ANNALS OF MATHEMATICS
Vol. 57, No. 3, May, 1953
Printed in U.S.A.

FREE DIFFERENTIAL CALCULUS. I

Derivation in the Free Group Ring

BY RALPH H. FOX

(Received November 4, 1952)

The free differential calculus grew up naturally out of an analysis that I began in the years 1944–45 of the basic idea of Alexander's knot polynomial [1]. My immediate objective was to solve the outstanding problem of the topological classification of the (3-dimensional) lens spaces by a scheme involving a generalization of Alexander's polynomial. This I have recently succeeded in doing, and a proof that Reidemeister's combinatorial classification [4] of the lens spaces is also the topological classification will appear in a later part of this paper.

As the calculus developed it became increasingly clear that *free differentiation is the fundamental tool for the study of groups defined by generators and relations.* It is closely connected with several of the significant modern developments of algebra and topology and, in fact, reveals hitherto unobserved relations between them.

A. In a cell-complex with a single vertex the 1-cells ξ_j and 2-cells ρ_i correspond to the generators x_j and relations r_i of a presentation of the fundamental group G. The Reidemeister homotopy boundary [3, 5] of ρ_i is $\sum c_{ij}\,\xi_j$, where the coefficients c_{ij} belong to some homomorph of the integral group ring of G. These "incidence numbers" c_{ij} turn out to be homomorphs of the free derivatives $\partial r_i/\partial x_j$. Thus the theory of 2-dimensional Reidemeister homotopy chains may be developed from the fundamental group and the free calculus. Conversely the free calculus, at least as regards its applications to group presentations, may be regarded as an application of the Reidemeister theory. The first viewpoint reduces the theory of 2-dimensional homotopy chains to algebraic algorithms; the second viewpoint gives a topological interpretation of the free calculus. Basically the free calculus and the theory of homotopy chains are founded on the same idea,—the systematic utilization of the "Decktransformationen." The first and the deepest investigations in this direction were those of Reidemeister; these pioneer results now take on new meaning.

B. In the Magnus theory [2] of representation by formal power series the coefficients are free derivatives of various orders evaluated "at the point $x = 1$." Thus in the free calculus the Magnus series plays a rôle analogous to that of the Maclaurin series.

C. From the point of view of the cohomology theory of groups [9] the derivatives are just the 1-dimensional cocycles, if one regards the group ring as operating on itself in a certain way. In particular the partial derivatives $\partial/\partial x_j$ constitute a basis for the 1-dimensional cohomology group of the free group ring (operating on itself in the peculiar way required). Conversely the calculation of the cohomology groups of a group, insofar as it is done algebraically and not by reference to topological models, is an application of the free calculus. Determination of the cohomology groups of dimensions $1, 2, 3, \cdots$ rests on the Jacobian matrices of the group, and may sometimes be carried through explicitly. This has been done for groups with only one defining relation by Lyndon [11] and for certain finite groups by E. Artin and J. Tate (publication forthcoming).

Through their connection with the free calculus the three theories—(A) Reidemeister homotopy chains, (B) Magnus series expansions, (C) Eilenberg-MacLane cohomology of groups—are seen to be related. Recently a connection between the free calculus and the theory of ends of groups has been discovered by B. Eckmann [25].

In Part I below is contained the definition of derivative in a group ring, the definition of the free partial derivatives in the free group ring, and various considerations in the free group ring. In particular the Magnus series expansion is considered in §3. Later parts of this paper will be concerned, among other things, with (a) the application of the free calculus to the isomorphism problem of groups, including the Jacobian matrices of a group and some invariants (like the Alexander polynomial) derived therefrom, Reidemeister homotopy chains, the cohomology groups of a group, algorithms for the calculation of various subgroups and quotient groups, (b) applications to topology, including algorithms for the calculation of the homology groups and other invariants of a covering space, a general "addition theorem" for Alexander polynomials of 3-dimensional manifolds, the topological classification of 3-dimensional lens spaces, and applications to knot theory.

Certain aspects of the free differential calculus were announced in [13], [14], [15], [16] and [18]. Essential use of the calculus has been made in [10], [11], [12] and [22]. In addition there are references to the results of this paper in [5], [17], [19], [20], [21] and [23].

1. Derivatives in a group ring

With any multiplicative group G there is associated its group ring JG with respect to the ring J of rational integers. An element of JG is a sum $\sum a_g g$, g ranging over the elements of G, where the integer a_g is equal to zero for all but a finite number of g. Addition and multiplication in JG are defined by $\sum a_g g + \sum b_g g = \sum (a_g + b_g) g$ and $(\sum a_g g) \cdot (\sum b_g g) = \sum (\sum_h a_{gh^{-1}} b_h) g$. The element a of J is identified with the element $a \cdot 1$ of JG and the element g of G is identified with the element $1 \cdot g$ of JG, so that J and G are to be regarded as subsets of JG.

A homomorphism ψ of a group G into a group H induces a ring-homomorphism of JG into JH. This ring-homomorphism, denoted by the same symbol ψ, is the linear extension of the group-homomorphism, $(\sum a_g g)^\psi = \sum a_g g^\psi$, and leaves fixed each element of J. The kernel of the group-homomorphism ψ is the normal subgroup N consisting of those elements of G that are mapped by ψ into the identity element 1 of H; the kernel of the ring-homomorphism ψ is the both-sided ideal $\mathfrak{N}$ consisting of those elements of JG that are mapped by ψ into the zero element 0 of JH. In this way a both-sided ideal $\mathfrak{N}$ is made to *correspond* to each normal subgroup N. (Note that N and $\mathfrak{N}$ have no elements in common.) Conversely each both-sided ideal $\mathfrak{M}$ in JG *determines* a normal subgroup of G, — the subgroup consisting of those elements of G that are mapped into 1 by the ring-homomorphism $JG \rightarrow JG/\mathfrak{M}$. Clearly *the ideal* $\mathfrak{N}$ *that corresponds to a given normal subgroup* N *determines* N, *and is the smallest ideal of* JG *that determines* N.

If n_1, n_2, *etc. generate* N *in* G *then* $n_1 - 1$, $n_2 - 1$, *etc. generate* N *in* JG. Suppose, in fact, that $\sum a_g g \in N$, so that $\sum a_g g^{\psi} = 0$. Then, for any element h of H, $\sum' a_g = 0$, where $\sum'$ is extended over those elements g for which $g^{\psi} = h$. Let g_0 be an element such that $g_0^{\psi} = h$. Then $\sum' a_g g = \sum' a_g(gg_0^{-1} - 1)g_0 + \sum' a_g g_0 = \sum' a_g(gg_0^{-1} - 1)g_0$. Thus $\sum a_g g$ is a linear combination of the elements $n - 1$, $n \in N$. That $\sum a_g g$ is a linear combination of $n_1 - 1$, $n_2 - 1$, etc. now follows from the identities

$$n^{-1} - 1 = -n^{-1}(n - 1),$$
$$nn' - 1 = (n - 1) + n(n' - 1),$$
$$gng^{-1} - 1 = g(n - 1)g^{-1}.$$

Of special importance is the retraction o of JG upon J induced by the trivial homomorphism $o: G \rightarrow 1$; an element $\sum a_g g$ of JG is mapped by o into its coefficient sum $(\sum a_g g)^o = \sum a_g g^o = \sum a_g$. The kernel of the ring-homomorphism o, i.e. the ideal $\mathfrak{G}$ corresponding to G itself, consists of all elements of coefficient sum zero; $\mathfrak{G}$ will be called the *fundamental ideal* of JG.

By a derivation in a group ring JG will be meant any mapping D of JG into itself which satisfies

$$D(u + v) = Du + Dv, \tag{1.1}$$

$$D(u \cdot v) = Du \cdot v^o + u \cdot Dv, \qquad u, v \in JG. \tag{1.2}$$

Note that, for elements of G, (1.2) takes the simpler form

$$D(gh) = Dg + gDh, \qquad g, h \in G. \tag{1.2$'$}$$

The following consequences of (1.1) and (1.2) are worth noting:

$$Da = 0, \qquad a \in J, \tag{1.3}$$

$$D(\sum a_g g) = \sum a_g Dg, \tag{1.4}$$

$$D(u_1 \cdot u_2 \cdots u_l) = \sum_{i=1}^{l} u_1 \cdots u_{i-1} \cdot Du_i \cdot u_{i+1}^o \cdots u_l^o, \tag{1.5}$$

$$D(g^{-1}) = -g^{-1}Dg, \qquad g \in G. \tag{1.6}$$

The derivations in JG form a right JG-module, where addition is defined by $(D_1 + D_2)u = D_1u + D_2u$ and right-multiplication by an element v of JG is defined by $(D \cdot v)(u) = Du \cdot v$.

2. Derivations in a free group ring

A free group X has a set[1] of generators $(x) = (x_1, x_2, \cdots)$. An element of X is an equivalence class u of words and is represented by a unique reduced word $\prod_{k=1}^{l} x_{j_i}^{\varepsilon_i}$, $\varepsilon_k = \pm 1$, $\varepsilon_k + \varepsilon_{k+1} \neq 0$ if $j_k = j_{k+1}$. By the *length* of u is meant the length l of the representative reduced word. The identity element 1 is represented by the empty word and is of length 0. The inverse u^{-1} of u is represented by the reduced word $\prod_{k=l}^{1} x_{j_i}^{-\varepsilon_i}$.

An element of the free group ring JK is a *free polynomial* $f(x) = \sum a_u u$, $u \in X$, $a_u \in J$, where almost all a_u are equal to zero. A homomorphism ϕ of X into a group G maps (x) into $(x^\phi) = (x_1^\phi, x_2^\phi, \cdots)$. The induced ring-homomorphism $\phi: JX \to JG$ maps $f(x)$ into $f(x^\phi) = \sum a_u u^\phi$. In particular the homomorphism $o: JX \to J$ maps $f(x)$ into $f(1) = \sum a_u x^o = \sum a_u$, the coefficient sum of $f(x)$. The fundamental ideal $\mathfrak{X}$ of JX consists of those polynomials $f(x)$ for which $f(1) = 0$.

The set of derivations in JX has a peculiarly simple structure.

THEOREM. *To each generator x_j of X there corresponds a derivation $f(x) \to D_j f(x) = f_{x_j}(x) = \partial f(x)/\partial x_j$, called the derivative with respect to x_j, which has the property*

$$\frac{\partial x_k}{\partial x_j} = \delta_{j,k} \quad \text{(Kroneker delta).} \tag{2.1}$$

Furthermore there is one and only one derivation $f(x) \to f'(x)$ mapping $x_1, x_2, \cdots$ into prescribed elements $h_1(x), h_2(x), \cdots$ of JX; it is given by the formula

$$f'(x) = \sum \frac{\partial f(x)}{\partial x_j} \cdot h_j(x). \tag{2.2}$$

PROOF. For each index j and element u of X define

$$\begin{aligned} \langle j, u \rangle &= 1 \text{ if } x_j \text{ is an initial segment of the reduced word representing } u, \\ &= 0 \text{ otherwise,} \end{aligned}$$

and extend this definition linearly to JX:

$$\langle j\ f(x) \rangle = \langle j, \sum a_u u \rangle = \sum a_u \langle j, u \rangle.$$

For each index j, element w of X and free polynomial $f(x)$ define

$$\langle j, w, f(x) \rangle = \langle j, w^{-1} f(x) \rangle - \langle j, w^{-1} \rangle f(1).$$

Then $\langle j, w, u \rangle = \langle j, w^{-1}u \rangle - \langle j, w^{-1} \rangle$ vanishes whenever w is not an initial segment of u, for in that case x_j is an initial segment of $w^{-1}u$ if and only if it is an initial segment of w^{-1}. It follows that, for given j and $f(x)$, the integer $\langle j, w, f(x) \rangle = \langle j, w, \sum a_u u \rangle = \sum a_u \langle j, w, u \rangle$ is equal to zero for all but a finite number of the

[1] The set of generators need not be enumerable, but it is convenient to write (x) as a sequence anyway.

elements w of X. The derivative of $f(x)$ with respect to x_j is now defined to be the finite sum

$$\frac{\partial f(x)}{\partial x_j} = \sum_{w \in X} \langle j, w, f(x) \rangle \, w.$$

It is clear that (1.1) is satisfied; it is therefore sufficient to prove the special case (1.2)′ of (1.2). Let $u, v \in X$. Then

$$\begin{aligned}
\frac{\partial (uv)}{\partial x_j} &= \sum_w (\langle j, w^{-1}uv\rangle - \langle j, w^{-1}\rangle)w \\
&= \sum_w (\langle j, w^{-1}u\rangle - \langle j, w^{-1}\rangle)w + \sum_w (\langle j, w^{-1}uv\rangle - \langle j, w^{-1}u\rangle)w \\
&= \sum_w (\langle j, w^{-1}u\rangle - \langle j, w^{-1}\rangle)w + u \sum_t (\langle j, t^{-1}v\rangle - \langle j, t^{-1}\rangle)t \\
&= \frac{\partial u}{\partial x_j} + u \frac{\partial v}{\partial x_j}.
\end{aligned}$$

To prove (2.1), we observe that the only initial segments of x_k are 1 and x_k, hence

$$\begin{aligned}
\frac{\partial x_k}{\partial x_j} &= \langle j, 1, x_k\rangle + \langle j, x_k, x_k\rangle x_k \\
&= (\langle j, x_k\rangle - \langle j, 1\rangle) + (\langle j, 1\rangle - \langle j, x_k^{-1}\rangle)x_k \\
&= (\delta_{jk} - 0) + (0 - 0)x_k.
\end{aligned}$$

Finally we prove (2.2). Since $\partial f(x)/\partial x_j$ vanishes for all but a finite number of indices j, the sum

$$\sum_j \frac{\partial f(x)}{\partial x_j} h_j(x)$$

is a finite sum. Since the derivations in JX form a right JX-module, $f(x) \to \sum_j (\partial f(x)/\partial x_j)h_j(x)$ is a derivation; furthermore $x_k \to h_k(x)$ for each index k. If $f(x) \to f'(x)$ is any derivation mapping $x_1, x_2, \cdots$ into $h_1(x), h_2(x), \cdots$ then $f(x) \to f'(x) - \sum_j (\partial f(x)/\partial x_j)h_j(x)$ is a derivation mapping each x_j into 0, hence each x_j^{-1} into $-x_j^{-1}\cdot 0 = 0$. From (1.1) and (1.2) it follows that every element of JX is mapped into 0; hence $f'(x) = \sum_j (\partial f(x)/\partial x_j)h_j(x)$.

It is easily verified that the mapping $f(x) \to f(x) - f(1)$ is a derivation mapping $x_1, x_2, \cdots$ into $x_1 - 1, x_2 - 1, \cdots$. Hence by (2.2), we obtain the *fundamental formula*

$$f(x) = f(1) + \sum_j \frac{\partial f(x)}{\partial x_j}(x_j - 1). \tag{2.3}$$

This formula shows that any element $f(x)$ of JX can be explicitly recovered from $f(1)$ and the derivatives $D_j f(x)$, $j = 1, 2, \cdots$, in particular any element u of the free group X can be explicitly recovered from its derivatives $\partial u/\partial x_1$, $\partial u/\partial x_2, \cdots$.

The derivative of a power of a generator is easily calculated from the fundamental formula (2.3).

$$(2.4)\quad D_j x_j^p = (x_j^p - 1)/(x_j - 1) = \begin{cases} 1 + x_j + \cdots + x_j^{p-1} & \text{for } p \geqq 1, \\ 0 & \text{for } p = 0, \\ -x_j^p - x_j^{p+1} - \cdots - x_j^{-1} & \text{for } p \leqq -1. \end{cases}$$

From this formula and (1.5) the following practical rule is evolved: Writing $u \in X$ in the form

$$u = u_0 x_j^{p_1} u_1 x_j^{p_2} \cdots u_{q-1} x_j^{p_q} u_q ,$$

where $p_1 \cdots , p_q$ are non-zero integers and the reduced words representing $u_0 , u_1 , \cdots , u_q$ do not involve the generator x_j, we get

$$(2.5)\qquad \frac{\partial u}{\partial x_j} = \sum\nolimits_{i=1}^{q} u_0 x_j^{p_1} \cdots u_{i-1} \frac{x_j^{p_i} - 1}{x_j - 1}.$$

For example, if $m, n > 0$,

$$\begin{aligned} D_1(x_1^m x_2^n x_1^{-m} x_2^{-n}) &= (1 + x_1 + \cdots + x_1^{m-1}) \\ &\qquad + x_1^m x_2^n(-x_1^{-m} - x_1^{-m+1} - \cdots - x_1^{-1}) \\ &= (1 - x_1^m x_2^n x_1^{-m})(1 + x_1 + \cdots + x_1^{m-1}). \end{aligned}$$

This example also illustrates the "chain rule of differentiation":

(2.6) *If λ is a homomorphism of a free group Y into a free group X then, for any $f \in JY$, $\partial f^{\lambda}/\partial x_j = \sum_k (\partial f/\partial y_k)^{\lambda}\, \partial y_k^{\lambda}/\partial x_j$.*

Although (2.5), in conjunction with (2.6), is the most practical way of calculating derivatives in individual cases, it will be useful to have an explicit formula for $\partial u/\partial x_j$ in terms of the representative reduced word of u. In deriving such a formula it becomes apparent that what one needs are not the *initial segments* of a word but a modification of them, which, to avoid confusion, I shall call *initial sections*. The k^{th} initial section of a word $\prod_{i=1}^{l} x_{j_i}^{\varepsilon_i}$, $\varepsilon_k = \pm 1$, is defined to be $\prod_{i=1}^{k-1} x_{j_i}^{\varepsilon_i}$ or $\prod_{i=1}^{k} x_{j_i}^{\varepsilon_i}$ according as $\varepsilon_k = +1$ or $\varepsilon_k = -1$. The k^{th} initial section $u_{(k)}$ of $u \in X$ is defined to be the k^{th} initial section of its representative reduced word. Thus

$$(2.7)\qquad u_{(k)} = \left(\prod_{i=1}^{l} x_{j_i}^{\varepsilon_i}\right)_{(k)} = \prod_{i=1}^{k-1} x_{j_i}^{\varepsilon_i} \cdot x_{j_k}^{(\varepsilon_k - 1)/2}.$$

With this notation for k^{th} section, $k = 1, \cdots , l$, we get the formula

$$(2.8)\qquad \frac{\partial u}{\partial x_j} = \sum \varepsilon_k u_{(k)} ,$$

the summation extended over those indices k for which $j_k = j$.

Because we have the fundamental formula (2.3) we may expect that every property of an element u of X is faithfully mirrored in the properties of its derivatives. For example, the *exponent sum* $\sum_k \varepsilon_k$, summed over those indices k for which $j_k = j$, *of a representative word* $\prod_{i=1}^{l} x_{j_i}^{\varepsilon_i}$ *of an element* u *of* X *is equal to* $(\partial u/\partial x_j)^\circ$, as is seen immediately from (2.8). In fact according to (2.8), $\partial u/\partial x_j$ may be regarded as a weighted exponent sum of x_j in u; the exponent ε_k of $x_{j_k} = x_j$ is "weighted" by the factor $u_{(k)}$. It is remarkable that u is determined by its derivatives $\partial u/\partial x_j$ even though the weighted exponents are added, in formula (2.8), by commutative addition.

It is worth while to note that *on the right hand side of* (2.8) *no cancelling or collecting is possible*. Suppose, on the contrary, that $u_{(k)} = u_{(i)}$ for indices $k < i$. It would follow that $x_{j_k}^{(\varepsilon_k+1)/2}\, x_{j_{k+1}}^{\varepsilon_{k+1}} \cdots x_{j_{i-1}}^{\varepsilon_{i-1}}\, x_{j_i}^{(\varepsilon_i-1)/2} = 1$. However $x_{j_i}^{\varepsilon_i} \cdots x_{j_l}^{\varepsilon_l}$ is, by hypothesis, a reduced word, so that this could only be possible if $i = k + 1$ and $\varepsilon_k = -1$, $\varepsilon_{k+1} = 1$. But this is also impossible for a reduced word $x_{j_1}^{\varepsilon_1} \cdots x_{j_l}^{\varepsilon_l}$. A consequence of this observation is that the x_j*-length* $l_j = \sum |\, \varepsilon_k \,|$, summed over those indices k for which $j_k = j$, *is equal to the number of terms in* $\partial u/\partial x_j$. Also it may now be observed that a free polynomial $f(x)$ can not be the derivative $\partial u/\partial x_j$ of any element u of X unless its coefficients are all $\leqq 1$ in absolute value. (But note that this condition is insufficient; $\partial u/\partial x_j = x_j$ is not possible for any $u \in X$.)

3. Derivatives in JX of higher order

The higher order derivatives are defined inductively

$$\frac{\partial^n f(x)}{\partial x_{j_n} \partial x_{j_{n-1}} \cdots \partial x_{j_1}} = \frac{\partial}{\partial x_{j_n}} \left(\frac{\partial^{n-1} f(x)}{\partial x_{j_{n-1}} \cdots \partial x_{j_1}} \right).$$

Alternative notations for $(\partial^n f(x)/\partial x_{j_n} \cdots \partial x_{j_1})$ are $f_{x_{j_n} \cdots x_{j_1}}(x)$ and $D_{j_n \cdots j_1} f(x)$. From (1.1) and (1.2) one obtains

$$(3.1) \qquad D_{j_n \cdots j_1}(f(x) + g(x)) = D_{j_n \cdots j_1} f(x) + D_{j_n \cdots j_1} g(x),$$

$$(3.2) \quad D_{j_n \cdots j_1}(f(x)g(x)) = \textstyle\sum_{p=1}^{n} D_{j_n \cdots j_p} f(x) D_{j_{p-1} \cdots j_1}\, g(1) + f(x) D_{j_n \cdots j_1}\, g(x).$$

By applying the fundamental formula (2.3) to $f_{x_{j_1}}(x)$ etc. one obtains

$$D_{j_1} f(x) = D_{j_1}\, f(1) + \textstyle\sum_j (D_{j j_1}\, f(x))(x_j - 1),$$

$$D_{j_2 j_1} f(x) = D_{j_2 j_1} f(1) + \textstyle\sum_j (D_{j j_2 j_1} f(x))(x_j - 1),$$

etc., and hence, for each positive integer n,

$$\begin{aligned}(3.3) \quad f(x) = f(1) &+ \textstyle\sum_{j_1}(D_{j_1} f(1))(x_{j_1} - 1) \\ &+ \textstyle\sum_{j_2, j_1}(D_{j_2 j_1} f(1))(x_{j_2} - 1)(x_{j_1} - 1) + \cdots \\ &+ \textstyle\sum_{j_{n-1}, \cdots, j_1}(D_{j_{n-1}, \cdots, j_1} f(1))(x_{j_{n-1}} - 1) \cdots (x_{j_1} - 1) \\ &+ \textstyle\sum_{j_n, \cdots, j_1}(D_{j_n, \cdots, j_1} f(x))(x_{j_n} - 1) \cdots (x_{j_1} - 1).\end{aligned}$$

From this "Taylor series with remainder" one obtains a formal "Taylor series" expansion

(3.4) $f(x)$

$$= f(1) + \sum_j (D_j f(1))(x_j - 1) + \sum_{j,k} (D_{jk} f(1))(x_j - 1)(x_k - 1) + \cdots .$$

It is easy to verify that $\partial^n x_j / \partial x_j^n = 0$ for $n > 1$ and $\partial^n x_j^{-1} / \partial x_j^n = (-1)^n x_j^{-}$, so that the formal expansions of $f(x) = x_j$ and $f(x) = x_j^{-1}$ are

$$x_j = 1 + (x_j - 1),$$
$$x_j^{-1} = 1 - (x_j - 1) + (x_j - 1)^2 - (x_j - 1)^3 + \cdots .$$

These expansions are identical with the expansions (1) of Magnus [2] if one writes a for x_j and s for $x_j - 1$. Furthermore, if the expansions

$$f(x) = f(1) + \sum_j (D_j f(1))(x_j - 1) + \cdots ,$$
$$g(x) = g(1) + \sum_j (D_j g(1))(x_j - 1) + \cdots$$

are formally multiplied together the result is

$$h(x) = f(x)g(x) = f(1)g(1) + \sum_j ((D_j f(1))g(1) + f(1)(D_j g(1)))(x_j - 1) + \cdots$$
$$= h(1) + \sum_j (D_j h(1))(x_j - 1) + \sum_{j,k} (D_{jk} h(1))(x_j - 1)(x_k - 1) + \cdots ,$$

by virtue of formula (3.2). Thus the expansion (3.4), applied to elements of X, is seen to be identical with the Magnus representation [2] of the free group by elements of the "free ring" on the quantities $s_j = x_j - 1$. (Magnus considered only the representation of X, but it is trivial to extend it to a representation of the whole ring JX.)

Using the coefficients $f(1)$, $f_{x_j}(1)$, $f_{x_j x_k}(1)$, etc. and the rules (3.1) and (3.2) would appear to be less cumbersome than using the formal expansion and formal multiplication of expansions, as is required in the Magnus theory.

By repeated application of (2.8) an analogous formula for the higher derivatives of $u \in X$ is obtained:

$$\frac{\partial^n u}{\partial x_{j_n} \cdots \partial x_{j_1}} = \sum_{\lambda_n, \cdots, \lambda_1} \varepsilon_{\lambda_n} \varepsilon_{\lambda_{n-1}} \cdots \varepsilon_{\lambda_1} u_{(\lambda_n)}, \tag{3.5}$$

where the summation is extended over all sequences of indices $\lambda_n , \cdots , \lambda_1$ such that $j_{\lambda_i} = j_i$ for $i = 1, \cdots , n$ and $1 \leqq \lambda_{i+1} \leqq \lambda_i - (1/2)(\varepsilon_{\lambda_i} + 1)$ for $i = 1, \cdots , n - 1$ and $\lambda_1 \leqq l$. Thus $D_{j_n \cdots j_1} u$ consists of the same terms $u_{(k)}$ as $D_j u$ does, but with different coefficients. The absolute values of these coefficients are, of course, $\leqq (l - 1)^{n-1}$.

From (3.2) we have

$$\frac{\partial^n x_j^p}{\partial x_j^n} = \frac{\partial^n x_j^{p-1}}{\partial x_j^n} + \frac{\partial^{n-1} x_j^{p-1}}{\partial x_j^{n-1}} \qquad \text{for} \qquad n \geqq 1.$$

Since

$$(\partial^n x_j^p / \partial x_j^n) = 1 \qquad \text{if } n = 0,$$
$$= 0 \qquad \text{if } p = 0 \text{ and } n > 0,$$

it follows that

$$\left(\frac{\partial^n x_j^p}{\partial x_j^n}\right)^o = \binom{p}{n} \qquad \text{for } n \geqq 0, \tag{3.6}$$

where, as usual, $\binom{p}{n}$ means $(-1)^n \binom{n-p-1}{n}$ if $p < 0$. From (3.3) and (3.6) it follows that

$$\frac{\partial^n x_j^p}{\partial x_j^n} = (x_j - 1)^{-n}\left(x_j^p - \sum_{k=0}^{n-1}\binom{p}{k}(x_j - 1)^k\right) \qquad \text{for} \qquad n \geqq 0. \tag{3.7}$$

It may also be proved, by induction on n, that

$$\begin{aligned} \frac{\partial^n x_j^p}{\partial x_j^n} &= \sum_{i=0}^{p-n}\binom{p-i-1}{n-1} x_j^i \qquad \text{if} \qquad p \geqq 0, n \geqq 1, \\ &= (-1)^n \sum_{i=0}^{-p-1}\binom{n+i-1}{n-1} x_j^{i+p} \qquad \text{if} \qquad p < 0, n \geqq 1. \end{aligned} \tag{3.8}$$

From (2.6), applied to the homomorphism $\lambda : x_j \to x_j$, $x_k \to 1$ for $k \neq j$ of X into the infinite cyclic group generated by x_j, we get

$$\left(\frac{\partial f}{\partial x_j}\right)^o = \left(\frac{\partial f^\lambda}{\partial x_j}\right)^o.$$

for any $f \in JX$. Consequently

$$\left(\frac{\partial^n f}{\partial x_j^n}\right)^o = \left(\frac{\partial^n f^\lambda}{\partial x_j^n}\right)^o.$$

Hence, for any element u of X,

$$\left(\frac{\partial^n u}{\partial x_j^n}\right)^o = \binom{\left(\frac{\partial u}{\partial x_j}\right)^o}{n}. \tag{3.9}$$

This is one of the many identities[2] relating the coefficients of the Magnus expansion of an element of X. The simplest one is

$$\left(\frac{\partial^2 u}{\partial x_j \partial x_k}\right)^o + \left(\frac{\partial^2 u}{\partial x_k \partial x_j}\right)^o = \left(\frac{\partial u}{\partial x_j}\right)^o\left(\frac{\partial u}{\partial x_k}\right)^o, \qquad j \neq k, \tag{3.10}$$

which follows easily from (3.5) and (2.8).

4. Structure of the free group ring

The powers of the fundamental ideal $\mathfrak{G}$ of a group ring JG form a descending chain of ideals

$$\mathfrak{G} \supset \mathfrak{G}^2 \supset G^3 \supset \cdots$$

[2] The complete set of such identities will be derived in [24].

that is closely related to the lower central series[3]

$$G = G_1 \supset G_2 \supset G_3 \supset \cdots$$

of the group G.

(4.1) *A free polynomial $f(x)$ belongs to $\mathfrak{X}^n$ if and only if all of its derivatives of orders* $0, 1, \cdots, n-1$ *vanish at* $x = 1$.

PROOF. If $f(x) \in \mathfrak{X}^n$ then $f(x) = \sum_k f_{1k}(x) f_{2k}(x) \cdots f_{nk}(x)$, where $f_{ik}(1) = 0$. From (3.2) it follows that $f(1) = D_j f(1) = \cdots = D_{j_{n-1} \cdots j_1} f(1) = 0$. Conversely if all the derivatives of orders $0, 1, \cdots, n-1$ vanish at $x = 1$ it follows from (3.3) that $f(x) = \sum_{j_{n-1} \cdots j_1} (D_{j_{n-1} \cdots j_1} f(x))(x_{j_{n-1}} - 1) \cdots (x_{j_1} - 1) \in \mathfrak{X}^n$.

By the *length* $l(f(x))$ of a non-zero free polynomial $f(x) = a_1 u_1 + \cdots + a_m u_m$ will be meant $\max_{i=1,\cdots,m}\{l(u_i)\}$, assuming that $u_i \neq u_k$ for $i \neq k$ and that all coefficients $a_1, \cdots, a_m$ are different from zero. The length of the free polynomial 0 is $l(0) = 0$.

(4.2) LEMMA. *The length of a non-zero free polynomial $f(x)$ of $\mathfrak{X}^n$ is not less than $n/2$.*

PROOF. This will be proved by induction on n. The truth of the statement for $n = 1, 2$, is obvious. Suppose $n \geqq 3$ and let $f(x)$ be an element of $\mathfrak{X}^n$ such that $l = l(f(x)) < n/2$. Each reduced word appearing in $f(x)$ must end in a generator or the inverse of a generator. Hence $f(x) = \sum_j (g^{(j)}(x) \cdot x_j + h^{(j)}(x) \cdot x_j^{-1})$, where $g^{(j)}(x)$ and $h^{(j)}(x)$ are free polynomials of length less than l. Thus $D_j f(x) = a^{(j)}(x) - h^{(j)}(x) \cdot x_j^{-1}$, where $a^{(j)}(x)$ is a free polynomial of length less than l. Therefore, if $i \neq j$, the length of $D_{ij} f(x)$ is less than l, and the length of $D_j((D_j f(x)) \cdot x_j) = D_{jj} f(x) + D_j f(x)$ is less than l. Since $f(x) \in \mathfrak{X}^n$, it follows from (4.1) that $D_{ij}(f(x))$, $i \neq j$, and $D_j((D_j f(x)) \cdot x_j)$ belong to $\mathfrak{X}^{n-2}$. It follows from the inductive hypothesis that $D_{ij} f(x) = 0$, $i \neq j$, and $D_{jj} f(x) = -D_j f(x)$. Therefore, for any j, it follows from (2.3) that

$$D_j f(x) = D_j f(1) + \textstyle\sum_i D_{ij} f(x) \cdot (x_j - 1) = -(D_j f(x)) \cdot (x_j - 1).$$

It follows that $D_j f(x) = (D_j f(x) + (D_j f(x))(x_j - 1)) x_j^{-1} = 0$, so that, by another application of (2.3), $f(x) = f(1) + \sum_j (D_j f(x)) \cdot (x_j - 1) = 0$. This completes the induction.

(4.3) UNIQUENESS THEOREM FOR FORMAL POWER SERIES EXPANSION. *If* $f(1) = g(1)$, $D_j f(1) = D_j g(1)$, $D_{ij} f(1) = D_{ij} g(1)$ *etc. then* $f(x) = g(x)$.

PROOF. Under the conditions stated $f(x) - g(x) \in \mathfrak{X}^n$ for every n. Hence, by (4.2), $f(x) - g(x)$ does not have finite positive length. Therefore $f(x) - g(x) = 0$.

(4.4) COROLLARY. $\bigcap_n \mathfrak{X}^n = 0$.

One can generalize (4.1), with virtually the same proof, to the following, $\mathfrak{R}$ being any ideal of JX that is contained in $\mathfrak{X}$:

(4.5) *A free polynomial $f(x)$ belongs to $\mathfrak{R}\mathfrak{X}^n$ if and only if it belongs to $\mathfrak{X}$ and all of its derivatives of order n belong to* $\mathfrak{R}$ (in which case its derivatives of order i belong to $\mathfrak{R}\mathfrak{X}^{n-1}$, $i = 0, 1, \cdots, n$).

[3] G_i is the subgroup of $G = G_1$ generated by the commutators

$$[g, h] = ghg^{-1}h^{-1}, \; g \in G_{i-1}, \; h \in G.$$

It is known[4] that an element u of X belongs to the n^{th} lower central group X_n if and only if it has "dimension" $\geqq n$, i.e. if and only if the derivatives of u of orders $1, \cdots, n-1$ vanish at $x = 1$. This means that $u \in X_n$ if and only if $u - 1 \in \mathfrak{X}^n$. Thus

(4.6) *The ideal $\mathfrak{X}^n$ determines the n^{th} lower central group X_n.*

Of course X_n is also determined by the ideal $\mathfrak{X}_n$ that corresponds to X_n. This ideal $\mathfrak{X}_n$ may be called the n^{th} *lower central ideal;* $\mathfrak{X}_1 = \mathfrak{X}$ and $\mathfrak{X}_n$ is generated by the ring-commutators $gh - hg$, $g \in \mathfrak{X}_{n-1}$, $h \in \mathfrak{X}$; $\mathfrak{X}_n \subset \mathfrak{X}^n$.

In view of (4.6) the theorem $\bigcap_n \mathfrak{X}^n = 0$ is equivalent to the known [2] fact that $\bigcap_n X_n = 1$. Furthermore, a homomorphism ϕ of X on a group G maps X_n on G_n and $\mathfrak{X}^n$ on $\mathfrak{G}^n$. Thus

(4.7) *The ideal $\mathfrak{G}^n$ determines the n^{th} lower central group G_n.* Consequently $\bigcap_n \mathfrak{G}^n$ determines $\bigcap_n G_n$. This shows that any invariant of G formed from its lower central series $G \supset G_2 \supset G_3 \supset \cdots$ must be calculable from the sequence of ideals $\mathfrak{G} \supset \mathfrak{G}^2 \supset \mathfrak{G}^3 \supset \cdots$. It is worth noting that this latter sequence may be easier to deal with because the quotient ring $\mathfrak{X}^n/\mathfrak{X}^{n-1}$ has an explicit basis $(x_{j_1} - 1)(x_{j_2} - 1) \cdots (x_{j_n} - 1)$, $j_1, j_2, \cdots, j_n = 1, \cdots, q$ of q^n elements while X_n/X_{n-1} is known [7] to be a free abelian group of $\psi_n = \frac{1}{n}\sum_{d|n} \mu(d) \cdot q^{n/d}$ rather elusive generators.[5]

By means of the uniqueness theorem (4.3) one can give a very simple proof of a theorem due to Higman [8]:

(4.8) *The free group ring JX has no divisors of zero.*

Proof. Let $h(x) = f(x) \cdot g(x)$, where $f(x) \neq 0$ and $g(x) \neq 0$. By (4.3) there exist integers m, n such that $f(x) \notin \mathfrak{X}^m$, $f(x) \in X^{m-1}$, $g(x) \notin \mathfrak{X}^n$, $g(x) \in \mathfrak{X}^{n-1}$ (where $\mathfrak{X}^o$ means JX). Then, by (3.2)

$$D_{j_{m+n}\cdots j_1}(h(1)) = D_{j_{m+n}\cdots j_{n+1}}(f(1)) \cdot D_{j_n \cdots j_1}(g(1)) \neq 0$$

for proper choice of the indices $j_{m+n}, \cdots, j_1$. Therefore $h(x) \neq 0$.

It seems reasonable to conjecture that a group ring JG can not have divisors of zero unless G has elements of finite order; this seems to be not an easy question.

In view of (4.5) it would be interesting to know what subgroup of X is determined by the ideal $\mathfrak{R}\mathfrak{X}^n$, if $\mathfrak{R}$ is the ideal corresponding to a normal subgroup R of X. This question is answered by (4.6) for the case $\mathfrak{R} = \mathfrak{X}$, and has a trivial answer for $\mathfrak{R} = 0$ or $n = 0$. Beyond this the only result is the following theorem of Schumann [6] and Blanchfield [10]:

(4.9) *The ideal $\mathfrak{R}\mathfrak{X}$ determines the commutator subgroup $[R, R]$ of R.*

Proof.[6] For any $u, v \in R$ we have

$$D_j([u, v]) = D_j(uvu^{-1}v^{-1}) = (1 - uvu^{-1})D_j u + u(1 - vu^{-1}v^{-1})D_j v \in \mathfrak{R}.$$

[4] See [7]; a simple proof using only the free calculus will appear in [24].

[5] A somewhat simplified system of generators which has a certain additional desirable property will be derived in [24].

[6] This proof differs from the proofs of Schumann [6] and Blanchfield [10]. Cf. also Lyndon [11].

It follows from (4.5) that $[u, v] - 1 \epsilon \mathfrak{R}\mathfrak{X}$. Since $[R, R]$ is generated by the commutators $[u, v]$, $u, v \epsilon R$, it is contained in the subgroup determined by $\mathfrak{R}\mathfrak{X}$.

It must be shown that, conversely, the subgroup determined by $\mathfrak{R}\mathfrak{X}$ is contained in $[R, R]$. Consider an element w such that $w - 1 \epsilon \mathfrak{R}\mathfrak{X}$; we shall prove by induction on the length l of w that $w \epsilon [R, R]$. This is trivially true for $l = 0$; suppose then that $l > 0$. By (4.5), $D_j w \epsilon \mathfrak{R}$ for every j. Since $(D_j w)^\phi = 0$, and since $JG = (JX)^\phi$ is a group ring, the indices $1, \cdots, l$ in the reduced word $x_{j_1}^{\varepsilon_1} \cdots x_{j_l}^{\varepsilon_l}$ representing w may be paired in such a way that to each pair $p < q$ of indices we have $j_p = j_q = j$, $\varepsilon_p + \varepsilon_q = 0$ and

$$(\varepsilon_p w_{(p)} + \varepsilon_q w_{(q)})^\phi = \varepsilon_q(x_{j_1}^{\varepsilon_1} \cdots x_{j_{q-1}}^{\varepsilon_{q-1}} x_{j_q}^{(\varepsilon_q - 1)/2} - x_{j_1}^{\varepsilon_1} \cdots x_{j_{p-1}}^{\varepsilon_{p-1}} x_{j_p}^{(\varepsilon_p - 1)/2})^\phi = 0$$

Thence it follows that $x_{j_{p+1}}^{\varepsilon_{p+1}} \cdots x_{j_{q-1}}^{\varepsilon_{q-1}} \epsilon R$. Note that, since $x_{j_1}^{\varepsilon_1} \cdots x_{j_l}^{\varepsilon_l}$ is a reduced word, q must be larger than $p + 1$, so that $x_{j_{p+1}}^{\varepsilon_{p+1}} \cdots x_{j_{q-1}}^{\varepsilon_{q-1}}$ can not be the empty word.

Now, of the pairs of indices, there must be one $p' < q'$ which is farthest to the left, i.e. if $p < q$ is any other pair of indices than $q' < q$. Then $p'' = q' - 1$ must be paired with an index $q'' > q'$. This shows that $x_{j_1}^{\varepsilon_1} \cdots x_{j_l}^{\varepsilon_l}$ is of the form $w = ax_j^\varepsilon bx_k^\eta x_j^{-\varepsilon} cx_k^{-\eta} d$, where a, b, c, d are reduced words, $bx_k^\eta \epsilon R$ and $x_j^{-\varepsilon} c \epsilon R$. (The indices j, k need not be distinct.) Then $w = ax_j^\varepsilon(bx_k^\eta)(x_j^{-\varepsilon}c)x_k^{-\eta} d \equiv ax_j^\varepsilon(x_j^{-\varepsilon}c)(bx_k^\eta)x_k^{-\eta} d \bmod [R, R]$, hence $w \equiv acbd \bmod [R, R]$. Since $[R, R]$ is contained in the subgroup determined by $\mathfrak{R}\mathfrak{X}$ we have $acbd - 1 \epsilon \mathfrak{R}\mathfrak{X}$. Since the length of $acbd$ is $< l$ it now follows from the inductive hypothesis that $acbd$ belongs to $[R, R]$. Therefore $w \epsilon [R, R]$, completing the induction.

(4.10)[7] *In order that an element $v - 1$ of JX belong to $\mathfrak{R}$ it is necessary and sufficient that there exist an element r of R such that $v - r$ belong to $\mathfrak{R}\mathfrak{X}$.*

Proof. If $r \epsilon R$ and $v - r \epsilon \mathfrak{R}\mathfrak{X}$ then $r \equiv 1 \pmod{\mathfrak{R}}$ and $v - r \equiv 0 \pmod{\mathfrak{R}}$, hence $v - 1 \equiv 0 \pmod{\mathfrak{R}}$. Suppose, conversely, that $v - 1 \epsilon \mathfrak{R}$, i.e. that $v - 1 = \sum_{i=1}^l \varepsilon_i c_i(s_i - 1)b_i$, where $\varepsilon_i = \pm 1$, c_i, $b_i \epsilon X$, $s_i \epsilon R$, hence that $v - 1 = \sum_{i=1}^l a_i(r_i - 1)b_i$ where a_i, $b_i \epsilon X$, $r_i \epsilon R (a_i = c_i s_i^{(1-\varepsilon_i)/2}$ and $r_i = s_i^{\varepsilon_i})$. Define $r = \prod_{i=1}^l a_i r_i a_i^{-1}$, so that $r \epsilon R$. Then $r - 1 = \sum_{i=1}^l a_i(r_i - 1)a_i^{-1}\prod_{k=i+1}^l a_k r_k a_k^{-1}$, so that $r \epsilon R$. Then $r - 1 = \sum_{i=1}^l a_i(r_i - 1)a_i^{-1}\prod_{k=i+1}^l a_k r_k a_k^{-1}$, so that $v - r = \sum_{i=1}^l a_i(r_i - 1)u_i$, where $u_i = b_i - a_i^{-1}\prod_{k=i+1}^l a_k r_k a_k^{-1} \epsilon \mathfrak{X}$. Thus $v - r \epsilon \mathfrak{R}\mathfrak{X}$.

5. Some commutator formulae

(5.1) *If $w \epsilon X_n$ then $(D_{j_r \cdots j_1} w)^\circ = 0$ for $r = 1, \cdots, n - 1$; furthermore if $u, v \epsilon X_n$ then $(D_{j_n \cdots j_1}(uv))^\circ = (D_{j_n \cdots j_1} u)^\circ + (D_{j_n \cdots j_1} v)^\circ$.*

This is a well-known theorem (it was used to deduce (4.6)) but the following proof recommends itself by its simplicity.

Proof. For $n = 1$ the first statement is trivially true and the second follows from (1.2). Let $n > 1$; by the inductive hypothesis it is sufficient to prove the first statement for $w = [u, v]$, where $u \epsilon X_{n-1}$, $v \epsilon X$, and $r = n - 1$. Since

---

[7] This result was proved by Lyndon ([11]. Corollary 4.4); Lyndon's statement is equivalent to ours by a simple application of (4.5). The present proof is much simpler.

$wvu = uv$ we have $(D_{j_{n-1}\cdots j_1}(wvu))^o = (D_{j_{n-1}\cdots j_1}(uv))^o$. By (3.2) and the inductive hypothesis we get from this that

$$(D_{j_{n-1}\cdots j_1}w)^o + (D_{j_{n-1}\cdots j_1}v)^o + (D_{j_{n-1}\cdots j_1}u)^o = (D_{j_{n-1}\cdots j_1}u)^o + (D_{j_{n-1}\cdots j_1}v)^o,$$

hence $(D_{j_{n-1}\cdots j_1}w)^o = 0$. The second statement now follows directly from (3.2).

(5.2) *If* $u \in X_m$ *and* $v \in X_n$ *then*

$$(D_{j_{m+n}\cdots j_1}[u, v])^o = \begin{vmatrix} (D_{j_{m+n}\cdots j_{m+1}}u)^o & (D_{j_m\cdots j_1}u)^o \\ (D_{j_{m+n}\cdots j_{n+1}}v)^o & (D_{j_n\cdots j_1}v)^o \end{vmatrix}.$$

PROOF. This rule follows easily if we write $w = [u, v]$, so that $wvu = uv$. For, by (3.2) and (5.1), we get

$$(D_{j_{m+1}\cdots j_1}w)^o + (D_{j_{m+n}\cdots j_{n+1}}v)^o(D_{j_n\cdots j_1}u)^o = (D_{j_{m+n}\cdots j_{m+1}}u)^o(D_{j_m\cdots j_1}v)^o.$$

Using the notations of §2 the well-known commutator formula

$$[u, vw] = [u, v][u, w]^v, \tag{5.3}$$

where b^a denotes aba^{-1}, may be generalized as follows:

(5.4) *If* $v = \prod_{i=1}^{l} x_{j_i}^{\epsilon_i}$ *then* $[u, v] = \prod_{i=1}^{l} [u, x_{j_i}]^{\epsilon_i v(i)}$.

PROOF. It follows from (5.3) that $[u, v^{-1}] = [u, v]^{-v^{-1}}$. Then $[u, x_{j_i}^{\epsilon_i}] = [u, x_{j_i}]^{\epsilon_i x_{j_i}}$. Hence

$$\prod_{i=1}^{l} [u, x_{j_i}]^{\epsilon_i v(i)} = \prod_{i=1}^{l} [u, x_{j_i}^{\epsilon_i}]^{x_{j_1}^{\epsilon_1}\cdots x_{j_{i-1}}^{\epsilon_{i-1}}} = [u, x_{j_1}^{\epsilon_1}\cdots x_{j_l}^{\epsilon_l}] = [u, v].$$

Since $[u, v]^{-1} = [v, u]$ it follows that $[v, u] = \prod_{i=l}^{1} [x_{j_i}, u]^{\epsilon_i v(i)}$. The two formulae may be combined to express $[u, v]$ as a product of transforms of $[x_j, x_k]$ but the order of the terms in the double product is rather awkward. This can be avoided by reducing modulo $[X_2, X_2]$. The result is the curious formula

$$[u, v] \equiv \prod_{j<k} [x_j, x_k]^{\partial(u,v)/\partial(x_j,x_k)} \bmod [X_2, X_2], \tag{5.5}$$

where $\partial(u, v)/\partial(x_j, x_k)$ means $\partial u/\partial x_j \partial v/\partial x_k - \partial u/\partial x_k\, \partial v/\partial x_j$. The exponent $\partial(u, v)/\partial(x_j, x_k) \in JX$ makes sense only because we are dealing with a congruence mod $[X_2, X_2]$; furthermore the exponent may be considered in X/X_2 so that the multiplication in the exponent is commutative.

PRINCETON UNIVERSITY

BIBLIOGRAPHY

[1] J. W. ALEXANDER, *Topological invariants of knots and links*. Trans. Amer. Math. Soc. 30 (1928), 275–306.

[2] W. MAGNUS, *Beziehungen zwischen Gruppen und Idealen in einem speziellen Ring*. Math. Ann. 111 (1935), 259–280.

[3] K. REIDEMEISTER, *Homotopiegruppen von Komplexen*. Abh. Math. Sem. Hamburgischen Univ. 10 (1934), 211–215.

[4] K. REIDEMEISTER, *Homotopieringe und Linsenräume*. Abh. Math. Sem. Hamburgischen Univ. 11 (1935), 102–109.

[5] K. Reidemeister, *Complexes and homotopy chains*. Bull. Amer. Math. Soc. 56 (1950), 297–307.

[6] H. G. Schumann, *Über Moduln und Gruppenbilder*. Math. Ann. 114 (1935), 385–413.

[7] E. Witt, *True Darstellung Liescher Ringe*. Journ. f. r. a. Math. 177 (1937), 152–160.

[8] G. Higman, *The units of group-rings*. Proc. London Math. Soc. 46 (1940), 231–248.

[9] S. Eilenberg and S. MacLane, *Cohomology theory in abstract groups*. Ann. of Math. 48 (1947), 51–78.

[10] R. C. Blanchfield, *Applications of free differential calculus to the theory of groups*. Senior Thesis, Princeton University, 1949.

[11] R. C. Lyndon, *Cohomology theory of groups with a single defining relation*. Ann. of Math. 52 (1950), 650–665.

[12] G. Torres, *On the Alexander polynomial*. Ann. of Math. 57 (1953), 57–89.

[13] R. H. Fox, *Free differential calculus and the Jacobian matrices of a group*. Bull. Amer. Math. Soc. 55 (1949) Abstract 276.

[14] R. H. Fox, *Free differential calculus and free groups*. Bull. Amer. Math. Soc. 55 (1949) Abstract 512.

[15] R. H. Fox, *On the asphericity of regions in a 3-sphere*. Bull. Amer. Math. Soc. 55 (1949) Abstract 521.

[16] R. H. Fox, *The order of the homology group of a cyclic covering of a knot*. Bull. Amer. Math. Soc. 55 (1949) Abstract 704.

[17] R. H. Fox, *On the total curvature of some tame knots*. Ann. of Math. 52 (1950), 258–260.

[18] R. H. Fox, *Recent development of knot theory at Princeton*. Proc. Int. Congress of Math. (1950), 453–457.

[19] K. T. Chen, *Integration in free groups*. Ann. of Math. 54 (1951), 147–162.

[20] K. T. Chen, *Commutator calculus and link invariants*. Proc. Amer. Math. Soc. 3 (1952), 44–55.

[21] K. T. Chen, *Isotopy invariants of links*. Ann. of Math. 56 (1952), 343–353.

[22] J. W. Milnor, *Link Groups*. Senior Thesis, Princeton University, 1951.

[23] E. Moise, *Affine structures in 3-manifolds V*. Ann. of Math. 56 (1952), 96–144.

[24] K. T. Chen, R. H. Fox and R. C. Lyndon, *On the quotient groups of the lower central series*. Publication forthcoming.

[25] B. Eckmann, *On complexes with operators*. Proc. Nat. Acad. Sci., U. S. A., 39 (1953), 35–42.