On V.I. Smirnov Thesis, Projective Structures with Real Holonomy, and Liouville Equation

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PDMI 70th Anniversary
Plan

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1. Uniformization of Riemann surfaces and Liouville equation

Let $X$ be a compact Riemann surface of genus $g$ with $n$ distinct marked points $P_1, \ldots, P_n$ and a sequence of integers $2 \leq k_1 \leq k_2 \leq \cdots \leq k_n \leq \infty$ satisfying

$$2g + \sum_{i=1}^{n} \left(1 - \frac{1}{k_i}\right) > 2,$$

and let

$$\mathbb{H} = \{w \in \mathbb{C} : w = u + iv, v > 0\}$$

be the upper half-plane model of the Lobachevsky plane with the hyperbolic metric

$$ds^2 = \frac{du^2 + dv^2}{v^2}$$

and the group of motions $\text{PSL}(2, \mathbb{R})$. 
The Uniformization Theorem (Klein, Poincaré, Koebe): There is a unique complex-analytic covering $J : \mathbb{H} \to X$, ramified over the marked points $P_i$ with ramification indices $k_i$, such that the group of deck transformations is isomorphic, up to a conjugation in $\text{PSL}(2, \mathbb{C})$, to the Fuchsian group $\Gamma$ of the first kind satisfying

$$X \simeq \Gamma \backslash \mathbb{H}^*.$$ 

$J^{-1} : X \to \mathbb{H}$ is a “linearly-polymorphic function” — a multi-valued function whose branches are related by Möbius transformations in $\Gamma \subset \text{PSL}(2, \mathbb{R})$. The pullback of the hyperbolic metric on $\mathbb{H}$ by $J^{-1}$ is well-defined and is a hyperbolic metric $e^{\varphi} |dz|^2$ on $X$, where

$$e^{\varphi(z, \bar{z})} = \frac{|(J^{-1})'(z)|^2}{(\text{Im} J^{-1}(z))^2},$$

and $z$ is a local complex coordinate on $X$. 

(1)
It satisfies the so-called Liouville equation

\[ \frac{\partial^2 \varphi}{\partial z \partial \bar{z}} = \frac{1}{2} e^\varphi \quad \text{on} \quad X \setminus \{ P_1, \ldots, P_n \} \]

(Gaussian curvature = -1), and has the following asymptotic behavior near the marked points:

\[ e^\varphi \simeq \frac{c_i}{|z|^{2\alpha_i}}, \quad \alpha_j = 1 - \frac{1}{k_i} \quad \text{and} \quad z(P_i) = 0. \quad (2) \]

(For the case \( \alpha = 1 \), the factor \(|z|^{-2\alpha}\) should be replaced by \((|z|^2 \log^2 |z|)^{-1}\)).

According to Picard (1893, 1905) and Poincaré (1898), the Liouville equation on \( X \setminus \{ P_1, \ldots, P_n \} \) with asymptotics (2) has a unique solution. (Locally, general solution of the Liouville equation is given by the Liouville formula (1), with arbitrary holomorphic function \( J^{-1} \).)
Conversely, the Liouville equation be used to prove the uniformization theorem. Namely, introduce the so-called stress-energy tensor (Poincaré 1898):

\[ T_\varphi = \frac{\partial^2 \varphi}{\partial z^2} - \frac{1}{2} \left( \frac{\partial \varphi}{\partial z} \right)^2. \]

Then \( T_{cl}(z) \) — the stress-energy tensor evaluated on the solution of the Liouville equation — defines a holomorphic projective connection on \( X \setminus \{P_1, \ldots, P_n\} \) such that second order differential equation

\[ \frac{d^2 y}{d z^2} + \frac{1}{2} T_{cl}(z)y = 0 \quad (3) \]

has regular singular points at \( P_1, \ldots, P_n \) and monodromy group \( \Gamma \), so that the inverse function to the ratio \( y_1/y_2 \) of its two linearly independent solutions defines to the covering map \( J : \mathbb{H} \to X \). The key fact is that \( e^{-\varphi/2} \) satisfies (3). The existence of such a projective connection on \( X \) — the Fuchsian projective connection — cannot be proved by complex-analytic methods only, as Klein and Poincaré were trying to prove.)
Definition

Let \( \{U_\alpha, z_\alpha\}_{\alpha \in A} \) be a complex-analytic covering of a Riemann surface, where \( z_\alpha \) are local coordinates and \( z_\alpha = f_{\alpha \beta}(z_\beta) \) are the transition functions. The set \( R = \{r_\alpha\}_{\alpha \in A} \), where \( r_\alpha \in \mathcal{O}(U_\alpha) \), is called a holomorphic projective connection, if

\[
r_\beta = r_\alpha \circ f_{\alpha \beta}(f'_{\alpha \beta})^2 + \mathcal{S}(f_{\alpha \beta}),
\]

one each intersection \( U_\alpha \cap U_\beta \), where

\[
\mathcal{S}(f) = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2
\]

is the Schwarzian derivative. With every projective connection there is an associated second order linear ordinary differential equation

\[
\frac{d^2 y_\alpha}{dz^2_\alpha} + \frac{1}{2} r_\alpha y_\alpha = 0, \quad \alpha \in A.
\]
Will consider only the special case of Riemann sphere $\mathbb{CP}^1$ with $n \geq 3$ marked points. Namely,

$$\frac{\partial^2 \varphi}{\partial z \partial \bar{z}} = \frac{1}{2} e^\varphi, \quad z \in \mathbb{C} \setminus \{z_1, \ldots, z_{n-3}, 0, 1\},$$

and

$$\varphi = \begin{cases} -2\alpha_i \log |z - z_i| + O(1), & z \to z_i, \ i = 1, \ldots, n - 1; \\ -2(2 - \alpha_n) \log |z| + O(1), & z \to \infty. \end{cases}$$

In this case

$$T_{cl}(z) = \sum_{i=1}^{n-1} \left( \frac{h_i}{2(z - z_i)^2} + \frac{c_i}{z - z_i} \right) + h_i = \alpha_i(2 - \alpha_i),$$

and

$$T_{cl}(z) = \frac{h_n}{2z^2} + O\left(|z|^{-3}\right), \quad z \to \infty,$$

so that

$$\sum_{i=1}^{n-1} c_i = 0 \quad \text{and} \quad \sum_{i=1}^{n-1} (h_i + 2c_i z_i) = h_n. \quad (4)$$

In modern terminology, $\frac{d^2}{dz^2} + \frac{1}{2} T_{cl}(z)$ is called “oper”.

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- The critical value $S_{\text{cl}}$ of the functional $S(\varphi)$ defines a smooth function on the configuration space on $n$ points

$$\mathcal{M}_{0,n} = \{(z_1, \ldots, z_{n-3}) \in \mathbb{C}^{n-3} | z_i \neq 0, 1; z_i \neq z_j, i \neq j\}$$

and

$$c_i = -\frac{1}{2\pi} \frac{\partial S_{\text{cl}}}{\partial z_i}, \quad i = 1, \ldots, n-3.$$

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$$c_i = -\frac{1}{2\pi} \frac{\partial S_{\text{cl}}}{\partial z_i}, \quad i = 1, \ldots, n-3.$$

- For every set of $\alpha_1, \ldots, \alpha_n$ satisfying $0 < \alpha_i \leq 1$ and $\sum_{i=1}^{n} \alpha_i > 2$ the function $-S_{\text{cl}}$ is a Kähler potential of a Kähler metric on the moduli space $\mathcal{M}_{0,n}$. For the case all $\alpha_i = 1$ this metric is the celebrated Weil-Petersson metric.

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- “The function” $\exp \left\{ \frac{1}{12\pi} S_{\text{cl}} \right\}$ defines an Hermitian metric on the holomorphic line bundle $\lambda_S$ over the moduli space $M_g$ of genus $g > 1$ compact Riemann surfaces, and the is an isomorphism of Hermitian line bundles over $M_g$,

$$\left( \lambda_S, \exp \left\{ \frac{1}{12\pi} S_{\text{cl}} \right\} \right) \simeq (\lambda_{\text{Hodge}}, \| \cdot \|_{\text{Quillen}}).$$
Questions

Q1. “Geometric meaning” of the monodromy group of the ordinary differential equation (3) for generic $\alpha_i$ (i.e., not of the form $1 - 1/n_i$, $n_i \geq 2$ an integer)?

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Q2. How to characterize the ordinary differential equation (3) with $T = T_{cl}$ among all equations of the same type?

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Answers

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Answers

A1. The answer is not known, though it should be related to A. Connes “non-commutative differential geometry”.

A2. Naive answer: “the monodromy group of the differential equation (3) should be a subgroup of $\text{PSL}(2, \mathbb{R})$” (as some physicists think) is wrong, as it was shown by V.I. Smirnov in 1918 for the case $n = 4$. 
2. V.I. Smirnov Thesis
V.I. Smirnov Thesis and Liouville Equation

Leon A. Takhtajan

Uniformization of Riemann surfaces and Liouville equation

V.I. Smirnov Thesis (Petrograd, January 1918)

Black-hole solutions of the Liouville equation
Consider the case $n = 4$ with $z_1 = 0$, $z_2 = a$, $z_3 = 1$, $z_4 = \infty$, where $0 < a < 1$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$. Writing the general solution of (4) as

$$
c_1 = 1 + \frac{1}{a} + \frac{2\lambda}{a}, \quad c_2 = -\frac{1}{a} + \frac{1}{a-1} + \frac{2\lambda}{a(a-1)},
$$

$$
c_3 = -1 - \frac{1}{a-1} - \frac{2\lambda}{a-1}
$$

and setting $y = \sqrt{z(z-a)(z-1)} u$, one transforms ordinary differential equation (3), where $z = x \in \mathbb{R}$, to the classical Sturm-Liouville form

$$
\frac{d}{dx} \left( p(x) \frac{du}{dx} \right) + (x + \lambda)u = 0, \quad p(x) = x(x-a)(x-1) \quad (5)
$$

considered, respectively, by Poincaré, Klein and Hilbert. Let $u_0(x, \lambda)$, $u_a(x, \lambda)$, $u_1(x, \lambda)$ and $u_\infty(x, \lambda)$ be solutions of (5) which are holomorphic, respectively, near $x = 0$, $x = a$, $x = 1$ and $x = \infty$, and given by power series with real coefficients.
Theorem (Klein, 1892, 1907; Hilbert 1912)

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I. Solution $u_0(x, \lambda)$ is also holomorphic at $x = a$.  

II. Solution $u_a(x, \lambda)$ is also holomorphic at $x = 1$.

III. Under real continuation through point $x = a$ solution $u_0(x, \lambda)$ is holomorphic at $x = 1$. (By definition, real continuation of $\log(a - x)$ from $x < a$ to $x > a$ is $\log(x - a)$.)

In case I the ratio $\eta = iu_1/u_0$ will transform by real Möbius transformations when continued around singular points $0, a, 1$ and $\infty$. In case II it will be the ratio $\eta = iu_a/u_\infty$, and in case III — the ratio $\eta = iu_0/u_a$. This leads to three Sturm-Liouville boundary problems for differential equation (5) on the intervals $[0, a]$, $[a, 1]$ and $[0, 1]$ with respective boundary conditions I, II and III.
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Theorem (V.I. Smirnov, 1918)

Each of these three Sturm-Liouville problems have simple discrete unbounded spectrum:

\[-a < \mu_1 < \mu_2 < \ldots\]

for the boundary condition I,

\[-a > \mu_{-1} > \mu_{-2} > \ldots\]

for the boundary condition II, and

\[\ldots < \lambda_{-1} < \lambda_0 < \lambda_1 < \ldots\]

for the boundary condition III. These eigenvalues are arranged as follows:

\[\ldots < \mu_{-2} < \lambda_{-1} < \mu_{-1} < \lambda_0 < \mu_1 < \lambda_1 < \mu_2 < \ldots\]
The case $\lambda = \lambda_0$ corresponds to the Fuchsian uniformization of the Riemann surface $X = \mathbb{C} \setminus \{0, a, 1\}$, where the image of $\mathbb{H}$ under the developing map $\eta$, which is 1-1 in this case, is the following geodesic quadrilateral with zero angles (see Fig. 1).

For cases $\lambda = \mu_{-1}$ and $\lambda = \mu_1$ the monodromy groups are real Schottky groups. Corresponding developing map $\eta$ is still 1-1 (these are the only three cases) and the image of $\mathbb{H}$ are, respectively, the following circular domains (see Fig. 2).
When $\lambda = \lambda_k$, the image of $\mathbb{H}$ is the annulus with the quadrilateral wrapping around itself $|k|$ times, and similarly for the cases $\lambda = \mu_k$ and $\lambda = \mu_{-k}$ (see Fig. 3 and Fig. 4).
Now setting $J^{-1} = \eta$, one can pull-back the hyperbolic metric from $\mathbb{C} = \mathbb{H} \cup \mathbb{R} \cup \bar{\mathbb{H}}$. We get a $\mathbb{Z}$-lattice of solutions of the Liouville equation with singularities at $z = 0, a, 1$ and $\infty$ and possible additional singularities along the union of finitely many non-intersecting simple, analytic, closed curves, the pre-image of $\mathbb{R}$ under the developing map $\eta$ — “black-hole” solutions. Namely

- When $\lambda = \lambda_k$, the set $\eta^{-1}(\mathbb{R})$ is the union of $2|k|$ non-intersecting simple, analytic, closed curves around points 0 and $a$ for $k > 0$, and around $a$ and 1 for $k < 0$. 


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- When $\lambda = \mu_k$, $k > 0$, the set $\eta^{-1}(\mathbb{R})$ is the union of $2k - 1$ non-intersecting simple, analytic, closed curves around points $0$ and $a$. 
Now setting $J^{-1} = \eta$, one can pull-back the hyperbolic metric from $C = \mathbb{H} \cup \mathbb{R} \cup \bar{\mathbb{H}}$. We get a $\mathbb{Z}$-lattice of solutions of the Liouville equation with singularities at $z = 0, a, 1$ and $\infty$ and possible additional singularities along the union of finitely many non-intersecting simple, analytic, closed curves, the pre-image of $\mathbb{R}$ under the developing map $\eta$ — “black-hole” solutions. Namely

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The V.I. Smirnov theorem was the first result on classification of projective connections with real monodromy on a given Riemann surface. This problem was discussed by Faltings in 1983 and by Goldman in 1987.

Initially Faltings have thought that there is a unique projective structure with real monodromy and later conjectured that there infinitely many real projective structures.

Goldman generalized prior results of Hejhal, Maskit and Sullivan-Thurston and gave a description of all real projective structures on Riemann surfaces in terms of half-integer Sullivan-Thurston laminations. He proved that this space (as a set) is isomorphic to

\[ T_{g,n} \times \text{“half-integer Sullivan-Thurston measured laminations”} \]

By inverting the so-called “grafting” map, Peter Zograf was able to describe all real projective structures on a given Riemann surface of finite type (work in progress).
3. Black-hole solutions of the Liouville equation

Given: $\mathbb{CP}^1$ with $n$-marked points $z_1, \ldots, z_n$ ($n \geq 4$)

Find: Solution of the Liouville equation with singularities

$$e^\varphi \approx \frac{1}{r_i^2 \log^2 r_i} \quad \text{as} \quad r_i = |z - z_i| \to 0$$

and with the following singular behavior near some simple analytic closed curve $C$,

$$e^\varphi \approx \frac{-4 S'(z_0)}{(S'(z_0)(z - z_0) - (\bar{z} - \bar{z}_0))^2} \quad \text{as} \quad z \to z_0 \in C,$$

where $S$ is the Schwarz function of the contour $C$:

$$C = \{z \in \mathbb{C} : \bar{z} = S(z)\}.$$ 

The curve $C$ needs to be determined as well.
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• Setting $\chi = e^{-\varphi/2}$, one gets the following equation

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- Here $\chi$ vanishes at the points $z_i$ and on the contour $C$, and is positive in the interior of $C$ and negative in the exterior domain. (This corresponds to the case $\varphi$ changes to $\varphi + 2\pi i$ when crossing the contour $C$).
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• Equation (6) is a nonlinear free-boundary problem.
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• Equation (6) is a nonlinear free-boundary problem.
• This type of singular solutions are important for quantum Liouville theory (for understanding the decomposition of the 4-point correlation function in terms of conformal blocks).
QUANTUM LIOUVILLE

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