MAT 127 Calculus C Fall 2002
Midterm II Solutions

1. (a) (10 points) Express the number \(3.212121\ldots\) as a ratio of integers.

   Solution

   \[ 3.212121\ldots = 3 + \frac{21}{100} + \frac{21}{100^2} + \cdots = 3 + \sum_{n=1}^{\infty} \frac{21}{100^n} \]

   \[ = 3 + \frac{21}{100} \cdot \frac{1}{1 - 1/100} = 3 + \frac{21}{99} = \frac{318}{99}, \]

   where we have used that the sum is geometric series with \(a = 21/100\) and \(r = 1/100\). The answer is represented as the ratio of integers (which is a fraction).

   (b) (10 points) For what value of \(q\) does

   \[ \sum_{n=1}^{\infty} q^n \]

   equal 2?

   Solution

   This is geometric series with \(a = r = q\). Assuming that the series converges, we find that its sum is \(q/(1 - q)\). Thus we have an equation

   \[ \frac{q}{1 - q} = 2, \text{ that is } q = 2 - 2q, \text{ or } 3q = 2, \]

   so that \( q = 2/3 \).

2. Determine whether each of the following sequences converges or diverges. If it converges, find the limit. In either case, justify your answer.

   Solutions

   (a) (4 points)

   \[ a_n = \frac{5n^2 + 3}{6n^3 + 2} \]

   It converges and

   \[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{5/n + 3/n^3}{6 + 2/n^3} = \frac{0}{6} = 0. \]

   (b) (3 points)

   \[ a_n = \cos \pi n \]

   It diverges since \( a_n = (-1)^n \), i.e., it is 1 for even \( n \) and \(-1\) for odd \( n \).
(c) (6 points)
\[ a_n = \ln(n^2 + 1) - \ln(n^2) \]
It converges and
\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} \ln \left( \frac{n^2 + 1}{n^2} \right) = \lim_{n \to \infty} \ln \left( 1 + \frac{1}{n^2} \right) = \ln 1 = 0. \]

(d) (6 points)
\[ a_n = \frac{n}{\ln n} \]
It diverges by the L’Hospital rule, since \( x' = 1 \), \((\ln x)' = 1/x\) and
\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{\frac{n}{\ln n}} = \lim_{n \to \infty} n = \infty. \]

(e) (6 points)
\[ a_n = \frac{\sin n}{\sqrt{n}} \]
It converges by the squeeze theorem: \(|\sin n| \leq 1\) and \( \lim_{n \to \infty} 1/\sqrt{n} = 0 \), so that the sequence converges to 0.

3. (30 points) Determine whether each of the following series is convergent or divergent. Justify your answer and state which test (Integral, Comparison, p-Series, etc.) you are using. NOTE: if the series converges, you do not need to find its sum.

Solutions
(a) (4 points)
\[ \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \]
It is convergent by the comparison test with convergent p-Series \( b_n = 1/n^2 \) with \( p = 2 \), because
\[ \frac{1}{n^2 + 1} < \frac{1}{n^2} \]
for all \( n \).

(b) (6 points)
\[ \sum_{n=1}^{\infty} \frac{6}{n^2 - 3n + 10} \]
It is convergent by the limit comparison test with convergent p-Series \( b_n = 1/n^2 \) with \( p = 2 \), because
\[ \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{6n^2}{n^2 - 3n + 10} = \lim_{n \to \infty} \frac{6}{1 - 3/n + 10/n^2} = 6 > 0 \]
(c) (3 points)

\[ \sum_{n=1}^{\infty} (-1)^n \]

It is divergent by the divergence test since \( \lim_{n \to \infty} a_n \) does not exist: \( a_n = 1 \) for even \( n \) and \( a_n = -1 \) for odd \( n \).

(d) (4 points)

\[ \sum_{n=10}^{\infty} \frac{\sin^2 n}{3^n + 5n} \]

It is convergent by comparison test with convergent geometric series \( b_n = 1/3^n \) with \( r = 1/3 \), because

\[ 0 < \frac{\sin^2 n}{3^n + 5n} < \frac{1}{3^n} \]

for all \( n = 1, 2, \ldots \)

(e) (7 points)

\[ \sum_{n=1}^{\infty} (2ne^{-n^2} + 3e^{-n}) \]

It is convergent: the second series is the geometric series with \( r = 1/e < 1 \), whereas the first series is subject to the integral test with \( f(x) = 2xe^{-x^2} \). Indeed, \( f(x) \) is continuous, positive and decreasing on \([1, \infty)\) since

\[ f'(x) = 2e^{-x^2} - 4xe^{-x^2} = 2(1 - 2x^2)e^{-x^2} < 0 \]

whenever \( x \geq 1 \). Using the substitution \( u = x^2 \), we get

\[ \int_{1}^{\infty} f(x)dx = \lim_{t \to \infty} \int_{1}^{t} 2xe^{-x^2}dx = \lim_{t \to \infty} \int_{1}^{t^2} e^{-u}du \]

\[ = \lim_{t \to \infty} \left( e^{-1} - e^{-t^2} \right) = e^{-1} \]

--- the improper integral is convergent.

NOTE One can also use the integral test applied to the function \( 2xe^{-x^2} + 3e^{-x} \), though it is an extra work since we already know that geometric series in question converges.

(f) (6 points)

\[ \sum_{n=1}^{\infty} \frac{\ln n}{n} \]
It is divergent by the comparison test with harmonic series \( b_n = 1/n \) (harmonic series is divergent), because
\[
\frac{\ln n}{n} > \frac{1}{n}
\]
whenever \( n \geq 3 \).

NOTE One can also use the integral test applied to the function \( f(x) = \ln x/x \) and use the substitution \( u = \ln x \), though it would require more work.

4. (15 points) Determine whether the series
\[
\sum_{n=3}^{\infty} \frac{2}{n(n-1)}
\]
is convergent or divergent. If it is convergent, find its sum.

Solution
It is convergent telescopic series. Indeed, by partial fractions,
\[
\frac{2}{n(n-1)} = \frac{2}{n-1} - \frac{2}{n}
\]
so that
\[
\sum_{n=3}^{\infty} a_n = \sum_{n=3}^{\infty} \left( \frac{2}{n-1} - \frac{2}{n} \right)
\]
\[
= \left( \frac{2}{2} - \frac{2}{3} \right) + \left( \frac{2}{3} - \frac{2}{4} \right) + \left( \frac{2}{4} - \frac{2}{5} \right) + \cdots = 1
\]
since \( s_n = 1 - 2/n \) converges to 1.

5. (10 points) It is given that
\[
\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.
\]
Give an estimate for the error \( |s_{100} - \pi^2/8| \). Is \( s_{100} \) greater or less than \( \pi^2/8 \)?

Solution
By the remainder estimate for the integral test,
\[
R_n \leq \int_{n}^{\infty} f(x)dx, \text{ where } f(x) = \frac{1}{(2x+1)^2}
\]
and \( |\pi^2/8 - s_{100}| \leq R_{100} \). We have, doing the standard integral,
\[
\int_{n}^{t} \frac{dx}{(2x+1)^2} = \frac{1}{2(2n+1)} - \frac{1}{2(2t+1)} \rightarrow \frac{1}{2(2n+1)}
\]
as \( t \to \infty \). Thus \( R_{100} \leq 1/2(201) = 1/402 \) and
\[
0 < \pi^2 - s_{100} \leq 1/402,
\]
since obviously \( s_{100} \) is less than \( s = \pi^2/8 \) — the sum of the series.