_{0,n}=\mathscr{M}_{0,n}/\mathrm{Symm}(n)$ be the moduli space of Riemann surfaces of type $(0,n)$. It is proved in [21, §1] that $\exp\{S/\pi\}$ determines a Hermitian metric in a holomorphic line bundle $\lambda_{0,n}$ over $\mathfrak{M}_{0,n}$ (see Sect. 2.2), so that

$$c_1(\lambda_{0,n},\exp\{S/\pi\})=\frac{1}{\pi^2}\omega_{\mathrm{WP}}. \tag{3.2}$$

3.2. Schottky domains

Let Σ be a marked normalized Schottky group of rank $g>1$. The classical Liouville action is a critical value of the Liouville action functional and is defined by the following formula [20] (see [14] for the cohomological interpretation),

$$S(\varphi)=\frac{\sqrt{-1}}{2}\iint\limits_D\omega(\varphi)+\frac{\sqrt{-1}}{2}\sum_{k=2}^{g}\int\limits_{C_k}\theta_{L_k^{-1}}(\varphi), \tag{3.3}$$

where

$$\omega(\varphi)=(|\varphi_w|^2+e^{\varphi})dw\wedge d\bar w$$

and for $\sigma\in\mathrm{PSL}(2,\mathbb{C})$

$$\theta_{\sigma^{-1}}(\varphi)=\left(\varphi-\frac{1}{2}\log|\sigma'|^2-\log|c(\sigma)|^2\right)\left(\frac{\sigma''}{\sigma'}dw-\frac{\overline{\sigma''}}{\overline{\sigma'}}d\bar w\right).$$

Here for $\sigma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ we put $c(\gamma)=c$, so that $\theta_{\sigma^{-1}}(\varphi)=0$ if $c(\sigma)=0$.

The classical Liouville action is independent of the choice of a fundamental domain D for the marked Schottky group Σ and determines a smooth function $S : \mathfrak{S}_g \to \mathbb{R}$. As in Sect. 3.1, denoting by ∂ and $\bar{\partial}$ the $(1,0)$ and $(0,1)$ components of de Rham differential on $\mathfrak{S}_g$ we have[4] (see [20, Theorems 1,2])

$$\partial S = -2\pi \sum_{l=1}^{3g-3} c_l P_l \quad \text{and} \quad \bar{\partial}\partial S = -2\sqrt{-1}\omega_{\mathrm{WP}},$$

so that $-S$ is a Kähler potential for the Weil–Petersson metric on $\mathfrak{S}_g$.

To define the classical Liouville action for the hyperbolic metric on $\Omega_0 = \Omega \setminus \Sigma \cdot \{w_1, \dots, w_n\}$ one needs to regularize the area integral in (3.3), which diverges due to the asymptotic behavior (2.9) of φ as $w \to w_i$. We do it in the same way as in genus 0 case. Namely, suppose that all $w_1, \dots, w_n \in \mathrm{Int}D$, the interior of D, and for sufficiently small $\delta > 0$ define $D_\delta = D \setminus \cup_{i=1}^n D_i(\delta)$, where $D_i(\delta) = \{|w - w_i| < \delta\} \subset D$, $i = 1, \dots, n$. It follows from (2.9) that the following limit exists

$$S_{\mathrm{bulk}}(\varphi) = \lim_{\delta \to 0^+} \left(\frac{\sqrt{-1}}{2} \iint\limits_{D_\delta} \omega(\varphi) + 2\pi n(\log\delta + 2\log|\log\delta|) \right). \tag{3.4}$$

Remark 5. Equivalently, one can define $S_{\mathrm{bulk}}(\varphi)$ by cutting out the interiors $D_i \subset D$ of arbitrary simple closed curves l_i around w_i such that $w_j \notin D_i$ for $i \neq j$. Namely, let

$$\frac{2}{\sqrt{-1}}\tilde{S}_l(\varphi) = \iint\limits_{D \setminus \cup_{i=1}^n D_i} \omega(\varphi) + \sum_{i=1}^n \int\limits_{l_i} \left(\frac{2\log|w - w_i|}{\bar{w} - \bar{w}_i} + \frac{2\log\left(\log|w - w_i|\right)^2}{\bar{w} - \bar{w}_i} \right) d\bar{w}.$$

Then it easily follows from Stokes' theorem and (2.9) that

$$S_{\mathrm{bulk}}(\varphi) = \lim_{r \to 0} \tilde{S}_l(\varphi),$$

where $r = \max\{\mathrm{diam}(l_1), \dots, \mathrm{diam}(l_n)\}$.

Now we define the regularized action[5] as

$$S = S(D; w_1, \dots, w_n) = S_{\mathrm{bulk}}(\varphi) + \frac{\sqrt{-1}}{2} \sum_{k=2}^{g} \int\limits_{C_k} \theta_{L_k^{-1}}(\varphi). \tag{3.5}$$

This completes the definition of S provided that fundamental domain D is such that $w_1, \dots, w_n \in \mathrm{Int}\, D$. As in the compact case, S does not depend on the choice of D

[4] See the previous footnote.

[5] It should be always clear from the context for which space the action S stands for.

with the property that $w_1, \dots, w_n \in \operatorname{Int} D$. However, $S(D; w_1, \dots, w_n)$ depends on the choice of representatives in $\Sigma \cdot \{w_1, \dots, w_n\}$ and no longer determines a function oh the Schottky space $\mathfrak{S}_{g,n}$. Its geometric meaning is the following (cf. Lemma 2).

Lemma 3. *The regularized Liouville action determines a Hermitian metric* $\exp\{S/\pi\}$ *in the holomorphic line bundle* $\mathscr{L} = \mathscr{L}_1 \otimes \cdots \otimes \mathscr{L}_n$ *over* $\mathfrak{S}_{g,n}$.

Proof. It is sufficient to prove that for $i = 1, \dots, n$,

$$S(\tilde{D}; w_1, \dots, L_k w_i, \dots, w_n) - S(D; w_1, \dots, w_n) = \pi \log |L_k'(w_i)|^2,$$

where $w_1, \dots, w_n \in \operatorname{Int} D$ and $w_1, \dots, w_{i-1}, L_k w_i, w_{i+1}, \dots, w_n \in \operatorname{Int} \tilde{D}$. Moreover, it is sufficient to consider the case when

$$\tilde{D} = (D \setminus D_0) \cup L_k(D_0)$$

and $D_0 \subset D$ is such that $\partial D_0 \cap \partial D \subset C_k$ and $w_i \in D_0$, while all other $w_j \in D \setminus D_0$, $j \neq i$. Indeed, any choice of a fundamental domain for Σ is obtained from D by a finite combination of such transformations.

Put

$$I_\delta(D; w_1, \dots, w_n) = \iint\limits_{D_\delta} \omega(\varphi) + \sum_{k=2}^{g} \int\limits_{C_k} \theta_{L_k^{-1}}(\varphi). \tag{3.6}$$

Since $\tilde{C}_j = C_j$ for $j \neq k$ and $\tilde{C}_k = C_k - \partial D_0$, we have

$$\begin{aligned} \Delta I_\delta &= I_\delta(\tilde{D}; w_1, \dots, L_k w_i, \dots, w_n) - I_\delta(D; w_1, \dots, w_n) \\ &= \iint\limits_{L_k(D_0) \setminus \tilde{D}_i(\delta)} \omega(\varphi) - \iint\limits_{D_0 \setminus D_i(\delta)} \omega(\varphi) - \int\limits_{\partial D_0} \theta_{L_k^{-1}}(\varphi). \end{aligned}$$

It follows from (2.24) that

$$L_k^*(\omega(\varphi)) = \omega(\varphi) \circ L_k |L_k'|^2 = \omega(\varphi) + d\theta_{L_k^{-1}}(\varphi),$$

and by Stokes theorem we get

$$\begin{aligned} \Delta I_\delta &= \iint\limits_{D_0 \setminus L_k^{-1}(\tilde{D}_i(\delta))} L_k^*(\omega(\varphi)) - \iint\limits_{D_0 \setminus D_i(\delta)} \omega(\varphi) - \int\limits_{\partial D_0} \theta_{L_k^{-1}}(\varphi) \\ &= \iint\limits_{D_0 \setminus L_k^{-1}(\tilde{D}_i(\delta))} \omega(\varphi) - \iint\limits_{D_0 \setminus D_i(\delta)} \omega(\varphi) - \int\limits_{\partial L_k^{-1}(\tilde{D}_i(\delta))} \theta_{L_k^{-1}}(\varphi) \end{aligned}$$

$$= \iint\limits_{D_0\backslash D(\tilde{\delta})} \omega(\varphi) - \iint\limits_{D_0\backslash D_i(\delta)} \omega(\varphi) + o(1) \quad \text{as} \quad \delta \to 0,$$

where $\tilde{\delta} = \delta/|L_k'(w_i)|$. Thus for $|L_k'(w_i)| < 1$ we have

$$\Delta I_\delta = -\iint\limits_{K_i} |\varphi_w|^2 dw \wedge d\bar{w} + o(1),$$

where K_i is the annulus $\delta \leq |w - w_i| \leq \tilde{\delta}$. It now follows from (2.9) that

$$\Delta I_\delta = -4\pi\sqrt{-1}\log|L_k'(w_i)| + o(1).$$

In case $|L_k'(w_i)| > 1$ we have

$$\Delta I_\delta = \iint\limits_{\tilde{K}_i} |\varphi_w|^2 dw \wedge d\bar{w} + o(1) = -4\pi\sqrt{-1}\log|L_k'(w_i)| + o(1),$$

where $\tilde{K}_i$ is the annulus $\tilde{\delta} \leq |w - w_i| \leq \delta$. □

Combining with Lemma 2 we obtain

Corollary 1. *Put $H = h_1 \cdots h_n$. Then*

$$\mathscr{S} = S - \pi \log H \tag{3.7}$$

determines a smooth real-valued function on $\mathfrak{S}_{g,n}$.

Remark 6. Let $D(w_i; \delta_i) = \{w \in \mathbb{C} : |w - w_i| < \delta_i\}$, where $\delta_i = |a_i(1)|\delta$. Since $a_i(1) \mapsto L_k'(w_i)a_i(1)$ under the transformation $w_i \mapsto L_k(w_i)$, we have that up to $O(\delta^2)$ terms $D(L_k w_i; \delta_i) = L_k\left(D(w_i; \delta_i)\right)$. This shows that (3.7) can be also defined as

$$\mathscr{S} = \lim_{\delta\to 0^+}\left(\frac{\sqrt{-1}}{2}\iint\limits_{D_\delta(h)} \omega(\varphi) + 2\pi n(\log\delta + 2\log|\log\delta|)\right) + \frac{\sqrt{-1}}{2}\sum_{k=2}^{g}\int\limits_{C_k} \theta_{L_k^{-1}}(\varphi), \tag{3.8}$$

where $D_\delta(h) = D \setminus \cup_{i=1}^n D(w_i; \delta_i)$.

4. Potentials for the WP and TZ metrics

Here using first Fourier coefficients of Klein's Hauptmodul we construct a global potential for the TZ metric on $\mathscr{M}_{0,n}$. For the Schottky space $\mathfrak{S}_{g,n}$ we prove that the first Chern forms of the line bundles $\mathscr{L}_i$ with Hermitian metrics h_i are $\frac{4}{3}\omega_{\mathrm{TZ},i}$. We also prove that $\frac{1}{\pi^2}\omega_{\mathrm{WP}}$ is the first Chern form of the line bundle $\mathscr{L} = \mathscr{L}_1 \otimes \cdots \otimes \mathscr{L}_n$ with the Hermitian metric $\exp\{S/\pi\}$, where S is the regularized classical Liouville action (3.7). As a corollary, the following combination $\omega_{\mathrm{WP}} - \frac{4\pi^2}{3}\omega_{\mathrm{TZ}}$ of WP and TZ metrics has a global Kähler potential on $\mathfrak{S}_{g,n}$.

4.1. Potential for the TZ metric on $\mathscr{M}_{0,n}$

As in Sect. 2.2, let Γ be marked normalized Fuchsian group of type $(0,n)$ uniformizing the Riemann surface $X = \mathbb{C} \setminus \{w_1, \ldots, w_{n-3}, 0, 1\}$, let $J : \mathbb{H} \to X$ be the normalized covering map, and let $h_i = |a_i(1)|^2$, $i = 1, \ldots, n-1$, and $h_n = |a_n(-1)|^2$ be smooth positive functions on $\mathscr{M}_{0,n}$. According to Remark 4 we have

$$\log h_i = \lim_{w \to w_i} \left(\log |w - w_i|^2 + \frac{2e^{-\varphi(w)/2}}{|w - w_i|} \right), \quad i = 1, \ldots, n-1,$$

and

$$\log h_n = \lim_{w \to \infty} \left(\log |w|^2 - \frac{2e^{-\varphi(w)/2}}{|w|} \right),$$

where the last formula follows from (2.10).

Lemma 4. *We have for all $i = 1, \ldots, n$,*

$$h_i^{-1} \frac{\partial h_i}{\partial w_k} = \dot{F}_w^k(w_i), \quad k = 1, \ldots, n-3.$$

Proof. For given $X = \mathbb{C} \setminus \{w_1, \ldots, w_{n-3}, 0, 1\} \simeq \Gamma\backslash\mathbb{H}$ there is an isomorphism $T_{g,n} \simeq T(\Gamma)$ (see Sect. 2.1.1). Consider first the case $i = n$. According to (2.15), it is sufficient to show that

$$\left(\frac{\partial \log h_n^{\varepsilon\mu}}{\partial \varepsilon} \right)\bigg|_{\varepsilon=0} = \dot{F}_w^k(\infty), \quad \text{where} \quad \mu = \mu_k.$$

Using that $F^{\varepsilon\mu}$ is holomorphic in ε at $\varepsilon = 0$ and formulas (2.10), (2.20), (2.21), we get

$$\begin{aligned}
&\left(\frac{\partial h_n^{\varepsilon\mu}}{\partial\varepsilon}\right)\bigg|_{\varepsilon=0} \\
&= \lim_{w\to\infty}\left\{\left(\frac{\partial}{\partial\varepsilon}\right)\bigg|_{\varepsilon=0}\left(\log|F^{\varepsilon\mu}|^2 - 2(F^{\varepsilon\mu})^*(e^{-\frac{1}{2}\varphi^{\varepsilon\mu}})\left|\frac{F_w^{\varepsilon\mu}}{F^{\varepsilon\mu}}\right|\right)(w)\right\} \\
&= \lim_{w\to\infty}\left(\frac{\dot F^k(w)}{w} - \frac{e^{-\varphi(w)/2}(w\dot F_w^k(w) - \dot F^k(w))|w|}{w^2\bar w}\right) \\
&= \dot F_w^k(\infty).
\end{aligned}$$

Interchanging the order of the limit $w \to \infty$ and differentiation is legitimate since convergence in the above formula and in the definition of h_n is uniform in a neighborhood of an arbitrary point $(w_1, \dots, w_{n-3}) \in \mathscr{M}_{0,n}$. The case $i \neq n$ is considered similarly. $\square$

Let ∂ and $\bar\partial$ be, respectively, $(1,0)$ and $(0,1)$ components of the de Rham differential d on $\mathscr{M}_{0,n}$. We have the following result.

Proposition 1. *The functions $-\log h_i : \mathscr{M}_{0,n} \to \mathbb{R}_{>0}$, $i = 1, \dots, n-1$, and $\log h_n$ are Kähler potential for the $4\pi/3$ multiples of TZ metrics,*

$$\bar\partial\partial\log h_i = -\frac{8\pi\sqrt{-1}}{3}\omega_{\mathrm{TZ},i}, \quad i \neq n \quad \textit{and} \quad \bar\partial\partial\log h_n = \frac{8\pi\sqrt{-1}}{3}\omega_{\mathrm{TZ},n}.$$

Proof. First consider the case $i = n$. We need to prove that

$$\frac{\partial^2\log h_n}{\partial w_j\partial\bar w_k} = \frac{4\pi}{3}\left\langle\frac{\partial}{\partial w_j}, \frac{\partial}{\partial w_k}\right\rangle_{\mathrm{TZ},n}, \quad j,k = 1,\dots,n-3.$$

By polarization, it is sufficient to consider the case $j = k$. According to Sect. 2.1.1, for given $X = \mathbb{C}\setminus\{w_1, \dots, w_{n-3}, 0, 1\} \simeq \Gamma\backslash\mathbb{H}$ we can use the isomorphism $T_{g,n} \simeq T(\Gamma)$. Thus we need to show that

$$\left(\frac{\partial^2\log h_n^{\varepsilon\mu}}{\partial\varepsilon\partial\bar\varepsilon}\right)\bigg|_{\varepsilon=0} = \frac{4\pi}{3}\|\mu\|^2_{\mathrm{TZ},n}, \quad \text{where} \quad \mu = \mu_k.$$

Using that $F^{\varepsilon\mu}$ is holomorphic in ε at $\varepsilon = 0$ and formulas (2.20), (2.21), (2.22), (2.10), we get

$$\begin{aligned}
&\left(\frac{\partial^2\log h_n^{\varepsilon\mu}}{\partial\varepsilon\partial\bar\varepsilon}\right)\bigg|_{\varepsilon=0} \\
&= \left(\frac{\partial^2}{\partial\varepsilon\partial\bar\varepsilon}\right)\bigg|_{\varepsilon=0}\left\{\lim_{w\to\infty}\left(\log|F^{\varepsilon\mu}|^2 - 2(F^{\varepsilon\mu})^*(e^{-\frac{1}{2}\varphi^{\varepsilon\mu}})\left|\frac{F_w^{\varepsilon\mu}}{F^{\varepsilon\mu}}\right|\right)(w)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= -2 \lim_{w\to\infty} \left\{ \frac{1}{|w|} \left(\frac{\partial^2}{\partial\varepsilon\partial\bar{\varepsilon}}\right)\bigg|_{\varepsilon=0} (F^{\varepsilon\mu})^*(e^{-\frac{1}{2}\varphi^{\varepsilon\mu}})(w) \right. \\
&\qquad\qquad\qquad\qquad \left. + e^{-\frac{1}{2}\varphi(w)} \left| \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \left(\frac{F_w^{\varepsilon\mu}(w)}{F^{\varepsilon\mu}(w)}\right)^{\frac{1}{2}} \right|^2 \right\} \\
&= \lim_{w\to\infty} \left\{ \frac{1}{2}\log|w| f_{\mu\bar{\mu}}(J^{-1}(w)) - \frac{1}{2} e^{-\frac{1}{2}\varphi(w)} \frac{\left|w\dot{F}_w(w) - \dot{F}(w)\right|^2}{|w|^3} \right\} \\
&= \pi \lim_{w\to\infty} y f_{\mu\bar{\mu}}(z) = \frac{4\pi}{3}\|\mu\|^2_{\mathrm{TZ},n}.
\end{aligned}
$$

The case $i \neq n$ is considered similarly. Here

$$
\lim_{w\to w_i} \frac{\log|w - w_i|}{\operatorname{Im}(\sigma_i^{-1}(J^{-1}(w)))} = -2\pi,
$$

and we get the different sign from the case of $i = n$. $\square$

Remark 7. One can also prove Proposition 1 by using Lemma 4 and another Wolpert's formula

$$
\frac{\partial}{\partial\bar{\varepsilon}}\bigg|_{\varepsilon=0} (f^{\varepsilon\mu})^*(\mu_i^{\varepsilon\mu})(z) = -\left(\frac{\partial}{\partial\bar{z}} y^2 \frac{\partial}{\partial\bar{z}}\right) f_{\mu\bar{\mu}_i}(z)
$$

(see [17, Theorem 2.9]).

Remark 8. Let $\mathscr{L}_i$ be the tautological line bundle on $\mathscr{M}_{0,n}$ — a holomorphic line bundle dual to the vertical tangent bundle of $\mathscr{M}_{0,n}$ along the fibers of the projection $p_i : \mathscr{M}_{0,n} \to \mathscr{M}_{0,n-1}$ which ‘forget’ the marked point z_i, $i = 1, \dots, n$. The line bundles $\mathscr{L}_i$ are holomorphically trivial over $\mathscr{M}_{0,n}$ (but not over $\overline{\mathscr{M}_{0,n}}$), and the functions h_i on $\mathscr{M}_{0,n}$ are trivializations of the Hermitian metrics in $\mathscr{L}_i$, introduced in [16,18].

By Lemma 1 and Proposition 1, we have

Corollary 2. *The function $-\log H = \log h_n - \log h_1 - \cdots - \log h_{n-1}$ is a potential for the $4\pi/3$ multiple of the TZ metric on $\mathscr{M}_{0,n}$. The first Chern form of the Hermitian line bundle $(\lambda_{0,n}, H)$ over $\mathfrak{M}_{0,n}$ is given by*

$$
c_1(\lambda_{0,n}, H) = \frac{4}{3}\omega_{\mathrm{TZ}}.
$$

For each marked Fuchsian group Γ denote by $r(z)$ the projection of the regular automorphic form $-\mathcal{S}(J)(z)$ of weight 4 to the subspace of cusp forms,

$$r(z)=\sum_{i=1}^{n-3}\alpha_i r_i(z),\quad \text{where}\quad \alpha_i=-\iint\limits_{\Gamma\backslash\mathbb{H}}\mathcal{S}(J)(z)\mu_i(z)d^2z.$$

According to Sect. 2.2, the family of cusp forms $r(z)$ for varying Γ determines a $(1,0)$-form r on $T_{0,n}$. Denote by $\vartheta=\sum_{i=1}^{n-3}\alpha_i dw_i$ the corresponding $(1,0)$-form on $\mathscr{M}_{0,n}$. It follows from (2.15), (2.16) that $p^*(\vartheta)=r$, where $p:T_{0,n}\to\mathscr{M}_{0,n}$.

Put $\mathscr{S}=S-\pi\log H$. Combining Lemma 4 with the proof of Theorem 1 in [19] and using Proposition 1 and Theorem 2 in [19], we obtain the following result.

Corollary 3. *The function $\mathscr{S}:\mathscr{M}_{0,n}\to\mathbb{R}$ satisfies*

$$\partial\mathscr{S}=2\vartheta$$

and

$$\bar\partial\partial\mathscr{S}=-2\sqrt{-1}\left(\omega_{\mathrm{WP}}-\frac{4\pi^2}{3}\omega_{\mathrm{TZ}}\right). \tag{4.1}$$

Remark 9. Since both H and $\exp\{S/\pi\}$ are Hermitian metrics in the line bundle $\lambda_{0,n}$ over $\mathfrak{M}_{0,n}$ (see Sects. 2.2 and 3.1), we conclude that $\mathscr{S}=S-\pi\log H$ determines a function on $\mathfrak{M}_{0,n}$. The combination $\omega_{\mathrm{WP}}-\dfrac{4\pi^2}{3}\omega_{\mathrm{TZ}}$, with the overall factor $1/12\pi$, appears in the local index theorem for families on punctured Riemann surfaces for $k=0,1$ (see [13, Theorem 1]). Equation (4.1) agrees with the fact that the analog of the Hodge line bundle λ_1 over $\mathfrak{M}_{0,n}$ is trivial. The function $\mathscr{S}$ plays the role of the Quillen metric in λ_1, defined in [13].

4.2. Chern forms and potential on $\mathfrak{S}_{g,n}$

As in Sect. 2.3, let $X=\Sigma\backslash\Omega$ be a compact Riemann surface of genus g with n marked points $x_1,\dots,x_n$, let Γ be a Fuchsian group of type (g,n) such that $X_0=X\setminus\{x_1,\dots,x_n\}\cong\Gamma\backslash\mathbb{H}$, and let $J:\mathbb{H}^*\to\Omega$ be the corresponding branched covering map. Similar to the previous section, denote by R the projection of the automorphic form $\mathcal{S}(J^{-1})$ of weight 4 for Σ to the subspace $\mathscr{H}^{2,0}(\Omega_0,\Sigma)\cong T_0^*\mathfrak{S}_{g,n}$. Using pairing (2.27), we get

$$R(w)=\sum_{j=1}^{3g-3+n}\beta_j P_j(w),\quad\text{where}\quad \beta_j=(\mathcal{S}(J^{-1}),M_j).$$

Corresponding automorphic forms over each point $(\Sigma^\mu,w_1^\mu,\dots,w_n^\mu)$ determine a $(1,0)$-form $\mathscr{R}$ on $\mathfrak{S}_{g,n}$.

Using (2.28) we have

$$R(w)=\pi R_0(w)+\sum_{i=1}^{n}R_i(w),$$

where

$$R_0(w)=-\sum_{j=1}^{3g-3+n}c_jP_j(w),\quad R_i(w)=\sum_{j=1}^{3g-3+n}(\mathscr{E}_i,M_j)P_j(w).$$

In the next theorem, using identification of cotangent spaces to $\mathfrak{S}_{g,n}$ at each point $(\Sigma^\mu,w_1^\mu,\dots,w_n^\mu)$ with $\mathscr{H}^{2,0}(\Omega_0^\mu,\Sigma^\mu)$ (see Sect. 2.3), we explicitly describe canonical connections on the Hermitian line bundles $\mathscr{L}_i$ and $\mathscr{L}$.

Theorem 1. *Let ∂ and $\bar\partial$ be $(1,0)$ and $(0,1)$ components of de Rham differential on $\mathfrak{S}_{g,n}$. The following statement holds.*

(i) *In a local holomorphic frame canonical connection on the Hermitian line bundle $(\mathscr{L}_i,h_i)$ is given by*

$$h_i^{-1}\partial h_i=-\frac{2}{\pi}R_i,\quad i=1,\dots,n.$$

(ii) *In a local holomorphic frame canonical connection on the Hermitian line bundle $(\mathscr{L},\exp\{S/\pi\})$ is given by*

$$\frac{1}{\pi}\partial S=2R_0.$$

(iii) *The function $\mathscr{S}:\mathfrak{S}_{g,n}\to\mathbb{R}$ given by (3.7) satisfies*

$$\partial\mathscr{S}=2\mathscr{R}.$$

Proof. To prove part (i), it is sufficient to show that

$$\left(\frac{\partial\log h_j^{\varepsilon\mu_i}}{\partial\varepsilon}\right)\bigg|_{\varepsilon=0}=-\frac{2}{\pi}(\mathscr{E}_j,M_i).$$

Repeating verbatim computation in the proof of Lemma 4 we get

$$\left(\frac{\partial\log h_j^{\varepsilon\mu_i}}{\partial\varepsilon}\right)\bigg|_{\varepsilon=0}=\dot F_w^i(w_j).$$

Now using (2.14) and (2.29) we obtain

$$\pi \dot{F}^i_w(w_j) = -\iint\limits_{\mathbb{C}} M_i(w)\left(\frac{1}{(w-w_j)^2} - \frac{1}{w(w-1)}\right) d^2w = -2(\mathscr{E}_j, M_i),$$

and the result follows.

To prove part (ii), it is sufficient to show that

$$\left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} S(\Sigma^{\varepsilon\mu_i}; w_1^{\varepsilon\mu_i}, \dots, w_n^{\varepsilon\mu_i}) = -2\pi c_i, \quad i = 1, \dots, 3g-3+n.$$

We have

$$\mathscr{I} = \left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} S(\Sigma^{\varepsilon\mu_i}; w_1^{\varepsilon\mu_i}, \dots, w_n^{\varepsilon\mu_i}) = \frac{\sqrt{-1}}{2} \lim_{\delta\to 0} \left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} I_\delta(\varepsilon)$$

and

$$I_\delta(\varepsilon) = \iint\limits_{D_\delta^{\varepsilon\mu_i}} \omega(\varphi^{\varepsilon\mu_i}) + \sum_{k=2}^{g} \int\limits_{C_k^{\varepsilon\mu_i}} \theta_{(L_k^{\varepsilon\mu_i})^{-1}}(\varphi^{\varepsilon\mu_i}).$$

The calculation of $\mathscr{I}$ almost verbatim repeats the corresponding computation in the proof of Theorem 1 in [20], where regularization at the punctures is treated as in the proof of Theorem 1 in [19]. Namely, using commutative diagram (2.25) and the change of variables $w \mapsto F^{\varepsilon\mu_i}(w)$, we get

$$I_\delta(\varepsilon) = \iint\limits_{D_\delta(\varepsilon)} (F^{\varepsilon\mu_i})^*(\omega(\varphi^{\varepsilon\mu_i})) + \sum_{k=2}^{g} \int\limits_{C_k} (F^{\varepsilon\mu_i})^*(\theta_{(L_k^{\varepsilon\mu_i})^{-1}}(\varphi^{\varepsilon\mu_i})),$$

where

$$D_\delta(\varepsilon) = D \setminus \cup_{j=1}^n \left\{w \in D \,|\, |F^{\varepsilon\mu_i}(w) - F^{\varepsilon\mu_i}(w_j)| < \delta\right\}.$$

To compute $\partial I_\delta(\varepsilon)/\partial\varepsilon|_{\varepsilon=0}$, we need to differentiate under the integral sign as well as over the variable integration domain $D_\delta(\varepsilon)$. The first computation repeats verbatim the one in [20, Theorem 1], with the only change that now integration goes over D_δ and ∂D_δ instead of D and ∂D as in [20]. For the second contribution we use an elementary formula for differentiating a given 2-form Ω over a smooth family of variable domains $\mathcal{D}(\varepsilon)$,

$$\left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} \iint\limits_{\mathcal{D}(\varepsilon)} \Omega = \int\limits_{\partial\mathcal{D}} i_V(\Omega),$$

where V is a vector field along $\partial\mathcal{D}$ corresponding to the family of curves $\partial\mathcal{D}(\varepsilon)$. In our case we readily obtain

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0}\iint\limits_{D_\delta(\varepsilon)}\omega = -\sum_{j=1}^{n}\int\limits_{\partial D_j(\delta)}|\varphi_w|^2\left(\dot{F}^i(w)-\dot{F}^i(w_j)\right)d\bar{w},$$

where $\partial D_j(\delta)$ are oriented as a boundary of $D_j(\delta)$ (which is opposite to the orientation from ∂D_δ).

Thus as in [20] we get

$$\frac{\partial I_\delta(\varepsilon)}{\partial\varepsilon}\bigg|_{\varepsilon=0} = 2\iint\limits_{D_\delta}\mathcal{S}(J^{-1})(w)M_i(w)dw\wedge d\bar{w}$$

$$-\sum_{j=1}^{n}\int\limits_{\partial D_j(\delta)}|\varphi_w|^2\left(\dot{F}^i(w)-\dot{F}^i(w_j)\right)d\bar{w}+I_1+I_2+I_3,$$

where

$$I_1=-2\sum_{j=1}^{n}\int\limits_{\partial D_j(\delta)}\varphi_w\dot{F}^i_{\bar{w}}d\bar{w},\quad I_2=-\sum_{j=1}^{n}\int\limits_{\partial D_j(\delta)}\varphi_w\dot{F}^i_w dw,$$

$$I_3=-\sum_{j=1}^{n}\int\limits_{\partial D_j(\delta)}\varphi_{\bar{w}}\dot{F}^i_w d\bar{w}.$$

As in the proof of Theorem 1 in [19], we obtain that I_1, I_2 and I_3 are $o(1)$ as $\delta\to 0$. Also,

$$\lim_{\delta\to 0}\iint\limits_{D_\delta}\mathcal{S}(J^{-1})(w)M_i(w)dw\wedge d\bar{w}=-2\sqrt{-1}(\mathcal{S}(J^{-1}),M_i)=-2\sqrt{-1}\beta_i,$$

and it follows from asymptotic behavior (2.9) that

$$\lim_{\delta\to 0}\sum_{j=1}^{n}\int\limits_{\partial D_j(\delta)}|\varphi_w|^2(\dot{F}^i(w)-\dot{F}^i(w_j))d\bar{w}$$

$$=\lim_{\delta\to 0}\sum_{j=1}^{n}\int\limits_{\partial D_j(\delta)}\left(\frac{\dot{F}^i(w)-\dot{F}^i(w_j)}{|w-w_j|^2}+\frac{2(\dot{F}^i(w)-\dot{F}^i(w_i))}{|w-w_j|^2\log|w-w_j|}\right)d\bar{w}$$

$$=2\pi\sqrt{-1}\sum_{j=1}^{n}\dot{F}^i_w(w_j).$$

Thus we have

$$\mathscr{I}=2\beta_i+\pi\sum_{j=1}^{n}\dot{F}^i_w(w_j)=-2\pi c_i.$$

Part (iii) immediately follows from (i) and (ii). $\square$

Remark 10. One can also restate the proof using cohomological methods developed in [14].

Theorem 2. *The following statements hold.*

(i) *The first Chern form of the Hermitian line bundle* $(\mathscr{L}_i, h_i)$ *is given by*

$$c_1(\mathscr{L}_i, h_i) = \frac{4}{3}\omega_{\mathrm{TZ},i}, \quad i = 1, \dots, n.$$

(ii) *The first Chern form of the Hermitian line bundle* $(\mathscr{L}, \exp\{S/\pi\})$ *is given by*

$$c_1(\mathscr{L}, \exp\{S/\pi\}) = \frac{1}{\pi^2}\omega_{\mathrm{WP}}.$$

(iii) *The function* $\mathscr{S}$ *given by* (3.7) *satisfies*

$$\bar{\partial}\partial\mathscr{S} = -2\sqrt{-1}\left(\omega_{\mathrm{WP}} - \frac{4\pi^2}{3}\omega_{\mathrm{TZ}}\right),$$

i.e., $-\mathscr{S}$ *is a potential for this special combination of WP and TZ metrics.*

Proof. Since

$$c_1(\mathscr{L}_i, h_i) = \frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial \log h_i,$$

the proof of part (i) is exactly the same as that of Proposition 1. Using part (ii) of Theorem 1, we obtain the proof of part (ii) by repeating, line by line, the computation in [20, Theorem 2]. Part (iii) immediately follows from (i) and (ii). □

Remark 11. As in case of the moduli space $\mathfrak{M}_{0,n}$ (see Remark 9), the combination $\omega_{\mathrm{WP}} - \frac{4\pi^2}{3}\omega_{\mathrm{TZ}}$, with the overall factor $1/12\pi$, appears in the local index theorem for families on punctured Riemann surfaces for $k = 0, 1$ (see [13, Theorem 1]). Part (iii) of Theorem 2 agrees with the fact that the Hodge line bundle λ_1 is holomorphically trivial over $\mathfrak{S}_{g,n}$. It would be interesting to relate the function $\mathscr{S}$ with the Quillen metric in λ_1, defined in [13] (see [21, §3]).

Remark 12. Let $\mathscr{M}_{g,n}$ be the moduli space of n-pointed algebraic curves of genus g. The Hermitian metrics h_i in the line bundles $\mathscr{L}_i$ provide explicit expressions for the pullbacks of the Hermitian metrics in tautological line bundles over $\mathscr{M}_{g,n}$, introduced in [16,18].

5. Generalization to quasi-Fuchsian deformation spaces

Here we define the Liouville action functional on the quasi-Fuchsian deformation spaces of punctured Riemann surfaces and prove that it is a Kähler potential for the

Weil–Petersson metric. The construction follows very closely our work [14] for compact Riemann surfaces, so here we just highlight the necessary modifications and refer to [14] for the details. For the convenience of the reader here we are using the same notations as in [14].

Let Γ be a marked, normalized, quasi-Fuchsian group of type (g, n) such that $3g-3+n>0$. Its region of discontinuity Ω has two invariant components Ω_1 and Ω_2 separated by a quasi-circle $\mathcal{C}$. There exists a quasiconformal homeomorphism J_1 of $\hat{\mathbb{C}}$ such that

QF1 J_1 is holomorphic on $\mathbb{U}$ and $J_1(\mathbb{U}) = \Omega_1$, $J_1(\mathbb{L}) = \Omega_2$, $J_1(\mathbb{R}) = \mathcal{C}$, where $\mathbb{U}$ and $\mathbb{L}$ are, respectively, upper and lower half-planes.
QF2 J_1 fixes $0, 1$ and ∞.
QF3 $\Gamma_1 = J_1^{-1} \circ \Gamma \circ J_1$ is a marked, normalized Fuchsian group.

Let $X \simeq \Gamma\backslash\Omega_1$ and $Y \simeq \Gamma\backslash\Omega_2$ be corresponding marked punctured Riemann surface of type (g, n) with opposite orientations. There is also a quasiconformal homeomorphism J_2 of $\hat{\mathbb{C}}$, holomorphic on $\mathbb{L}$ with a Fuchsian group $\Gamma_2 = J_2^{-1} \circ \Gamma \circ J_2$ so that $X \simeq \Gamma_1\backslash\mathbb{U}$ and $Y \simeq \Gamma_2\backslash\mathbb{L}$. The hyperbolic metric $e^{\phi_{\text{hyp}}(w)}|dw|^2$ on $\Omega = \Omega_1 \sqcup \Omega_2$ is explicitly given by

$$e^{\phi_{\text{hyp}}(w)} = \frac{|(J_i^{-1})_w(w)|^2}{|\operatorname{Im}(J_i^{-1}(w))|^2} \quad \text{if} \quad w \in \Omega_i, \quad i = 1, 2, \tag{5.1}$$

and is a pull-back by the map $J^{-1} : \Omega_1 \sqcup \Omega_2 \to \mathbb{U} \sqcup \mathbb{L}$ of the hyperbolic metric on $\mathbb{U} \sqcup \mathbb{L}$, where $J|_{\mathbb{U}} = J_1|_{\mathbb{U}}$ and $J|_{\mathbb{L}} = J_2|_{\mathbb{L}}$.

Denote by $\mathfrak{D}(\Gamma)$ the deformation space of the quasi-Fuchsian group Γ. It is a complex manifold of complex dimension $6g-6+2n$ with the Weil–Petersson Kähler form (see [14, Sect. 3] and references therein). As in [14], we define the smooth function $S : \mathfrak{D}(\Gamma) \to \mathbb{R}$, the critical value of the Liouville action functional, using homology and cohomology double complexes associated with the Γ-action on Ω.

5.1. Homology construction

Start with marked normalized Fuchsian group Γ of type (g, n) with $2g$ hyperbolic generators $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ and n parabolic generators $\lambda_1, \dots, \lambda_n$ satisfying the single relation

$$\gamma_1 \cdots \gamma_g \lambda_1 \cdots \lambda_n = \text{id},$$

where $\gamma_k = [\alpha_k, \beta_k] = \alpha_k \beta_k \alpha_k^{-1} \beta_k^{-1}$. Here the attracting and repelling fixed points of α_1 are, respectively, 0 and ∞, and the attracting fixed point of β_1 is 1.

The double homology complex $\mathsf{K}_{\bullet,\bullet}$ is defined as $\mathsf{S}_\bullet \otimes_{\mathbb{Z}\Gamma} \mathsf{B}_\bullet$, a tensor product over the integral group ring $\mathbb{Z}\Gamma$, where $\mathsf{S}_\bullet = \mathsf{S}_\bullet(\mathbb{U})$ is the singular chain complex of $\mathbb{U}$ with the differential ∂', considered as a right $\mathbb{Z}\Gamma$-module, and $\mathsf{B}_\bullet = \mathsf{B}_\bullet(\mathbb{Z}\Gamma)$ is the standard bar resolution complex for Γ with the differential ∂''. The associated total complex $\operatorname{Tot}\mathsf{K}$ is equipped with the total differential $\partial = \partial' + (-1)^p\partial''$ on $\mathsf{K}_{p,q}$.

The analog of the total 2-cycle that represents the fundamental class of the compact Riemann surface in [14, Sect. 2.2.1] is the following 2-chain

$$\Sigma = F + L - V,$$

satisfying

$$\partial\Sigma = -\sum_{i=1}^{n} z_i \otimes [\lambda_i], \tag{5.2}$$

where $z_i \in \mathbb{R}$ are fixed points of the parabolic generators λ_i.

Remark 13. Note that $\Sigma - \bar{\Sigma}$, where $\bar{\Sigma} = \bar{F} + \bar{L} - \bar{V}$, is a total 2-cycle in the double complex associated with $\mathbb{U} \sqcup \mathbb{L}$,

$$\partial(\Sigma - \bar{\Sigma}) = 0.$$

Here the elements $F \simeq F \otimes [\,] \in \mathsf{K}_{2,0}$, $L \in \mathsf{K}_{1,1}$ and $V \in \mathsf{K}_{0,2}$ are defined as follows. The element F is a standard fundamental domain for Γ in $\mathbb{U}$ — a closed non-Euclidean polygon with $4g+2n$ edges labeled by a_k, a_k', b_k', b_k, $k = 1,\dots,g$, and c_i, c_i', $i = 1,\dots,n$, satisfying $\alpha_k(a_k') = a_k$, $\beta_k(b_k') = b_k$ and $\lambda_i(c_i') = c_i$. The orientation of the edges is such that

$$\partial' F = \sum_{k=1}^{g}(a_k + b_k' - a_k' - b_k) + \sum_{i=1}^{n}(c_i - c_i').$$

Set $\partial' a_k = a_k(1) - a_k(0)$, $\partial' b_k = b_k(1) - b_k(0)$, $\partial' c_i = c_i(1) - c_i(0)$, so that $a_k(0) = b_{k-1}(0)$, $k = 2,\dots,g$, $a_1(0) = c_n'(0)$, $c_i(0) = c_{i-1}'(0)$, $i = 2,\dots,n$, $c_1(0) = b_g(0)$. The elements $L \in \mathsf{K}_{1,1}$ and $V \in \mathsf{K}_{0,2}$ are given by

$$L = \sum_{k=1}^{g}(b_k \otimes [\beta_k] - a_k \otimes [\alpha_k]) - \sum_{i=1}^{n} c_i \otimes [\lambda_i] \tag{5.3}$$

and

$$\begin{aligned} V = &\sum_{k=1}^{g}\left(a_k(0) \otimes [\alpha_k|\beta_k] - b_k(0) \otimes [\beta_k|\alpha_k] + b_k(0) \otimes \left[\gamma_k^{-1}|\alpha_k\beta_k\right]\right) \\ &- \sum_{k=1}^{g-1} b_g(0) \otimes \left[\gamma_g^{-1}\dots\gamma_{k+1}^{-1}|\gamma_k^{-1}\right] + \sum_{i=1}^{n-1} c_1(0) \otimes [\lambda_1\cdots\lambda_i|\lambda_{i+1}]\,. \end{aligned} \tag{5.4}$$

Finally let P_k be Γ-contracting paths in $\mathbb{U}$ connecting 0 to $b_k(0)$ (see [14, Definition 2.3]), and let

$$W = \sum_{k=1}^{g} \left(P_{k-1} \otimes [\alpha_k|\beta_k] - P_k \otimes [\beta_k|\alpha_k] + P_k \otimes \left[\gamma_k^{-1}|\alpha_k\beta_k\right]\right)$$
$$- \sum_{k=1}^{g-1} P_g \otimes \left[\gamma_g^{-1} \cdots \gamma_{k+1}^{-1}|\gamma_k^{-1}\right] + \sum_{i=1}^{n-1} P_g \otimes [\lambda_1 \cdots \lambda_i|\lambda_{i+1}] .$$

Now let Γ be a marked, normalized, quasi-Fuchsian group of type (g, n) with the $2g$ loxodromic generators $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ and n parabolic generators $\lambda_1, \ldots, \lambda_n$, and let Γ_1 be the Fuchsian group such that $\Gamma_1 = J_1^{-1} \circ \Gamma \circ J_1$. The double complex associated with $\Omega = \Omega_1 \sqcup \Omega_2$ and the group Γ is a push-forward by the map J_1 of the double complex associated with $\mathbb{U} \sqcup \mathbb{L}$ and the group Γ_1. The corresponding total 2-cycle for this complex is given by

$$\Sigma_1 - \Sigma_2 = J_1(\Sigma) - J_1(\bar{\Sigma}) = F_1 - F_2 + L_1 - L_2 - V_1 + V_2,$$

where $F_1 = J_1(F)$, $F_2 = J_1(\bar{F})$, $L_1 = J_1(L)$, $L_2 = J_1(\bar{L})$, $V_1 = J_1(V)$, $V_2 = J_1(\bar{V})$, and we continue to denote by $\Sigma - \bar{\Sigma}$ the total 2-cycle for the double complex associated with $\mathbb{U} \sqcup \mathbb{L}$ and the group Γ_1.

5.2. Cohomology construction

The corresponding double complex in cohomology $\mathsf{C}^{\bullet,\bullet}$ is defined as $\mathsf{C}^{p,q} = \mathrm{Hom}_{\mathbb{C}}(\mathsf{B}_q, \mathsf{A}^p)$, where $\mathsf{A}^\bullet$ is the complexified de Rham complex on $\Omega = \Omega_1 \sqcup \Omega_2$. The associated total complex $\mathrm{Tot}\,\mathsf{C}$ is equipped with the total differential $D = d + (-1)^p\delta$ on $\mathsf{C}_{p,q}$, where d is the de Rham differential and δ is the group coboundary. The natural pairing $\langle\, , \rangle$ between $\mathsf{C}^{p,q}$ and $\mathsf{K}_{p,q}$ is given by the integration over chains (see [14] for details).

Put $\varphi = \phi_{\mathrm{hyp}}$. As in [14], starting from the 2-form

$$\omega[\varphi] = \left(|\varphi_w|^2 + e^\varphi\right) dw \wedge d\bar{w} \in \mathsf{C}^{2,0}$$

(cf. the corresponding 2-form in Sect. 3.2), one constructs the total 2-cocycle $\Psi[\varphi]$ and defines the Liouville action as

$$S_\Gamma = \frac{i}{2}\langle\Psi[\varphi], \Sigma_1 - \Sigma_2\rangle,$$

provided that integrals over F_1 and F_2 exist (as we will show below). Moreover, S_Γ does not depend on the choice of the fundamental domains F_1 and F_2 for Γ in Ω_1 and Ω_2. Simplifying as in [14, Sect. 2.3.3], we finally obtain

$$S_\Gamma = \frac{i}{2}\Big(\langle\omega[\varphi], F_1 - F_2\rangle - \langle\check{\theta}[\varphi], L_1 - L_2\rangle + \langle\check{u}, W_1 - W_2\rangle\Big), \tag{5.5}$$

(cf. [14, formula (2.27)]). Here $W_1 = J_1(W)$, $W_2 = J_1(\bar{W})$, and

$$\check{\theta}_{\gamma^{-1}}[\varphi] = \left(\varphi - \frac{1}{2}\log|\gamma'|^2 - 2\log 2 - \log|c(\gamma)|^2\right)\left(\frac{\gamma''}{\gamma'}dw - \frac{\overline{\gamma''}}{\overline{\gamma'}}d\bar{w}\right) \tag{5.6}$$

(cf. the corresponding 1-form in Sect. 3.2) and

$$\begin{aligned}\check{u}_{\gamma_1^{-1},\gamma_2^{-1}} =& -\left(\frac{1}{2}\log|\gamma_1'|^2 + \log\frac{|c(\gamma_2)|^2}{|c(\gamma_2\gamma_1)|^2}\right)\left(\frac{\gamma_2''}{\gamma_2'}\circ\gamma_1\,\gamma_1'\,dw - \frac{\overline{\gamma_2''}}{\overline{\gamma_2'}}\circ\gamma_1\,\overline{\gamma_1'}\,d\bar{w}\right)\\ &+\left(\frac{1}{2}\log|\gamma_2'\circ\gamma_1|^2 + \log\frac{|c(\gamma_2\gamma_1)|^2}{|c(\gamma_1)|^2}\right)\left(\frac{\gamma_1''}{\gamma_1'}dw - \frac{\overline{\gamma_1''}}{\overline{\gamma_1'}}d\bar{w}\right).\end{aligned}$$

Denote by $z_{1i}, z_{2i} \in \mathbb{R}$, $i = 1, \dots, n$, the fixed points of the parabolic generators of Γ_1 and Γ_2, and by $w_i = J_1(z_{1i}) = J_2(z_{2i}) \in \mathcal{C}$ — the fixed points of the parabolic generators λ_i of Γ. Let $\sigma_{1i}, \sigma_{2i} \in \mathrm{PSL}(2,\mathbb{R})$ be such that $\sigma_{1i}\infty = z_{1i}$ and $\sigma_{2i}\infty = z_{2i}$, $i = 1, \dots, n$.

Lemma 5. *Let $e^{\varphi(w)}|dw|^2$ be the hyperbolic metric on $\Omega = \Omega_1 \sqcup \Omega_2$. Then*

$$\begin{aligned}(\varphi\circ J_1\circ\sigma_{1i})(z) &= 2\log y + O(1) \quad \textit{as} \quad y = \operatorname{Im} z \to \infty,\\ (\varphi\circ J_2\circ\sigma_{2i})(z) &= 2\log|y| + O(1) \quad \textit{as} \quad y = \operatorname{Im} z \to -\infty.\end{aligned}$$

Proof. It is sufficient to prove the first formula. By definition,

$$(\varphi\circ J_1)(z) + \log|J_1'(z)|^2 = -2\log y.$$

Let $\sigma_i \in \mathrm{PSL}(2,\mathbb{C})$ be such that $\sigma_i(\infty) = w_i$. The map $\tilde{J}_1 = \sigma_i^{-1}\circ J_1\circ\sigma_{1i}$ is univalent and preserves ∞, so that in the neighborhood of ∞

$$\tilde{J}_1(z) = a_1 z + a_0 + a_{-1}z^{-1} + a_{-2}z^{-2} + \dots, \quad \text{where} \quad a_1 \neq 0.$$

Whence

$$(J_1\circ\sigma_{1i})(z) = w_i + b_{-1}z^{-1} + b_{-2}z^{-2} + b_{-3}z^{-3} + \dots, \quad \text{where} \quad b_{-1} \neq 0.$$

Thus as $y \to \infty$ we obtain

$$\begin{aligned}(\varphi\circ J_1)(\sigma_{1i}z) &= -\log|J_1'(\sigma_{1i}z)|^2 - \log\operatorname{Im}(\sigma_{1i}z)^2\\ &= -\log|(J_1\circ\sigma_{1i})'(z)|^2 - \log y^2\\ &= 4\log y - 2\log y + O(1). \quad \square\end{aligned}$$

Corollary 4. *The integrals in definition (5.5) of S_Γ are convergent.*

Proof. Since S_Γ does not depend on the choices of fundamental domains for Γ in Ω_1 and Ω_2, we can choose F_1 to be the push-forward by J_1 of a fundamental domain for Γ_1 in $\mathbb{U}$ and F_2 — the push-forward by J_2 of a fundamental domain for Γ_2 in $\mathbb{L}$. It immediately follows from Lemma 5 that the pullback $J_1^*(\omega[\varphi])$ is integrable over the fundamental domain for Γ_1 in $\mathbb{U}$, and $J_2^*(\omega[\varphi])$ — over the fundamental domain for Γ_2 in $\mathbb{L}$. The line integrals in the definition of S_Γ converge as well. □

Using formula (5.5), we define a function $S : \mathfrak{D}(\Gamma) \to \mathbb{R}$ by setting $S(\Gamma') = S_{\Gamma'}$ for every $\Gamma' \in \mathfrak{D}(\Gamma)$.

5.3. Potential for the WP metric on $\mathfrak{D}(\Gamma)$

Let

$$\vartheta(z) = 2\varphi_{zz} - \varphi_z^2 = \begin{cases} 2\mathcal{S}\left(J_1^{-1}\right)(z), & \text{if} \quad z \in \Omega_1 \\ 2\mathcal{S}\left(J_2^{-1}\right)(z), & \text{if} \quad z \in \Omega_2. \end{cases}$$

It follows from Lemma 5 that an automorphic form ϑ of weight 4 for Γ vanishes at the cusps $w_1, \dots, w_n$. As in [14, Sect. 4], the family of automorphic forms ϑ for every $\Gamma' \in \mathfrak{D}(\Gamma)$ determines a $(1,0)$-form $\boldsymbol{\vartheta}$ on $\mathfrak{D}(\Gamma)$. Denote by $d = \partial + \bar\partial$ the decomposition of de Rham differential on $\mathfrak{D}(\Gamma)$ into $(1,0)$ and $(0,1)$ components.

The following result is an exact analog of Theorem 4.1 in [14].

Theorem 3. *On $\mathfrak{D}(\Gamma)$,*

$$\partial S = \boldsymbol{\vartheta}.$$

The proof repeats that of Theorem 4.1 in [14]. The only modification is a δ-truncation of fundamental domains F_1 and F_2 near the cusps $w_1, \dots, w_n$, needed for the application of Stokes' theorem. Lemma 5 shows that in the limit $\delta \to 0$ the corresponding boundary terms vanish.

The next result is exact analog of Theorem 4.2 in [14].

Theorem 4. *The following formula holds on $\mathfrak{D}(\Gamma)$,*

$$d\boldsymbol{\vartheta} = \bar\partial\partial S = -2i\omega_{\mathrm{WP}},$$

so that $-S$ is a Kähler potential of the WP metric on $\mathfrak{D}(\Gamma)$.

The proof repeats that of Theorem 2 and uses Lemma 5. We leave details to the interested reader.

6. Holography and renormalized volume

6.1. Renormalized volume of Schottky 3-manifolds

Here we prove the holography principle, a precise relation between the renormalized hyperbolic volume of the corresponding Schottky 3-manifold and the function $\mathscr{S} = S - \pi \log H$, where S is the regularized Liouville action and H is the Hermitian metric in the line bundle $\mathscr{L}$ over $\mathfrak{S}_{g,n}$ (see Sects. 3.2 and 4.2). In case of the classical Liouville action on $\mathfrak{S}_g$, this relation was proved in [7] for classical Schottky groups and in [14, Remark 6.2] for the general case.

As in Sect. 2.3, let $\Sigma \subset \mathrm{PSL}(2,\mathbb{C})$ be marked normalized Schottky group with the region of discontinuity $\Omega \subset \hat{\mathbb{C}}$, and let $M = \Sigma \backslash \mathbb{U}^3$ be the corresponding hyperbolic 3-manifold with the conformal boundary at infinity $X = \Sigma \backslash \Omega$. Here $\mathbb{U}^3 = \{(z,t) : z \in \mathbb{C},\, t > 0\}$ is the Lobachevsky (hyperbolic) space.

As in [14, Sect. 5], let $\mathsf{K}_{\bullet,\bullet} = \mathsf{S}_\bullet \otimes_{\mathbb{Z}\Sigma} \mathsf{B}_\bullet$ be the corresponding double homology complex, where $\mathsf{S}_\bullet = \mathsf{S}_\bullet(\mathbb{U}^3)$ is the singular chain complex of $\mathbb{U}^3$ with the differential ∂' and $\mathsf{B}_\bullet = \mathsf{B}_\bullet(\mathbb{Z}\Sigma)$ is the standard bar resolution complex for Σ with the differential ∂''.

Let $R \subset \mathbb{U}^3$ be the fundamental region for the marked Schottky group Σ in $\mathbb{U}^3$, identified with $R \otimes [\,] \in \mathsf{K}_{3,0}$. We have $\partial'' R = 0$ and

$$\partial' R = -D + \sum_{i=1}^{g} (H_i - L_i(H_i)) = -D + \partial'' S \tag{6.1}$$

where D is the fundamental domain for Σ in Ω as in Sect. 2.3, H_i is a topological hemisphere[6] with the boundary C_i, and $S \in \mathsf{K}_{2,1}$ is defined by

$$S = -\sum_{i=1}^{g} H_i \otimes L_i^{-1}.$$

Putting $L = \sum_{i=1}^{g} C_i \otimes L_i^{-1} \in \mathsf{K}_{1,1}$, we have $\partial' S = -L$ and

$$\begin{aligned} \partial\,(R - S) &= \partial' R - \partial' S - \partial'' S \\ &= -D + \partial'' S + L - \partial'' S = -D + L. \end{aligned} \tag{6.2}$$

Let $e^{\varphi(w)}|dw|^2$ be the hyperbolic metric on $\Omega_0 = \Omega \setminus \Sigma \cdot \{w_1, \dots, w_n\}$ (see Sect. 2.3). As in [14, Lemma 5.1], there is a Σ automorphic function $f \in C^\infty(\mathbb{U}^3 \cup \Omega_0)$ which is positive on $\mathbb{U}^3$ and uniformly on a compact subsets of Ω_0 satisfies

$$f(Z) = t e^{\varphi(z)/2} + O(t^3) \quad \text{as} \quad t \to 0,$$

[6] It is a Euclidean hemisphere when Σ is a classical Schottky group.

where $Z = (z, t)$. However near $(w_i, 0)$, as it follows from (2.9), the function f satisfies

$$f(Z) = te^{\varphi(z)/2} + O(t^3|z - w_i|^{-2}) \qquad \text{as} \quad t \to 0, \tag{6.3}$$

so that the level surface $f = \varepsilon$ meets $(w_i, 0)$ and is non-compact. Hence in order to use f as a level defining function for the truncated fundamental region $R \cap \{f \geq \varepsilon\}$, one also needs to remove a neighborhoods in $\mathbb{U}^3$ of the points $(w_1, 0), \ldots, (w_n, 0)$. Define

$$R_\varepsilon = R \cap \{f \geq \varepsilon\} \setminus \bigcup_{i=1}^{n} \left\{(z, t) \in \mathbb{U}^3 \,|\, \|(z, t) - (w_i, 0)\| \leq \varepsilon |a_i(1)|\right\},$$

where $\| \ \|$ is the Euclidean distance in $\overline{\mathbb{U}}^3$ (cf. the definition of $S(\Sigma; w_1, \ldots, w_n)$ in Remark 6). As in (6.1),

$$\partial' R_\varepsilon = -D_\varepsilon + \sum_{i=1}^{g} (H_{i,\varepsilon} - L_i(H_{i,\varepsilon})), \tag{6.4}$$

where D_ε is the complement in a level surface $f(Z) = \varepsilon$ of its intersection with the set $\cup_{i=1}^{n}\{\|Z - (w_i, 0)\| \leq \varepsilon |a_i(1)|\}$, and $H_{i,\varepsilon} = R_\varepsilon \cap H_i$.

Following [14], we define the regularized volume of the Schottky 3-manifold M (the regularized on-shell Einstein–Hilbert action) by

$$V_{\text{reg}}(M) = \lim_{\varepsilon \to 0} \left(V_\varepsilon - \frac{1}{2} A_\varepsilon - \frac{1}{2} \pi n \left(\log \varepsilon + 2 \log |\log \varepsilon|\right) - \pi\, \chi(X) \log \varepsilon \right),$$

where V_ε is the hyperbolic volume of R_ε and

$$A_\varepsilon = \iint\limits_{D_\varepsilon} dA,$$

where dA is the area form on D_ε induced by the hyperbolic metric on $\mathbb{U}^3$. Note that the only difference with the [14, Def. 5.1] is the extra subtraction of $\frac{1}{2}\pi n \left(\log \varepsilon + 2 \log |\log \varepsilon|\right)$, which is due the fact that $f(Z)$ blows up as $Z \to (w_i, 0)$.

Repeating almost verbatim computations in [14, Sect. 5.2] and using (6.3), we arrive at the following statement.

Theorem 5. *Let $e^{\varphi(w)}|dw|^2$ be hyperbolic metric on $\Omega \setminus \Sigma \cdot \{w_1, \ldots, w_n\}$. The regularized hyperbolic volume $V_{\text{reg}}(M)$ of the Schottky 3-manifold $M = \Sigma \backslash \mathbb{U}^3$ is well-defined and*

$$V_{\text{reg}}(M) = -\frac{1}{4} \mathscr{S} + \pi(g - 1),$$

where $\mathscr{S}$ is given by (3.7).

Remark 14. Equivalently, the regularized volume $V_{\mathrm{reg}}(M)$ is $-1/4$ times the function $\check{\mathscr{S}} = \check{S} - \pi \log H$, where $\check{S}$ is the Liouville action without the area term.

6.2. Renormalized volume of quasi-Fuchsian 3-manifolds

Here we define the renormalized hyperbolic volume of quasi-Fuchsian 3-manifolds and establish its relation with the classical Liouville action in Sect. 5. For the renormalized hyperbolic volume, another approach to the case of geometrically finite hyperbolic 3-manifolds was developed in [4].

6.2.1. Rank one cusps

Let Γ be marked, normalized, quasi-Fuchsian group of type (g,n) and let $\lambda_1,\dots,\lambda_n$ be its parabolic generators with fixed points $v_1,\dots,v_n \in \mathcal{C}$ (see Sect. 5). Since the stabilizer of a parabolic fixed point v_k in Γ is a cyclic subgroup $\langle\lambda_k\rangle$, it is a *rank one cusp*. Denote by $M = \Gamma\backslash\mathbb{U}^3$ the corresponding quasi-Fuchsian 3-manifold and let $X \sqcup Y = \Gamma\backslash\Omega_1 \sqcup \Omega_2$ be its conformal boundary at infinity.

If $\lambda(z) = z+1$, there exists a $s_0 > 0$ such that the image of the projection $\pi : \mathbb{U}^3 \to M$ of an open horoball

$$\mathcal{H}_s = \{(z,t) \in \mathbb{U}^3 \,|\, t > s\}$$

is embedded into M for $s \geq s_0$. In this case, $\pi(\mathcal{H}_s) \subset M$ is homeomorphic to $\{0 < |z| < 1\} \times \mathbb{R}$ and $\pi(\{(z,t) \in \mathbb{U}^3 \,|\, t = s\})$ corresponds to $\{|z| = 1\} \times \mathbb{R}$. The set $\pi(\mathcal{H}_s)$ is called a *solid cusp tube*.

In general, if a rank one cusp $v = \infty$ is associated to the parabolic subgroup generated by $\lambda = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}$, we have

$$\sigma^{-1}\lambda\,\sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{where} \quad \sigma = \begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix}, \tag{6.5}$$

and σ maps an open horoball $\mathcal{H}_s$ onto $\mathcal{H}_{|q|s}$. In this case, the corresponding solid cusp tube is $\pi(\mathcal{H}_{|q|s})$. When a rank one cusp v_i is finite and is associated with the parabolic subgroup generated by

$$\lambda_i = \begin{pmatrix} 1 + q_i v_i & -q_i v_i^2 \\ q_i & 1 - q_i v_i \end{pmatrix},$$

we have

$$\sigma_i^{-1}\lambda_i\sigma_i = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad \text{where} \quad \sigma_i = \begin{pmatrix} q_i^{\frac{1}{2}} v_i & -q_i^{-\frac{1}{2}} \\ q_i^{\frac{1}{2}} & 0 \end{pmatrix}. \tag{6.6}$$

It is easy to see that $\sigma_i(\mathcal{H}_s)$ is an open horoball tangent to $\mathbb{C}$ at $\sigma_i(\infty) = v_i$, which is an Euclidean ball with radius of $(2|q_i|s)^{-1}$, and the corresponding solid cusp tube is

$\pi(\sigma_i(\mathcal{H}_s))$. In our case the normalization of Γ is such that all cusps are finite and the solid cusp tubes corresponding to v_i can be chosen to be mutually disjoint in M. We denote $\mathcal{H}_{i,\varepsilon} = \sigma_i(\mathcal{H}_{1/\varepsilon^2})$, $i = 1, \ldots, n$.

6.2.2. Truncation of a fundamental region

Let $R \subset \mathbb{U}^3$ be a fundamental region for Γ in $\mathbb{U}^3$. Put $\mathcal{H}_{\varepsilon_0} = \cup_{i=1}^n (\mathcal{H}_{i,\varepsilon_0} \setminus \overline{\mathcal{P}}_{i,\varepsilon_0})$, where $\mathcal{P}_{i,\varepsilon} = \sigma_i(\mathcal{P}_\varepsilon)$ and

$$\mathcal{P}_\varepsilon := \{(z,t) \in \mathbb{U}^3 \mid z = x + iy, |y| > \varepsilon^{-1}\}.$$

The proof of Γ-automorphic partition of unity in [6, Lemma V.3.1] can be easily adapted to the case of Kleinian groups with parabolic elements. As in [14, Lemma 5.1], we conclude that there exist $\varepsilon_0 > 0$ and a Γ-automorphic function $f \in C^\infty(\mathbb{U}^3_0)$, where $\mathbb{U}^3_0 = \cup_{\gamma\in\Gamma}\gamma(R \setminus \mathcal{H}_{\varepsilon_0})$, satisfying

$$f(Z) = te^{\varphi(z)/2} + O(t^3) \quad \text{as} \quad t \to 0, \tag{6.7}$$

uniformly on compact subsets of $\mathbb{U}^3_0$. Here $e^{\varphi(z)}|dz|^2$ is the hyperbolic metric on $\Omega_1 \sqcup \Omega_2$ (see Sect. 5).

Using the level defining function f we truncate a non-compact fundamental region R as follows:

$$R_\varepsilon = R \setminus \left(\{Z \in R \setminus \mathcal{H}_{\varepsilon_0} : f(Z) \le \varepsilon\} \cup \bigcup_{k=1}^n \mathcal{H}_{k,\varepsilon} \right).$$

Similar to the Schottky case, we define a renormalized hyperbolic volume of the quasi-Fuchsian 3-manifold M by

$$V_{\mathrm{reg}}(M) = \lim_{\varepsilon\to 0} \left(V_\varepsilon - \frac{1}{2}A_\varepsilon - \pi\,\chi(X \sqcup Y)\log\varepsilon \right),$$

where V_ε is the hyperbolic volume of R_ε and A_ε is the area of the surface $-F_\varepsilon = \partial' R_\varepsilon \cap \{f = \varepsilon\}$ in the induced metric. Repeating computation in [14] and analyzing the extra terms due to the removal of a solid cusp tubes from M, on can show that their contribution vanishes as $\varepsilon \to 0$. Thus we arrive at the following statement.

Theorem 6. *Let $e^{\varphi(z)}|dz|^2$ be the hyperbolic metric on $\Omega_1 \sqcup \Omega_2$. The regularized hyperbolic volume $V_{\mathrm{reg}}(M)$ of the quasi-Fuchsian 3-manifold $M = \Gamma\backslash\mathbb{U}^3$ is well-defined and*

$$V_{\mathrm{reg}}(M) = -\frac{1}{4}\check{S}_\Gamma,$$

where $\check{S}_\Gamma$ is the Liouville action (5.5) without the area term,

$$\check{S}_\Gamma = S_\Gamma - \iint_F e^\varphi d^2z + 4\pi\chi(X \sqcup Y)\log 2.$$

Note that in this case the statement of the theorem is exactly the same as in the compact case [14, Theorem 5.1]. We leave details to the interested reader.

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