

The Symplectic Nature of Fundamental Groups of Surfaces

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A symplectic structure on a manifold is a closed nondegenerate exterior 2-form. The most common type of symplectic structure arises on a complex manifold as the imaginary part of a Hermitian metric which is Kähler. Many moduli spaces associated with Riemann surfaces have such Kähler structures: the Jacobi variety, Teichmüller space, moduli spaces of stable vector bundles and even the first real cohomology group have such structures. In all of these examples the topology of the associated spaces depends, remarkably, only on the topology of the Riemann surface, while often their complex structures vary as the complex structure of the Riemann surface changes. However, the symplectic structure of these spaces depends only on the underlying topological surface.

The purpose of this paper is to present a general explanation for this phenomenon. We present a single construction which unifies all of the above examples and interprets their symplectic structures in terms of the intersection pairing on the surface.

Our setup is as follows. Consider a closed oriented topological surface S with fundamental group π and let G be a connected Lie group. The space $\text{Hom}(\pi, G)$ consisting of representations $\pi \rightarrow G$ (given the compact-open topology) is a real analytic variety (which is an algebraic variety if G is an algebraic group). There is a canonical G -action on $\text{Hom}(\pi, G)$ obtained by composing representations with inner automorphisms of G . The resulting quotient space $\text{Hom}(\pi, G)/G$ is a space canonically associated with S (or π) and G .

When G is an abelian group $\text{Hom}(\pi, G)/G = \text{Hom}(\pi, G) = H^1(S; G)$ has a natural (abelian) group structure. The present paper addresses the question of what sort of natural structure $\text{Hom}(\pi, G)/G$ possesses when G is not necessarily abelian. We find that under fairly general conditions on G (e.g., if it is reductive) $\text{Hom}(\pi, G)/G$ admits in a natural way a symplectic structure which generalizes the Kähler forms on all of the spaces mentioned above.

Perhaps the main specific new result in this paper is that the Weil-

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Petersson Kähler form on Teichmüller space naturally extends to a symplectic structure on the space of equivalence classes of representations of the fundamental group of a surface into $PSL(2, \mathbb{R})$ (and also $PSL(2, \mathbb{C})$). Section 2 discusses how this symplectic structure is a special case of our general constructions (discussed in Section 1) and Section 3 gives an explicit formula in terms of cocycles.

One criterion for naturality of the symplectic geometry on $\text{Hom}(\pi, G)/G$ is invariance under any natural symmetries of $\text{Hom}(\pi, G)/G$. Note that the G -action on $\text{Hom}(\pi, G)$ is really part of an action of $\text{Aut}(\pi) \times G$, where $\text{Aut}(\pi)$ (the group of automorphisms of π) acts on representations $\pi \rightarrow G$ by composition on the left. The action of the normal subgroup $\text{Inn}(\pi)$ consisting of inner automorphisms is absorbed into the G -action so that $\text{Inn}(\pi)$ acts trivially on $\text{Hom}(\pi, G)/G$. It follows that the *outer automorphism group* $\text{Out}(\pi)$ acts on $\text{Hom}(\pi, G)/G$. By a theorem of J. Nielsen, $\text{Out}(\pi)$ is isomorphic to the mapping class group $\pi_0 \text{Diff}(S)$ of S , also called the Teichmüller modular group. If $G = PSL(2, \mathbb{R})$ then the Teichmüller space \mathcal{E}_S of S is a connected component of $\text{Hom}(\pi, G)/G$ [G1] and $\text{Out}(\pi)$ acts properly discontinuously on \mathcal{E}_S ; the quotient $\mathcal{E}_S/\text{Out}(\pi)$ is the Riemann moduli space of complex structures on S .

The space $\text{Hom}(\pi, G)$ is generally a singular algebraic variety and generally the action of G on $\text{Hom}(\pi, G)$ makes $\text{Hom}(\pi, G)/G$ even more singular. Thus we should perhaps mention what we mean by symplectic structure on $\text{Hom}(\pi, G)/G$. The simple points of $\text{Hom}(\pi, G)$ are representations whose images have centralizers of minimum dimension and it is easy to see that G acts freely on these points. The quotient $\text{Hom}(\pi, G)^-/G$ is a (possibly non-Hausdorff) manifold, and a symplectic structure on a manifold is a closed nondegenerate exterior 2-form. In general $\text{Hom}(\pi, G)/G$ will have a Zariski tangent space and we shall require that the symplectic structure on $\text{Hom}(\pi, G)^-/G$ extend continuously (in fact analytically) over $\text{Hom}(\pi, G)/G$ to give a closed 2-form on each singular structure which is nondegenerate on the Zariski tangent space. In [G3] we discuss the singularities of $\text{Hom}(\pi, G)$ further and show, under very general conditions, that they are at worst quadratic.

With these preparatory remarks we can now state the main result.

THEOREM. *Let \mathcal{G} denote the category of Lie groups G with a nonsingular symmetric bilinear form B on the Lie algebra. Let π be the fundamental group of a closed oriented surface of genus 1 and denote by \mathcal{S} the category of symplectic $\text{Out}(\pi)$ -spaces as defined above. Then the correspondence $(G, B) \mapsto \text{Hom}(\pi, G)/G, \omega^{(B)}$ defines a functor $\mathcal{G} \rightarrow \mathcal{S}$ where $\omega^{(B)}$ is the symplectic structure on $\text{Hom}(\pi, G)/G$ defined by B .*

General discussion of the spaces $\text{Hom}(\pi, G)/G$ and their local structure is

presented in Section 1. It is there that the symplectic structure is constructed. The proof that the form is closed is adapted from Atiyah and Bott [AB], where the case $G = U(n)$ is discussed; there $\text{Hom}(\pi, G)/G$ is the space of stable vector bundles. The topological invariance of the symplectic structure in this case has been noticed by Narasimhan and Atiyah and Bott [AB] and I am grateful to R. Bott, V. Guillemin and D. Mumford for helping me to understand this.

In Section 2 is discussed the Weil–Petersson symplectic geometry on Teichmüller spaces as described to me by S. Wolpert and its extension to $\text{Hom}(\pi, PSL(2, \mathbb{R}))/PSL(2, \mathbb{R})$. In Section 3 is given a formula for the fundamental 2-cycle on a surface in terms of the free differential calculus of R. H. Fox. This enables us to write a formula for the symplectic structure, thus verifying that the 2-form is an algebraic tensor in the coordinates coming from the algebraic structure on $\text{Hom}(\pi, G)$. Together Sections 2 and 3 rework the formula of Shimura [SH] for the Weil–Petersson metric in a more understandable manner. In particular we obtain a new proof that the Weil–Petersson metric on Teichmüller space is Kähler (Ahlfors [AH1, AH2]) using the *periods* of quadratic differentials. Note that from this point of view the question of the Weil–Petersson metric being Kähler is rendered equivalent to the same question for the canonical Hermitian metric on the space of stable rank 2 vector bundles of Chern class 0.

In forthcoming papers the symplectic geometry is further discussed. Given any function $f: G \rightarrow \mathbb{R}$ invariant under inner automorphisms and $\gamma \in \pi$, there is an associated function f_γ on $\text{Hom}(\pi, G)/G$ which assigns to an equivalence class $\{\phi\}$ of representations the real number $f \circ \phi(\gamma)$. In [G4], following a suggestion of J. Millson, we compute Poisson brackets on such functions, generalizing and re-proving formulas of Wolpert [WO1, WO2, WO3]. In particular we describe Lie algebra structures on spaces based on closed curves in S for which this Poisson algebra of functions on $\text{Hom}(\pi, G)/G$ is a homomorphic image. For specific choices of G we interpret geometrically the Hamiltonian flows. For example, for $G = PSL(2, \mathbb{C})$ we find Hamiltonian flows on $\text{Hom}(\pi, G)/G$ corresponding to “bending” quasi-Fuchsian groups along simple geodesics à la Thurston. In [G5] the symplectic geometry is used to construct effective ergodic actions of the mapping class group with finite invariant measure.

1. LOCAL PROPERTIES OF $\text{Hom}(\pi, G)/G$

1.1. Let S denote a closed oriented topological surface of genus $p > 1$. Fix a universal covering $\tilde{S} \rightarrow S$ and let π denote the fundamental group of S acting by deck transformations on \tilde{S} . Let G denote a connected Lie group and $\text{Hom}(\pi, G)$ the set of homomorphisms $\pi \rightarrow G$.

$\text{Hom}(\pi, G)$ can be topologized with the compact-open topology; representations $\phi_n: \pi \rightarrow G$ converge to $\phi \in \text{Hom}(\pi, G)$ if and only if for each $\gamma \in \pi$, $\phi_n(\gamma)$ converges to $\phi(\gamma)$. In particular this convergence need only be checked for γ in any preassigned set of generators.

In the special case at hand more can be said explicitly: π admits a presentation with $2p$ generators $A_1, B_1, \dots, A_p, B_p$ subject to a single relation $R(A, B) = A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_p B_p A_p^{-1} B_p^{-1}$. Then $\text{Hom}(\pi, G)$ may be identified with the collection of all $(2p)$ -tuples $(A_1, B_1, \dots, A_p, B_p)$ (henceforth abbreviated (A, B)) of elements of G satisfying $R(A, B) = 1$. This is a "polynomial equation" in the variables (A, B) so that if G is an algebraic group $\text{Hom}(\pi, G)$ is an algebraic variety. Thus there are algebraic local coordinates on $\text{Hom}(\pi, G)$ inherited from coordinates on G^{2p} .

There is a canonical action of $\text{Aut}(\pi) \times G$ on $\text{Hom}(\pi, G)$ where $(\sigma, \gamma) \in \text{Aut}(\pi) \times G$ acts on $\phi \in \text{Hom}(\pi, G)$ by $\phi^{(\sigma, \gamma)}(x) = \gamma(\phi \circ \sigma(x)) \gamma^{-1}$ for $x \in \pi$. If $\sigma_y \in \text{Inn}(\pi)$ is the inner automorphism of π induced by $y \in \pi$ then $(\sigma_y, \phi(y))$ acts identically on ϕ . Since we shall mainly be concerned with the quotient $\text{Hom}(\pi, G)/G$ we may pass to the quotient $\text{Out}(\pi) = \text{Aut}(\pi)/\text{Inn}(\pi)$ to obtain a group which acts more effectively on $\text{Hom}(\pi, G)/G$.

1.2. We wish to analyze the local structure of $\text{Hom}(\pi, G)$ and $\text{Hom}(\pi, G)/G$ near a representation $\phi \in \text{Hom}(\pi, G)$. We begin by finding the Zariski tangent space to $\text{Hom}(\pi, G)$ at ϕ . To this end consider a path ϕ_t in $\text{Hom}(\pi, G)$, depending differentiably on a real parameter t . To first order ϕ_t is a crossed homomorphism of π into the π -module $\mathfrak{G}_{\text{Ad}\phi}$ (where \mathfrak{G} is the Lie algebra of G) obtained by the composition $\pi \xrightarrow{\phi} G \xrightarrow{\text{Ad}} \text{Aut}(\mathfrak{G})$. This can be seen easily by writing

$$\phi_t(x) = \exp(tu(x) + O(t^2)) \phi(x) \tag{1.1}$$

for $x \in \pi$, and for t in some interval about 0 depending on x . The homomorphism condition

$$\phi_t(xy) = \phi_t(x) \phi_t(y) \tag{1.2}$$

implies the cocycle condition

$$u(xy) = u(x) + \text{Ad } \phi(x) u(y). \tag{1.3}$$

Conversely, if u is a 1-cocycle $\pi \rightarrow \mathfrak{G}_{\text{Ad}\phi}$, i.e., satisfies (1.3), then any ϕ_t satisfying (1.3) satisfies (1.2) to first order. Thus the Zariski tangent space to $\text{Hom}(\pi, G)$ at ϕ is the space $Z^1(\pi; \mathfrak{G}_{\text{Ad}\phi})$ of 1-cocycles with values in $\mathfrak{G}_{\text{Ad}\phi}$. In the special case when π is the fundamental group of a surface of genus p , there is the following, which will be proved in 3.7:

PROPOSITION. *Let G be a Lie group which preserves a nondegenerate bilinear form on its Lie algebra, e.g. a reductive Lie group.¹ Then the dimension of $Z^1(\pi; \mathfrak{G}_{\text{Ad } \phi})$ equals $(2p - 1) \dim G + \dim \zeta(\phi)$ where $\zeta(\phi)$ is the centralizer of $\phi(\pi)$ in G . In particular $Z^1(\pi; \mathfrak{G}_{\text{Ad } \phi})$ has minimum for representations ϕ satisfying*

$$\dim \zeta(\phi)/\zeta(G) = 0. \quad (1.4)$$

Here $\zeta(G)$ denotes the center of G .

It follows that $\phi \in \text{Hom}(\pi, G)$ is a simple point of $\text{Hom}(\pi, G)$ if and only if it satisfies (1.4). Accordingly we denote by $\text{Hom}(\pi, G)^-$ the manifold consisting of all ϕ satisfying (1.4). Note that when $G = U(n)$, condition (1.4) is equivalent to irreducibility of ϕ and for $G = SL(2, \mathbb{C})$, condition (1.4) is equivalent to $\phi(\pi)$ being nonabelian, thereby recovering well-known results (see, e.g., [GU, NS1]) in two important special cases.

Even when ϕ is a singular point of $\text{Hom}(\pi, G)$, there is a natural submanifold of $\text{Hom}(\pi, G)$ containing ϕ : namely, ϕ is a simple point of the subvariety $\text{Hom}(\pi, G_1)$ where $G_1 = \zeta_G(\zeta(\phi))$ is the centralizer of the centralizer of $\phi(\pi)$. It turns out that the natural stratification consists of all the G -orbits of such submanifolds $\text{Hom}(\pi, G_1)^-$ and indeed the singular strata of $\text{Hom}(\pi, G)/G$ are themselves all spaces of the form $\text{Hom}(\pi, G_1)^-/G_1$. In a forthcoming paper [G3], we will prove that under fairly general conditions on $\{\phi\}$ in the singular set, the singularity of $\text{Hom}(\pi, G)$ is at worst quadratic.

1.3. Next we turn to the action of G on $\text{Hom}(\pi, G)$ by inner automorphisms. To compute the tangent spaces of orbits G_ϕ consider a deformation ϕ_t which is "trivial" in the sense that it is induced by a path g_t in G :

$$\phi_t(x) = g_t^{-1} \phi(x) g_t.$$

Then it is easy to see that if $g_t = \exp(tu_0 + O(t^2))$, then the cocycle $u: \pi \rightarrow \mathfrak{G}_{\text{Ad } \phi}$ corresponding to ϕ_t is

$$u(x) = \text{Ad } \phi(x) u_0 - u_0,$$

that is, the coboundary δu_0 . It follows immediately that the space $B^1(\pi; \mathfrak{G}_{\text{Ad } \phi})$ of 1-coboundaries is isomorphic as a vector space to the quotient $\mathfrak{G}/\mathfrak{Z}(\phi)$ where $\mathfrak{Z}(\phi)$ is the Lie algebra of $\zeta(\phi)$. Thus

$$\dim G_\phi = \dim B^1(\pi, \mathfrak{G}_{\text{Ad } \phi}) = \dim G = \dim \zeta(\phi).$$

It follows that $G/\zeta(G)$ acts locally freely precisely on $\text{Hom}(\pi, G)^-$.

¹ *Note added in proof.* V. Kac and R. Coley have informed me that a Lie algebra which admits an Ad-invariant nondegenerate bilinear form admits one which is symmetric. Such a Lie algebra is not necessarily reductive.

Unfortunately it is not generally true that $\text{Hom}(\pi, G)^-/G$ is Hausdorff if G is noncompact. For a simple example we may take $G = SL(2, \mathbb{R})$ and π the fundamental group of a surface of genus 2. For simplicity let us consider representations ϕ which satisfy

$$\phi(A_2) = \phi(B_1) = g = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$$

where $a > 1$ is fixed and $\phi(B_2) = \phi(A_1)$. Under these conditions ϕ is completely determined by $\phi(A_1)$, which may be an arbitrary element of G ; moreover $\phi \in \text{Hom}(\pi, G)^-$ precisely when $\phi(A_1)$ is not a diagonal matrix. Consider representations ϕ_1, ϕ_2 with $\phi_1(A_1) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\phi_2(A_1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$; it is easy to check that ϕ_1 and ϕ_2 lie in different G -orbits. However these orbits cannot be separated. For let ϕ_n be the representation determined by

$$\phi_n(A_1) = \begin{bmatrix} (1 + a^{-2n})^{1/2} & a^{-2n} \\ 1 & (1 + a^{-2n})^{1/2} \end{bmatrix}.$$

Then $\phi_n \rightarrow \phi_1$ as $n \rightarrow \infty$. On the other hand $g^n(\phi_n) \rightarrow \phi_2$ proving that the equivalence classes of ϕ_1 and ϕ_2 in $\text{Hom}(\pi, G)/G$ do not have disjoint neighborhoods.

However, if we remove some more representations we may avoid this pathology. For let $\text{Hom}(\pi, G)^{-}$ be the open subset of $\text{Hom}(\pi, G)^-$ consisting of representations whose image does not lie in a subgroup of the form $P \times \zeta(G)$ where P is a parabolic subgroup of the semisimple $[G, G]$. It can be proved that $G/\zeta(G)$ acts properly on $\text{Hom}(\pi, G)^{-}$ (i.e., Gunning [GU] for the case $G = SL(n, \mathbb{C})$ and [JM] for the general case) so that $\text{Hom}(\pi, G)^{-}/G$ is an analytic manifold of dimension $(2p - 2) \dim G + 2 \dim \zeta(G)$.

1.4. It is somewhat remarkable that the points of $\text{Hom}(\pi, G)$ where the G -action is not locally free are precisely the singularities of $\text{Hom}(\pi, G)$. By recasting the preceding discussion in a more general setting, we may reduce this ‘‘coincidence’’ to Poincaré duality.

For the moment let π be an arbitrary finitely generated discrete group and G an arbitrary Lie group. Then the ‘‘Zariski tangent space’’ to $\text{Hom}(\pi, G)/G$ at an equivalence class $\{\phi\}$ is represented as a cohomology group $H^1(\pi; \mathfrak{G}_{\text{Ad}\phi})$. We seek necessary and sufficient conditions that a cohomology class $\xi \in H^1(\pi; \mathfrak{G}_{\text{Ad}\phi})$ be tangent to a path $\{\phi_t\}$ in $\text{Hom}(\pi, G)/G$, or equivalently, that a cocycle $u \in Z^1(\pi; \mathfrak{G}_{\text{Ad}\phi})$ representing ξ is tangent to a differentiable path ϕ_t in $\text{Hom}(\pi, G)$, as in 1.2. Writing

$$\phi_t(x) = \exp(tu(x) + t^2u_2(x) + O(t^3))\phi(x) \tag{1.5}$$

and applying the homomorphism condition (1.2) one obtains

$$u_2(x) - u_2(xy) + \text{Ad } \phi(x) u_2(y) = \frac{1}{2}[u(x), \text{Ad } \phi(x) u(y)]. \tag{1.6}$$

It is easy to check that for any $u, v \in Z^1(\pi; \mathfrak{G}_{\text{Ad } \phi})$ the map

$$(x, y) \mapsto \frac{1}{2}[u(x), \text{Ad } \phi(x) u(y)]$$

defines a 2-cocycle in $Z^2(\pi; \mathfrak{G}_{\text{Ad } \phi})$, and on the level of cohomology this operation is just the product $[\xi, \zeta]: H^1(\pi; \mathfrak{G}_{\text{Ad } \phi}) \times H^1(\pi; \mathfrak{G}_{\text{Ad } \phi}) \rightarrow H^2(\pi; \mathfrak{G}_{\text{Ad } \phi})$ which is cup-product in π using Lie product in \mathfrak{G} as coefficient homomorphism. (In general this operation $H^p(\pi; \mathfrak{G}_{\text{Ad } \phi}) \times H^q(\pi; \mathfrak{G}_{\text{Ad } \phi}) \rightarrow H^{p+q}(\pi; \mathfrak{G}_{\text{Ad } \phi})$ turns $H^*(\pi; \mathfrak{G}_{\text{Ad } \phi})$ into a graded Lie algebra, but we shall not need this extra structure.) Thus finding a second-order term u_2 so that (1.5) defines a homomorphism (to second order) means solving (1.6), which in turn means that the product $[\xi, \zeta] = 0$. It turns out that $[\xi, \zeta]$ is merely the first in an infinite sequence of obstructions (one for each coefficient in the Taylor series for $\phi(x)^{-1} \phi_t(x)$), each defined in terms of the preceding solution on the cochain level, and each taking values in $H^2(\pi; \mathfrak{G}_{\text{Ad } \phi})$. Moreover if each successive obstruction vanishes, then ξ is tangent to a deformation ϕ_t defined for small values of t ; we refer to Nijenhuis and Richardson [NR] for details. In particular if $H^2(\pi; \mathfrak{G}_{\text{Ad } \phi}) = 0$ then every $\xi \in H^1(\pi; \mathfrak{G}_{\text{Ad } \phi})$ is unobstructed and is tangent to a smooth path in $\text{Hom}(\pi, G)/G$.

In general $H^2(\pi; \mathfrak{G}_{\text{Ad } \phi})$ is difficult to compute and this machinery is not as useful as it may first appear. In the case of interest, when π is the fundamental group of a closed oriented surface, Poincaré duality allows an effective calculation.

If V is any π -module (where $\pi = \pi_1(S)$ as before), and V^* its dual, there are natural dual pairings

$$H^i(\pi; V) \times H^{2-i}(\pi; V^*) \rightarrow H^2(\pi; \mathbb{R}) = \mathbb{R},$$

given by cup-product in π using the dual pairing $V \times V^* \rightarrow \mathbb{R}$ as coefficient homomorphism. It follows (taking $i = 2$) that $H^2(\pi; V) \cong H^0(\pi; V^*)^*$ and $H^0(\pi; V^*)$ is just the space of π -invariants in V^* .

We may apply this to the π -module $V = \mathfrak{G}_{\text{Ad } \phi}$ for any $\phi \in \text{Hom}(\pi, G)$. Let us assume that the adjoint G -module $\mathfrak{G}_{\text{Ad } \phi}$ is isomorphic as a G -module to its dual. For example, if G is reductive (i.e., $\text{Ad}(G)$ is semisimple) then there is a nondegenerate symmetric bilinear form $B: \mathfrak{G} \times \mathfrak{G} \rightarrow \mathbb{R}$ invariant under the adjoint representation, which defines an isomorphism $\mathfrak{G}_{\text{Ad } \phi} \cong \mathfrak{G}_{\text{Ad } \phi}^*$. Under this assumption on G , there are isomorphisms

$$H^2(\pi; \mathfrak{G}_{\text{Ad } \phi}) \cong H^0(\pi; \mathfrak{G}_{\text{Ad } \phi}^*)^* \cong H^0(\pi; \mathfrak{G}_{\text{Ad } \phi})^*.$$

Furthermore $H^0(\pi; \mathfrak{G}_{\text{Ad}\phi})$, the space of invariants in $\mathfrak{G}_{\text{Ad}\phi}$, is the infinitesimal centralizer $\mathfrak{Z}(\phi)$ of $\phi(\pi)$ in \mathfrak{G} . It follows that

$$\dim H^2(\pi; \mathfrak{G}_{\text{Ad}\phi}) = \dim H^0(\pi; \mathfrak{G}_{\text{Ad}\phi}) = \dim \zeta(\phi).$$

1.5. We may use this formula to compute the dimension of $\text{Hom}(\pi, G)/G$ at $\{\phi\}$ as follows. Let $\chi(\mathfrak{G}_{\text{Ad}\phi})$ be the alternating sum $\sum (-1)^i \dim H^i(\pi; \mathfrak{G}_{\text{Ad}\phi})$, as usual. Then $\chi(\mathfrak{G}_{\text{Ad}\phi})$ may be computed on the level of cochains as an alternating sum of the dimensions of the cochain groups, provided they are finite dimensional. Working with simplicial cohomology with local coefficients, for example, we can achieve such finite dimensionality. Since the cochains for twisted cohomology are just tensor products $c \otimes \eta$ where c is an ordinary cochain and η an element of the coefficient module (i.e., the structure of the coefficient module appears only in the differential), the Euler characteristic $\chi(\mathfrak{G}_{\text{Ad}\phi})$ is independent of ϕ .

Thus $\chi(\mathfrak{G}_{\text{Ad}\phi}) = \chi(S) \dim \mathfrak{G}$ by replacing ϕ by the trivial representation. It follows that

$$\begin{aligned} \dim H^1(\pi; \mathfrak{G}_{\text{Ad}\phi}) &= 2 \dim H^0(\pi; \mathfrak{G}_{\text{Ad}\phi}) + \chi(S) \dim \mathfrak{G} \\ &= (2p - 2) \dim G + 2 \dim \zeta(\phi). \end{aligned}$$

In particular these dimensions are always even.

1.6. EXAMPLE. In general if the adjoint representation $\mathfrak{G}_{\text{Ad}\phi}$ is not isomorphic to its dual, $\text{Hom}(\pi, G)^-/G$ need not be even dimensional. For let G be the 3-dimensional Heisenberg group of upper unitriangular 3×3 real matrices. Then $\text{Hom}(\pi, G)$ can be naturally identified with a product $\mathbb{R}^{2p} \times Q$ where Q is the quadric in \mathbb{R}^{2p} defined by

$$\sum_{i=1}^p x_i y'_i - x'_i y_i = 0.$$

Thus $\text{Hom}(\pi, G)$ has dimension $3p - 1$. Since G has a 1-dimensional center, $\dim \text{Ad } G = 2$ and $\dim \text{Hom}(\pi, G)^-/G = 3p - 3$.

1.7. For the remainder of the paper we assume that G preserves a nondegenerate symmetric bilinear form B on its Lie algebra \mathfrak{G} . In 1.4 we found a dual pairing

$$H^1(\pi; \mathfrak{G}_{\text{Ad}\phi}) \times H^1(\pi; \mathfrak{G}_{\text{Ad}\phi}) \rightarrow H^2(\pi; \mathbb{R}) = \mathbb{R}$$

defined by the cup-product on π and with B as a coefficient pairing. Regarding $H^1(\pi; \mathfrak{G}_{\text{Ad}\phi})$ as the Zariski tangent space to $\text{Hom}(\pi, G)/G$ at $\{\phi\}$ we may regard this pairing as a 2-tensor $\omega^{(B)} = \omega_{\phi}^{(B)}$ on $\text{Hom}(\pi, G)/G$.

THEOREM. $\omega^{(B)}$ is a closed nondegenerate exterior 2-form on $\text{Hom}(\pi, G)/G$.

Proof. Nondegeneracy follows from Poincaré duality on S and nondegeneracy of B . Similarly that $\omega^{(B)}$ is alternating follows from the fact that cup-product in dimension 1 is alternating and B is symmetric.

1.8. The most insightful proof that $\omega^{(B)}$ is closed follows the approach taken by Atiyah and Bott in their study [AB] of Yang–Mills equations on Riemann surfaces. Our philosophy is that the question of whether a form is closed is best suited for de Rham cohomology even when the *form* is defined in group cohomology! Thus we shall represent $H^1(\pi; \mathfrak{G}_{\text{Ad}\phi})$ as de Rham cohomology of S with coefficients in the flat vector bundle corresponding to $\mathfrak{G}_{\text{Ad}\phi}$.

The de Rham cohomology of a smooth manifold S with coefficients in a flat vector bundle V is the cohomology of the complex $\mathcal{A}^*(S; V)$ of V -valued exterior differential forms on S with a differential d_V arising from canonical local trivializations of V , or what is the same from the flat connection giving the flat structure on V . In local coordinates a V -valued exterior k -form η is a tensor product $\omega \otimes \theta$ where ω is an exterior k -form and θ is a section of V ; its differential is defined to be

$$d\omega \otimes \theta + (-1)^k \omega \otimes d_V \theta$$

where $d_V \theta$ is defined in terms of a local trivialization as follows. Let $x \in S$ and let U be a coordinate neighborhood of x . Then θ is a map $U \rightarrow U \times V_x$ which is the identity on the first factor. Its derivative at x , $T_x \theta: T_x U \rightarrow T_x U \times T_{\theta(x)}(V_x)$, is the identity on $T_x U$ and, using the canonical identification of $T_{\theta(x)}(V_x)$ with V_x , may be identified with a V -valued 1-form $d_V \theta$. For more details see Raghunathan [R, pp. 105–108]. Alternatively $d_V \theta$ is the covariant differential of θ with respect to the canonical connection defining the flat structure on V ; the flatness of the connection easily implies $(d_V)^2 = 0$ so the cohomology of $\mathcal{A}^*(S; V)$ may be defined.

The de Rham theorem for local coefficients implies that the cohomology of $\mathcal{A}^*(S; V)$ is isomorphic to the singular cohomology of S with coefficients in V . When S is a $K(\pi, 1)$ (e.g., a surface) this is isomorphic to the group cohomology $H^*(\pi; V)$.

To rephrase our setup in de Rham theory it is useful to introduce the principal G -bundle P_ϕ associated to ϕ . Recall that P_ϕ may be identified with the quotient of $\tilde{S} \times G$ by the π -action defined by $\gamma: (\tilde{s}, x) \mapsto (\gamma\tilde{s}, \phi(\gamma)x)$; then the projection $\tilde{S} \times G \rightarrow \tilde{S}$ defines a fibration $P_\phi \rightarrow S$. Note that P_ϕ has a canonical flat connection we denote by D_ϕ . Let $\text{ad } P_\phi$ be the \mathfrak{G} -bundle associated to P_ϕ and the adjoint action $G \rightarrow \text{Aut}(\mathfrak{G})$; then $\text{ad } P_\phi$ has a canonical flat connection associated to D_ϕ , which we denote by d_ϕ . Clearly

ad P_ϕ with the flat structure defined by d_ϕ is the flat vector bundle over S corresponding to the π -module $\mathfrak{G}_{\text{Ad } \phi}$. Thus we may identify $H^1(\pi; \mathfrak{G}_{\text{Ad } \phi})$ with the quotient $Z^1(S; \text{ad } P)/(B^1(S; \text{ad } P)$ where $Z^1(S; \text{ad } P) = \text{Ker } d_\phi: \mathcal{A}^1(S; \text{ad } P) \rightarrow \mathcal{A}^2(S; \text{ad } P)$ and $B^1(S; \text{ad } P) = \text{Image } d_\phi: \mathcal{A}^0(S; \text{ad } P) \rightarrow \mathcal{A}^1(S; \text{ad } P)$.

Now let η, θ be two $(\text{ad } P)$ -valued 1-forms on S . Their product $\eta \wedge \theta$ is a 2-form on S taking values in the $\mathfrak{G} \otimes \mathfrak{G}$ -bundle $\text{ad } P \otimes \text{ad } P$ associated to P . The bilinear form $B: \mathfrak{G} \times \mathfrak{G} \rightarrow \mathbb{R}$ defines a bundle map B_* from $\text{ad } P \otimes \text{ad } P$ into the trivial \mathbb{R} -bundle over S . Thus $B_*(\eta \wedge \theta)$ is an exterior 2-form on S and its integral $\int_S B_*(\eta \wedge \theta)$ defines an alternating pairing $\mathcal{A}^1(S; \text{ad } P) \times \mathcal{A}^1(S; \text{ad } P) \rightarrow \mathbb{R}$ which induces the symplectic pairing $\omega_\phi^{(B)}: H^1(S; \text{ad } P) \times H^1(S; \text{ad } P) \rightarrow \mathbb{R}$ on cohomology.

To prove that $\omega^{(B)}$ is a closed form on $\text{Hom}(\pi, G)/G$ we use this description and, following Atiyah and Bott [AB], show that $\omega^{(B)}$ arises from a 2-form on a larger space where its closedness is more transparent. Following [AB], let \mathfrak{a} denote the space of all connections on the principal G -bundle P . Then \mathfrak{a} is an affine space with $\mathcal{A}^1(S; \text{ad } P)$ as its group of translations. In particular each tangent space $T_A \mathfrak{a}$ is identified with $\mathcal{A}^1(S; \text{ad } P)$. Then the pairing

$$(\eta, \theta) \mapsto \int_S B_*(\eta \wedge \theta)$$

$(\eta, \theta \in \mathcal{A}^1(S, \text{ad } P) \cong T_A \mathfrak{a})$ defines an exterior 2-form on the infinite-dimensional affine space \mathfrak{a} . Since its definition does not involve A explicitly, it is invariant under the translations of \mathfrak{a} and is thus closed.

Now let \mathcal{F} denote the subset of \mathfrak{a} consisting of flat connections. If d_A is the covariant differential corresponding to A , then a necessary and sufficient condition for A to be flat is that $d_A \circ d_A = 0$. Differentiating this equation with respect to a tangent vector η in $\mathcal{A}^1(S, \text{ad } P)$ one finds that the tangent vectors to \mathcal{F} are precisely those $\eta \in \mathcal{A}^1(S; \text{ad } P)$ with $d_A \eta = 0$, i.e., $T_A \mathcal{F} = Z^1(S; \text{ad } P)$.

The exterior 2-form $\omega^{(B)}$ on \mathfrak{a} restricts to a closed 2-form on \mathcal{F} . However, on \mathcal{F} this form is degenerate. In fact the subspace of $T_A \mathcal{F}$ which annihilates $\omega^{(B)}$ is precisely $B^1(S; \text{ad } P) \subset Z^1(S; \text{ad } P)$. First let us prove that $B^1(S; \text{ad } P)$ annihilates $\omega^{(B)}$ restricted to $Z^1(S; \text{ad } P)$, that is, if $\eta \in B^1(S; \text{ad } P)$ and $\theta \in Z^1(S; \text{ad } P)$, then $\omega^{(B)}(\eta, \theta) = 0$. This follows from the "Leibnitz rule" in twisted de Rham theory: Let $\sigma \in \mathcal{A}^0(S; \text{ad } P)$ and $\theta \in \mathcal{A}^1(S; \text{ad } P)$; then $\sigma \wedge \theta \in \mathcal{A}^1(S; \text{ad } P \otimes \text{ad } P)$ and $B_*(\sigma \wedge \theta) \in \mathcal{A}^1(S; \mathbb{R})$ is defined. Then for any connection $A \in \mathfrak{a}$, the equation

$$d(B_*(\sigma \wedge \theta)) = B_*(d_A \sigma \wedge \theta) + B_*(\sigma \wedge d_A \theta)$$

is valid in $\mathcal{A}^2(S; \mathbb{R})$. Taking $\eta = d_A \sigma$ and $\theta \in Z^1(S; \text{ad } P)$ we find

$$\omega^{(B)}(\eta, \theta) = \int_S B_*(d_A \sigma \wedge \theta) = \int_S d(\sigma \wedge \theta) = 0$$

since S is closed. It follows that $B^1(S; \text{ad } P)$ lies in the annihilator of $\omega^{(B)}$ restricted to $Z^1(S; \text{ad } P)$.

The subspace $B^1(S; \text{ad } P)$ plays another important role. It is the image under d_A of $\mathcal{A}^0(S; \text{ad } P)$, the space of sections of $\text{ad } P$. This space $\mathcal{A}^0(S; \text{ad } P)$ has a Lie algebra structure coming from the Lie bracket on the fibers. Indeed $\mathcal{A}^0(S; \text{ad } P)$ is the Lie algebra of the group \mathcal{G} of sections of the G -bundle $\text{Ad } P$ associated to P by the action of G on itself by inner automorphisms. \mathcal{G} is the *gauge group* of all “inner” bundle automorphisms $P \rightarrow P$ covering the identity map $S \rightarrow S$. It is not difficult to see that the differential of the evaluation map $\mathcal{G} \rightarrow \mathfrak{a}$ of the action of \mathcal{G} at $A \in \mathfrak{a}$ is precisely $d_A: \mathcal{A}^0(S; \text{ad } P) \rightarrow \mathcal{A}^1(S; \text{ad } P)$. Thus $B^1(S; \text{ad } P)$ is the tangent space to the \mathcal{G} -orbit of A in \mathfrak{a} .

Now consider the quotient \mathcal{F}/\mathcal{G} , i.e., the space of G -equivalence classes of flat connections on P . It is well known (see [ST, GU]) that this space is canonically identified with an open and closed subset of $\text{Hom}(\pi, G)/G$; its tangent space is the de Rham cohomology $H^1(S; \text{ad } P)$, canonically identified with $H^1(\pi; \mathfrak{G}_{\text{Ad } \phi})$. The symplectic structures $\omega_A^{(B)}$ and $\omega_\phi^{(B)}$ correspond under this identification; since $\omega_\phi^{(B)}$ is nondegenerate, so is $\omega_A^{(B)}$ and the annihilator of $\omega^{(B)}$ on $Z^1(S; \text{ad } P)$ actually equals $B^1(S; \text{ad } P)$. Furthermore it follows from the fact that the 2-form on \mathfrak{a} is closed that the *reduced* 2-form $\omega^{(B)}$ on $\mathcal{F}/\mathcal{G} \cong \text{Hom}(\pi, G)/G$ is closed. Thus the proof of Theorem 1.7 is complete.

1.9. Remark. The argument in the last paragraph may be avoided by observing the following beautiful fact, as in [AB]: The action of the gauge group \mathcal{G} on \mathfrak{a} is a Hamiltonian action and its moment map is the function $F: \mathfrak{a} \rightarrow A^2(S; \text{ad } P) \cong \mathcal{A}^0(S; \text{ad } P)^*$ which assigns to a connection A its curvature $F(A)$, which is an $\text{ad } P$ -valued 2-form. That is, for any $X \in \mathcal{A}^0(S; \text{ad } P)$ the 1-parameter group $\exp(tX)$ of gauge transformations defines a Hamiltonian flow on \mathfrak{a} whose Hamiltonian function $H: \mathfrak{a} \rightarrow \mathbb{R}$ is $H(A) = \langle F(A), X \rangle$ (i.e., the vector field on \mathfrak{a} which generates this flow is symplectic dual to the 1-form dH on \mathfrak{a}). It is a general principle in symplectic geometry [WE] that the level surface $\mathcal{F} = F^{-1}(0)$ of the moment map is coisotropic and its reduction \mathcal{F}/\mathcal{G} has an induced symplectic structure.

2. THE WEIL–PETERSSON KÄHLER FORM ON
TEICHMÜLLER SPACE: THE SHIMURA ISOMORPHISM

2.1. The purpose of this section is to prove that the exterior 2-form $\omega^{(B)}$ constructed in the previous section restricts to the Weil–Petersson Kähler form on Teichmüller space. Here Teichmüller space is an open subset of $\text{Hom}(\pi, G)/G$ where $G = PSL(2, \mathbb{R})$ and B is the Killing form on $\mathfrak{G} = \mathfrak{sl}(2, \mathbb{R})$.

The tangent and cotangent bundles of Teichmüller space are usually described in terms of automorphic forms for Fuchsian groups and the Weil–Petersson Kähler form is just the imaginary part of the Petersson Hermitian pairing of automorphic forms. For exposition of this theory, we refer to Bers [BE], Earle [E] or Kra [K]. In this section, we show following Shimura [SH] (see also Gunning [GU]) that the Petersson pairing can be computed in terms of a coefficient pairing of the “period” cohomology classes associated to automorphic forms. Although almost everything we say is contained in [SH], we hope that our more invariant and coordinate-free description clarifies the role of such topological concepts as the cup-product in group cohomology, the Killing form on the Lie algebra as a coefficient pairing and the fundamental cycle on the surface.

2.2. Let S be a closed orientable surface. The Teichmüller space \mathcal{E}_S of S is defined as the space of equivalence classes of marked complex structures on S . A *marked complex structure* on S is a homotopy-equivalence $f: S \rightarrow M$ where M is a Riemann surface; two marked complex structures $f: S \rightarrow M, f': S \rightarrow M'$ are equivalent if and only if there is a conformal isomorphism $h: M \rightarrow M'$ with $h \circ f \simeq f'$. \mathcal{E}_S has a natural structure as a finite-dimensional real analytic manifold. The tangent space to \mathcal{E}_S at a point $f: S \rightarrow M$ is the infinitesimal deformation space of complex structures as defined by Kodaira and Spencer [KS] and is identified as the first cohomology group of M with coefficients in the sheaf \mathcal{O} of germs of holomorphic vector fields. Since $\dim_{\mathbb{C}} M = 1$, sections of \mathcal{O} are holomorphic sections of the holomorphic line bundle \mathcal{K}^{-1} , the line bundle dual to the canonical bundle. Thus the tangent space to \mathcal{E}_S is $H^1(M, \mathcal{O}) = H^1(M; \mathcal{K}^{-1})$, which by Serre duality is dual to $H^0(M, \mathcal{K}^2)$. Thus the cotangent space to \mathcal{E}_S is $H^0(M; \mathcal{K}^2)$ the space of holomorphic quadratic differentials on M .

The Klein–Poincaré–Koebe uniformization theorem asserts that M has a representation as H/Γ where $H \subset \mathbb{C}$ is the upper-half-plane $\{z \in \mathbb{C} \mid y = \text{Im } z > 0\}$ and Γ is a discrete subgroup (unique up to conjugacy) of $G = PSL(2, \mathbb{R})$ acting on H by isometries of the Poincaré metric $ds = |dz|/y$. In particular \mathcal{E}_S may be identified with the subset of $\text{Hom}(\pi, G)/G$ consisting of all $\{\phi\}$ where ϕ is an orientation-preserving isomorphism of $\pi = \pi_1(S)$ onto a discrete subgroup of G . (We say that ϕ is *orientation-preserving* if there

exists an orientation-preserving homeomorphism $S \rightarrow H/\Gamma = M$ inducing $\phi: \pi_1(S) \rightarrow \Gamma$.) Necessarily the image $\Gamma = \phi(\pi)$ is a torsion-free cocompact Fuchsian group. The set of all such ϕ as above forms a connected component of $\text{Hom}(\pi, G)$ and its Zariski closure in $\text{Hom}(\pi, G)$ is the irreducible component of $\text{Hom}(\pi, G)$ consisting of representations $\pi \rightarrow G$ which possess lifts $\pi \rightarrow SL(2, \mathbb{R})$ to the double covering $SL(2, \mathbb{R}) \rightarrow G$ (Goldman [G1]). Since \mathcal{E}_S is open in $\text{Hom}(\pi, G)/G$ its tangent space admits a description as a cohomology group $H^1(\pi; \mathfrak{G}_{\text{Ad}\phi})$.

The Poincaré area form on H is the 2-form $dA = y^{-2} dx \wedge dy = -2i(z - \bar{z})^{-2} dz \wedge d\bar{z}$; its dual current is $(dA)^{-1} = y^2(\partial/\partial x) \wedge (\partial/\partial y) = -(z - \bar{z})^2/2i (\partial/\partial z) \wedge (\partial/\partial \bar{z})$. If $M = H/\Gamma$ with covering projection $p: H \rightarrow M$ then both dA and $(dA)^{-1}$ define tensors on M . A holomorphic quadratic differential $\mathcal{E} \in H^0(M; \mathcal{K}^2)$ lifts to a Γ -invariant holomorphic quadratic differential $p^*\mathcal{E}$ on H which must be of the form $\xi(z) dz^2$ where $\xi(z)$ is a holomorphic function on H satisfying

$$\xi(\gamma z) = \gamma'(z)^{-2} \xi(z)$$

for all $\gamma \in \Gamma, z \in H$. If Ψ is another holomorphic quadratic differential, $p^*\Psi = \psi(z) dz^2$, then the Weil–Petersson Hermitian product of \mathcal{E} and Ψ is defined as

$$\langle \mathcal{E}, \Psi \rangle = -\frac{1}{4} \int_M \mathcal{E} \cdot (dA)^{-1} \cdot \bar{\Psi}.$$

(Here the integrand is an exterior 2-form (of type (1, 1)) and the dots indicate tensor contraction: \mathcal{E} is a section of \mathcal{K}^2 , $\bar{\Psi}$ a section of $\bar{\mathcal{K}}^2$, and $(dA)^{-1}$ a section of $\mathcal{K}^{-1} \otimes \bar{\mathcal{K}}^{-1}$.) Thus $\mathcal{E} \otimes (dA)^{-1} \otimes \bar{\Psi}$ is a section of $\mathcal{K}^2 \otimes (\mathcal{K}^{-1} \otimes \bar{\mathcal{K}}^{-1}) \otimes \bar{\mathcal{K}}^2$, which admits a canonical mapping (contraction) to $\mathcal{K} \otimes \bar{\mathcal{K}}$. In what follows we shall use dots for contraction in this way, suppressing the (obvious) tensor mapping involved. In coordinates on H the above integral reduces to the classical expression

$$\langle \mathcal{E}, \Psi \rangle = \int_{\mathcal{F}} \xi(z) \bar{\psi}(z) y^2 dx dy$$

where \mathcal{F} is a fundamental domain for Γ acting on H .

The above pairing is a Hermitian metric on the cotangent bundle of \mathcal{E}_S and provides a nice description of the tangent space of \mathcal{E}_S : the dual of a holomorphic quadratic differential \mathcal{E} is the harmonic Beltrami differential $-4\bar{\mathcal{E}} \cdot (dA)^{-1}$, which in coordinates has the expression

$$2i\bar{\xi}(z) y^2 d\bar{z} \otimes (\partial/\partial z).$$

Using the explicit duality, we easily recover the Hermitian metric on the tangent bundle of \mathcal{E}_S . This metric was first shown to be Kähler by Ahlfors [AH1] and Weil.

2.3. Our immediate goal is to write the Hermitian pairing in terms of the cup-product pairing on cohomology, as in Shimura [SH] or Gunning [GU]. Actually this is completely general and everything we say, apart from relations with deformation theory, applies to the Petersson pairing on holomorphic q -differentials (sections of \mathcal{K}^q), with minor and obvious modifications (such as integrating $\mathcal{E} \cdot (dA)^{1-q} \cdot \bar{\mathcal{P}}$). Rather than discuss this more general formula, we leave the details to the reader (referring to [GU] and [SH]) and content ourselves to the case relevant to deformation theory ($q = 2$).

The key step for the passage from holomorphic sections of the line bundle \mathcal{K}^2 to cocycles of the fundamental group with coefficients in the flat vector bundle $\mathfrak{G}_{\text{Ad}\phi}$ arises from the representation of the Lie algebra $\mathfrak{G} = \mathfrak{sl}(2, \mathbb{R})$ as quadratic vector fields $(az^2 + bz + c)(\partial/\partial z)$ on H , where $a, b, c \in \mathbb{R}$. Since the symplectic pairing $\omega^{(B)}$ arises from a Hermitian pairing, we might as well consider the complexified Lie algebra $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{G} \otimes \mathbb{C}$ represented as complex quadratic vector fields on the Riemann sphere $\mathbb{C}P^1$. We decompose $\mathbb{C}P^1$ as $H_+ \cup H_- \cup \mathbb{R}P^1$ where $H_+ = H$ is the upper-half-plane and $\mathbb{R}P^1 = \mathbb{R} \cup \{\infty\}$ is the equator, etc. Then any holomorphic tensor field Φ on H may be uniquely extended to H_- so that it is invariant under complex conjugation $z \mapsto \bar{z}$. Since $\Gamma \subset PSL(2, \mathbb{R})$, complex conjugation induces an involution of the π -module $\mathfrak{sl}(2, \mathbb{C})$.

Let $\mathcal{S}(\mathfrak{G}^{\mathbb{C}})$ denote the sheaf of germs of quadratic vector fields. Then $\mathcal{S}(\mathfrak{G}^{\mathbb{C}})$ is a locally free sheaf and is just the sheaf of parallel (i.e., flat) sections of the flat vector bundle $\mathfrak{G}_{\text{Ad}\phi}^{\mathbb{C}}$ over S . Let $\mathcal{o}(\mathcal{K}^{-1})$ be the sheaf of germs of holomorphic sections of \mathcal{K}^{-1} , the tangent complex line bundle to M . Since every locally defined parallel section of $\mathfrak{G}_{\text{Ad}\phi}^{\mathbb{C}}$ is a locally defined holomorphic vector field, there is a canonical sheaf homomorphism $\hat{Z}: \mathcal{S}(\mathfrak{G}^{\mathbb{C}}) \rightarrow \mathcal{o}(\mathcal{K}^{-1})$ which associates to a germ of a quadratic vector field its value as a holomorphic tangent vector. It will be more convenient to regard \hat{Z} as a section Z of $(\mathfrak{G}_{\text{Ad}\phi}^{\mathbb{C}})^* \otimes \mathcal{K}^{-1}$ which is defined over all of $\mathbb{C}P^1$. (Here $(\mathfrak{G}_{\text{Ad}\phi}^{\mathbb{C}})^*$ denotes the flat vector bundle dual to $\mathfrak{sl}(2, \mathbb{C})_{\text{Ad}\phi}$.) Clearly Z is a holomorphic section and invariant under the full group $G_{\mathbb{C}} = PSL(2, \mathbb{C})$ of complex linear fractional transformations of $\mathbb{C}P^1$.

To discuss all of this in coordinates we may describe sections of $(\mathfrak{G}_{\text{Ad}\phi}^{\mathbb{C}})^*$ in terms of the basis of $(\mathfrak{G}^{\mathbb{C}})^*$ dual to $\{z^2(\partial/\partial z), z(\partial/\partial z), (\partial/\partial z)\}$. In other words the row vector (a, b, c) represents the element $(az^2 + bz + c)(\partial/\partial z)$ of $\mathfrak{G}^{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$. Then Z is represented by the column vector field

$$\begin{bmatrix} z^2 & \partial/\partial z \\ z & \partial/\partial z \\ & \partial/\partial z \end{bmatrix}$$

and the $G^{\mathbb{C}}$ -invariance of Z is merely the transformation law

$$\begin{bmatrix} \gamma_* & (z^2(\partial/\partial z)) \\ \gamma_* & (z(\partial/\partial z)) \\ \gamma_* & (\partial/\partial z) \end{bmatrix} = \begin{bmatrix} d^2 & -2bd & b^2 \\ -cd & 1 & ab \\ c^2 & -2ac & a^2 \end{bmatrix} Z$$

where $\gamma(z) = (az + b)/(cz + d)$ is an element of $G^{\mathbb{C}}$, and the matrix on the right is the transpose of $\text{Ad } \gamma$ in the above basis.

Now let $B: \mathfrak{G}^{\mathbb{C}} \times \mathfrak{G}^{\mathbb{C}} \rightarrow \mathbb{C}$ denote the complex Killing form on $G^{\mathbb{C}}$; then B induces a nondegenerate symmetric pairing of the flat vector bundles $\mathfrak{G}_{\text{Ad } \phi}^{\mathbb{C}}$ and $(\mathfrak{G}_{\text{Ad } \phi}^{\mathbb{C}})^*$ also denoted by B . In the vector coordinates above, B is represented by the matrix

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Consider the diagonal action of $G^{\mathbb{C}}$ on the product $\mathbb{C}P^1 \times \mathbb{C}P^1$. Letting Z_1 and Z_2 denote the canonical $\mathfrak{G}^{\mathbb{C}}$ -valued vector fields on each of the two copies of $\mathbb{C}P^1$, their product $Z_1 \wedge Z_2$ is a holomorphic 2-vector field in $\mathbb{C}P^1 \times \mathbb{C}P^1$ taking values in $\mathfrak{G}^{\mathbb{C}} \otimes \mathfrak{G}^{\mathbb{C}}$ and is invariant under the diagonal action of $G^{\mathbb{C}}$. The contraction $B(Z_1 \wedge Z_2)$ under the coefficient pairing $B: \mathfrak{G}^{\mathbb{C}} \otimes \mathfrak{G}^{\mathbb{C}} \rightarrow \mathbb{C}$ is a complex-valued 2-vector field on $\mathbb{C}P^1 \times \mathbb{C}P^1$ invariant under $G^{\mathbb{C}}$. In coordinates $B(Z_1 \wedge Z_2) = -(z_1 - z_2)^2 (\partial/\partial z_1) \wedge (\partial/\partial z_2)$; its invariance is just the familiar formula

$$\left(\frac{\gamma(z_1) - \gamma(z_2)}{z_1 - z_2} \right)^2 = \gamma'(z_1) \gamma'(z_2)$$

valid for all $\gamma \in G^{\mathbb{C}}$.

We may introduce a Hermitian product $\mathfrak{G}^{\mathbb{C}} \times \mathfrak{G}^{\mathbb{C}} \rightarrow \mathbb{C}$ by $(X, Y) \mapsto B(X, \bar{Y})$; this Hermitian pairing is no longer invariant under all of $G^{\mathbb{C}}$ but is invariant under G . Using this Hermitian coefficient pairing we may substitute $Z_1 = Z, Z_2 = \bar{Z}$ into the above formula and we recover the G -invariance of the dual Poincaré area current $(dA)^{-1} = (i/2)(z - \bar{z})^2 (\partial/\partial z) \wedge (\partial/\partial \bar{z}) = (i/2) B(Z \wedge \bar{Z})$.

2.4. Now suppose that $\Xi \in H^0(M; \mathcal{K}^2)$ is a holomorphic quadratic differential on M , which is represented by the Γ -invariant holomorphic quadratic differential $\xi(z) dz^2$ on H . The tensor field $Z \cdot \Xi$ (where the dot

indicates contraction) is a holomorphic section of $\mathcal{H} \otimes (\mathbb{G}_{\text{Ad}\phi}^{\mathbb{C}})^*$ and is represented by the Γ -invariant holomorphic $\mathbb{G}_{\mathbb{C}}^*$ -valued $(1, 0)$ -form on H

$$\xi(z) dx = \begin{bmatrix} \xi(z) z^2 \\ \xi(z) z \\ \xi(z) \end{bmatrix} dz$$

considered by Shimura. (Here dx means the $\mathbb{G}_{\mathbb{C}}^*$ -valued 1-form on H defined by

$$\begin{bmatrix} z^2 & dz \\ z & dz \\ & dz \end{bmatrix}$$

which is *not* invariant.) Using the tensor Z , we may express the Weil–Peterson Hermitian product of two quadratic differentials Ξ, Ψ as the integral over M of the complex $(1, 1)$ -form obtained by contracting the exterior product of $Z \cdot \Xi \in H^0(M; \mathcal{H} \otimes \mathbb{G}_{\mathbb{C}}^*)$ and $\bar{Z} \cdot \bar{\Psi} \in H^0(M; \bar{\mathcal{H}} \otimes \bar{\mathbb{G}}_{\mathbb{C}}^*)$ using B as coefficient pairing:

$$\langle \Xi, \Psi \rangle = -i/8 \int_M B((Z \cdot \Xi) \wedge (\bar{Z} \cdot \bar{\Psi})).$$

Since $p^*(Z \cdot \Xi)$ (resp. $p^*(\bar{Z} \cdot \bar{\Psi})$) is a holomorphic $(1, 0)$ -form (resp. antiholomorphic $(0, 1)$ -form) with values in a flat (and hence trivial) vector bundle over H , these 1-forms may be integrated to give holomorphic (resp. antiholomorphic) sections of the trivial $\mathbb{G}_{\mathbb{C}}^*$ -bundle over H . That is, we may write

$$p^*(Z \cdot \Xi) = \begin{bmatrix} \xi(z) & z^2 dz \\ \xi(z) & z dz \\ \xi(z) & dz \end{bmatrix} = \begin{bmatrix} df_1 \\ df_2 \\ df_3 \end{bmatrix} = d\mathbf{f}$$

where f_1, f_2, f_3 are holomorphic functions $H \rightarrow \mathbb{C}$. Similarly we may write $p^*(\bar{Z} \cdot \bar{\Psi}) d\mathbf{h}$ where $\mathbf{h} = (h_1, h_2, h_3)$ is a holomorphic section of $\bar{\mathbb{G}}_{\mathbb{C}}^*$. For example we may take for \mathbf{f} the primitive

$$\begin{aligned} \mathbf{f}(z) &= \int_{z_0}^z p^*(z \cdot \Xi) \\ &= \left(\int_{z_0}^z \xi(z) z^2 dz, \int_{z_0}^z \xi(z) z dz, \int_{z_0}^z \xi(z) dz \right) \end{aligned}$$

where $z_0 \in H$ is an arbitrary but fixed basepoint. As f is generally not Γ -invariant, its collection of periods

$$u_{\underline{z},z}(\gamma) = f(\gamma z) - f(z) = \int_z^{\gamma z} p^*(Z \cdot \Xi)$$

defines a cocycle on Γ values in $(\mathbb{G}^{\mathbb{C}})^*$. The cohomology class $[u_{\underline{z},z}] \in H^1(\Gamma; (\mathbb{G}^{\mathbb{C}})^*)$ is independent of z , and we henceforth take $z = z_0$.

Let \tilde{B} be the isomorphism $(\mathbb{G}^{\mathbb{C}})^* \rightarrow \mathbb{G}^{\mathbb{C}}$ induced by B . Then $\tilde{B} \circ Z \cdot \Xi$ is a holomorphic $\mathbb{G}^{\mathbb{C}}$ -valued 1-form on M and has period cocycle $\tilde{B} \circ u_{\underline{z},z}$ with cohomology class in $H^1(\Gamma; \mathbb{G}^{\mathbb{C}})$. Applying the isomorphism $\phi: \pi \rightarrow \Gamma$ we obtain a cohomology class in $H^1(\pi; \mathbb{G}_{\text{Ad}\phi}^{\mathbb{C}})$. It has been shown by various authors (Ahlfors [AH2], Hejhal [H], Wolpert [WO2]) that this cohomology class corresponds to the deformation by the harmonic Beltrami differential $\Xi \cdot (dA)^{-1}$, thus showing these two methods of defining tangent vectors to $\mathcal{E}_{\mathcal{S}}$ are equivalent.

2.5. Now we come to the main result of this section.

PROPOSITION. *Let Ξ, Ψ be two holomorphic quadratic differentials on $M = H/\Gamma$ and $u_{\underline{z}}$ and $u_{\underline{\psi}}$ be the corresponding $(\mathbb{G}^{\mathbb{C}})^*$ -valued cocycles on π . Then the Weil–Petersson Hermitian product of Ξ and Ψ is given by the formula*

$$\langle \Xi, \Psi \rangle = (-i/8) B_*([u_{\underline{z}}] \cup [\overline{u_{\underline{\psi}}}] \cdot [M])$$

where \cup is the cup product pairing $H^1(\pi; (\mathbb{G}^{\mathbb{C}})^*) \times H^1(\pi; (\mathbb{G}^{\mathbb{C}})^*) \rightarrow H^2(\pi; (\mathbb{G}^{\mathbb{C}})^* \otimes (\mathbb{G}^{\mathbb{C}})^*)$, B_* is the coefficient pairing $(\mathbb{G}^{\mathbb{C}})^* \otimes (\mathbb{G}^{\mathbb{C}})^* \rightarrow \mathbb{C}$ and $[M]$ is the image of the fundamental homology class under the isomorphism $H_2(M; \mathbb{Z}) \rightarrow H_2(\pi; \mathbb{Z})$.

2.6. As this result involves the interplay between differential forms and group cohomology, its proof will require a geometric model for the fundamental cycle of M in group cohomology. That is, we need the notions of simplex, chain, cycle, etc., which are compatible both with the Eilenberg–MacLane complex of group cohomology and the geometry of M . Since M is a $K(\pi, 1)$ its universal cover \tilde{M} is a contractible space upon which π acts freely and properly discontinuously. Let $E\pi$ be the infinite join $\pi * \pi * \dots$ considered by Milnor [M]; then the diagonal action of π by left-multiplication is also free and properly discontinuous. $E\pi$ is an infinite simplicial complex whose k -simplices correspond to $(k + 1)$ -tuples $[g_0, \dots, g_k]$ whose vertices are elements of π . The image of $[g_0, \dots, g_k]$ under $g \in \pi$ is the k -simplex $[gg_0, \dots, gg_k]$. In particular every k -simplex is equivalent under π to

one whose first vertex g_0 is the identity element 1. Let $B\pi$ be the quotient $E\pi/\pi$; then the simplicial chain complex of $B\pi$ is the Eilenberg–MacLane complex of π whose cohomology is the cohomology $H^*(\pi)$ of the group π . By lifting each k -simplex σ in $B\pi$ to the unique k -simplex in $E\pi$ with $g_0 = 1$ we may unambiguously represent simplices of $B\pi$ by k -tuples (g_1, \dots, g_k) of elements of π . That is, the *inhomogeneous* k -simplex (g_1, \dots, g_k) corresponds to the equivalence class of homogeneous k -simplices $[g_0, g_0 g_1, \dots, g_0 g_1, \dots, g_k]$. For example, the boundary of the inhomogeneous 2-simplex (a, b) is $(a) - (ab) + b$ since $\partial[g, ga, gab] = [g, ga] - [g, gab] + [ga, gab] \sim [1, a] - [1, ab] + [1, b]$.

Representing M as H/Γ , we may relate chains on $B\pi$ to singular chains on M . The Γ -orbit of a basepoint $z_0 \in H$ is an isomorphic replica of $\pi \cong \Gamma$. For every k -simplex $[g_1, g_1, \dots, g_k]$ in $E\pi$ there is an affine map onto the geodesic k -simplex $[g_0 z_0, \dots, g_k z_0]$ in H with vertices $g_0 z_0, g_1 z_0, \dots, g_k z_0$ where z_0 is a fixed basepoint. This defines a map $E\pi \rightarrow H$ which is equivariant with respect to $\phi: \pi \rightarrow \Gamma \subset G$. The induced map $B\pi \rightarrow M$ is a homotopy equivalence and defines an isomorphism of the homology of π with the homology of M .

Let $c = \sum_{i=1}^m (a_i, b_i)$ be a 2-cycle in $B\pi$ representing the fundamental homology class $[M] \in H_2(M; \mathbb{Z}) \cong H_2(\pi)$. Then c corresponds to the singular 2-chain $\tilde{c}z_0$ in H defined by

$$\tilde{c}z_0 = \sum_{i=1}^m \sigma[z_0, a_i z_0, a_i b_i z_0].$$

Now $\langle \Phi, \Psi \rangle = (-i/8) B(Z \cdot \Phi \wedge \bar{Z} \cdot \bar{\Psi}) \cdot [M]$ can be expressed as the integral of the exterior 2-form $p^*B(Z \cdot \Phi \wedge \bar{Z} \cdot \bar{\Psi})$ over the 2-chain $\tilde{c}z_0$ in H . Since $p^*(Z \cdot \Phi) = d\ell$, it follows that $p^*B(Z \cdot \Phi \wedge \bar{Z} \cdot \bar{\Psi}) = d(B(\ell \otimes p^*(\bar{Z} \cdot \bar{\Psi})))$. Thus Stokes' theorem implies that $2i\langle \Psi, \Phi \rangle$ equals the integral of $B(\ell \otimes p^*(\bar{Z} \cdot \bar{\Psi}))$ over the 1-chain

$$\partial(\tilde{c}z_0) = \sum_{i=1}^m (\sigma[z_0, a_i z_0] + \sigma[z_0, a_i b_i z_0] - \sigma[a_i z_0, a_i b_i z_0]).$$

Thus $i/2\langle \Phi, \Psi \rangle$ is a sum of integral of $B(\ell \otimes p^*(\bar{Z} \cdot \bar{\Psi}))$ over $3m$ geodesic 1-simplices. At least $2m$ of these 1-simplices are of the form $\sigma[z_0, g z_0]$ for $g \in \Gamma$ and m of these 1-simplices are of the form $\sigma[a_i z_0, a_i b_i z_0]$. Since c is a cycle, for each 1-simplex $[a_i, a_i b_i]$ in this sum there exists a 1-simplex $[1, g]$ in the sum which is equivalent under π to it, and cancels it. (The remaining 1-simplices with vertex z_0 must cancel in pairs.) Thus we may rearrange the sum representing $\langle \Phi, \Psi \rangle$ to obtain

$$-8i\langle \Phi, \Psi \rangle = \sum_{i=1}^m \int_{\sigma[z_0, b_i z_0]} B(\ell \otimes p^*(\bar{Z} \cdot \bar{\Psi})) - \int_{\sigma[a_i z_0, a_i b_i z_0]} B(\ell \otimes p^*(\bar{Z} \cdot \bar{\Psi})).$$

Now $p^*(\bar{Z} \cdot \bar{\Psi}) = d\bar{h}$ is a Γ -invariant antiholomorphic $(0, 1)$ -form with values in $(\mathfrak{G}^C)^*$. If $a, b \in \pi$ then

$$\begin{aligned} \int_{\sigma[a z_0, a b z_0]} B(\not{f} \otimes p^*(\bar{Z} \cdot \bar{\Psi})) &= \int_{\sigma[a z_0, a b z_0]} B(\not{f} \otimes d\bar{h}) \\ &= \int_{\sigma[z_0, b z_0]} B((a^{-1})^* \not{f} \otimes (a^{-1})^* d\bar{h}) \\ &= \int_{\sigma[z_0, b z_0]} B((a^{-1})^* \not{f} \otimes d\bar{h}). \end{aligned}$$

Applying this identity to each term in the above sum with $a = a_i, b = b_i$, we obtain

$$\begin{aligned} -8i \langle \mathcal{E}, \Psi \rangle &= \sum_{i=1}^m \int_{\sigma[z_0, b_i z_0]} B(((a^{-1})^* \not{f} - \not{f}) \otimes d\bar{h}) \\ &= \sum_{i=1}^m \int_{\sigma[z_0, b_i z_0]} B(u_\phi(a_i^{-1}) \otimes d\bar{h}) \\ &= \sum_{i=1}^m B(u_\Xi(a_i^{-1}) \otimes \int_{\sigma[z_0, b_i z_0]} d\bar{h}) \\ &= \sum_{i=1}^m B(u_\Xi(a_i^{-1}) \otimes \overline{u_\Psi(b_i)}) \end{aligned}$$

since $u_\Xi(a_i^{-1})$ is constant. Since $u_\Xi(a^{-1}) = -a^{-1}u_\Xi(a)$, we obtain

$$\begin{aligned} 8i \langle \mathcal{E}, \Psi \rangle &= \sum_{i=1}^m B(a_i^{-1} u_\Xi(a_i) \otimes \overline{u_\Psi(b_i)}) \\ &= \sum_{i=1}^m B(u_\Xi(a_i) \otimes a_i \overline{u_\Psi(b_i)}) \\ &= \sum_{i=1}^m B_I([u_\Xi] \cup \overline{[u_\Psi]}) \cdot (a_i, b_i) \\ &= B_I([u_\Xi] \cup \overline{[u_\Psi]}) \cdot [M]. \end{aligned}$$

as desired. The proof of Proposition 2.5 is concluded. Q.E.D.

2.7. Proposition 2.5 implies that if $\beta, \eta \in H^1(\pi; \mathfrak{G}_{\text{Ad}\phi})$ are cohomology classes with coefficients in $\mathfrak{G} = \mathfrak{sl}(2, \mathbb{R})$ then the symplectic product $\omega^{(B)}(\beta, \eta) = B_*(\beta \cup \eta) \cdot [M]$ equals (-8) the imaginary part of the Hermitian product $\langle \mathcal{E}, \Psi \rangle$ where $\beta = [u_\Xi]$ and $\eta = [u_\Psi]$.

3. APPLICATIONS OF THE FREE DIFFERENTIAL CALCULUS

3.1. In a series of papers starting with [F], R. H. Fox developed a noncommutative differential calculus for words in a free group and applied his free differential calculus to various problems in knot theory. It turns out that Fox's calculus is exactly the tool needed in calculations with spaces of representations $\text{Hom}(\pi, G)$. The purpose of this section is to apply the Fox calculus to various general problems: (i) calculate the dimensions of the strata of $\text{Hom}(\pi, G)$ when π is a surface group; (ii) calculate the Zariski tangent spaces of $\text{Hom}(\pi, G)$; (iii) describe the fundamental cycle of a surface in group cohomology. Although each of these problems has been solved in the literature, their solutions often are not quite as transparent or general as one might like. Still the calculations are quite formidable but I hope that the systematic use of the Fox calculus in these matters serves to unify and explain the complicated formulae which arise.

3.2. We begin by summarizing the rudiments of Fox's theory; see Fox [F] (and also Birman [BI]) for more details.

Let Π denote a free group on generators x_1, \dots, x_n , and let $\mathbb{Z}\Pi$ denote its integral group ring. The augmentation homomorphism is a ring homomorphism $\varepsilon: \mathbb{Z}\Pi \rightarrow \mathbb{Z}$ which maps a general element $\sum_{\sigma \in \Pi} m_\sigma \sigma$ to the coefficient sum $\sum_{\sigma \in \Pi} m_\sigma \in \mathbb{Z}$. Let $M = \mathbb{Z}\Pi$ but with the structure of a (nonassociative) $\mathbb{Z}\Pi$ -bimodule where $\mathbb{Z}\Pi$ acts on the left by ordinary left-multiplication (i.e., the left regular action) and on the right by ε . That is, if $(m) \in M$ corresponds to the element $\sum_{\sigma \in \Pi} m_\sigma \sigma$ of $\mathbb{Z}\Pi$ and $g, h \in \Pi$, then $g(m)h$ is the element of M corresponding to $\sum m_\sigma g\sigma$. A derivation of M then corresponds to a \mathbb{Z} -linear map $D: \mathbb{Z}\Pi \rightarrow \mathbb{Z}\Pi$ satisfying $D(m_1 m_2) = D(m_1) \varepsilon(m_2) + m_1 D(m_2)$. In particular taking m_1, m_2 to be elements of Π , we see that a derivation is just a 1-cocycle on Π with coefficients in $\mathbb{Z}\Pi$.

3.3. Let $\text{Der}(\Pi)$ be the set of all derivations. If $D \in \text{Der}(\Pi)$, and $u \in \mathbb{Z}\Pi$, then $x \mapsto D(x)u$ is also a derivation. Thus $\text{Der}(\Pi)$ is a right $\mathbb{Z}\Pi$ -module.

PROPOSITION (Fox [F]). *Der(Π) is freely generated as a right $\mathbb{Z}\Pi$ -module by n elements $\partial_i = \partial/\partial x_i$, $i = 1, \dots, n$, which satisfy $(\partial/\partial x_i)(x_j) = \delta_{ij}$.*

The formula $D(ab) = D(a) + aD(b)$ for $a, b \in \Pi$ makes the calculation of the Fox derivations $\partial_i w$ for any word $w \in \Pi$ quite mechanical. For example, in the free group on two generators A, B we have:

$$\begin{aligned} (\partial/\partial A)(A^{-1}) &= -A^{-1}; & (\partial/\partial A)(AB) &= I; & (\partial/\partial B)(AB) &= A; \\ (\partial/\partial A)(ABA^{-1}) &= I - ABA^{-1}; & (\partial/\partial B)(ABA^{-1}) &= A; \\ (\partial/\partial A)(ABA^{-1}B^{-1}) &= I - ABA^{-1}; & (\partial/\partial B)(ABA^{-1}B^{-1}) &= A - ABA^{-1}B^{-1}. \end{aligned}$$

It is also easy to prove that for any $w \in \Pi$, $\varepsilon(\partial_i w)$ equals the total exponent sum of the letter x_i in the word w .

Fox proves that these free derivations satisfy many of the usual rules of differential calculus: chain rule, mean value theorem, Taylor series, etc. For example, for any $u \in \mathbb{Z}\Pi$, the "mean value theorem" states that

$$u - \varepsilon(u) = \sum (\partial_i u)(x_i - 1) \quad (3.1)$$

and the $\partial_i u$ are the unique elements of $\mathbb{Z}\Pi$ which satisfy the equations above.

3.4. Since a Fox derivation is a cocycle on a free group Π with values in its group ring, the Fox calculus is intimately connected to group cohomology. Let $\phi: \Pi \rightarrow GL(V)$ be a linear representation of the free group on n generators x_1, \dots, x_n on a vector space V . Obviously such representations consist of nothing more than a choice of n linear automorphisms $\phi(x_1), \dots, \phi(x_n)$ of V . The representation ϕ determines by linearity a ring homomorphism $\mathbb{Z}\Pi \rightarrow \text{End}(V)$ which we also denote by ϕ (or suppress it completely when the context is clear).

Suppose that $u: \Pi \rightarrow V$ is a cocycle. By linearity, u extends to a linear map $u: \mathbb{Z}\Pi \rightarrow V$ which satisfies the cocycle identity

$$u(ab) = u(a) \varepsilon(b) + \phi(a) u(b)$$

for all $a, b \in \mathbb{Z}\Pi$. Applying this identity to Fox's formula (3.1) we obtain

$$\begin{aligned} u(w) &= u(w - 1) = u \left(\sum_{i=1}^n (\partial_i w)(x_i - 1) \right) \\ &= \sum_{i=1}^n (u(\partial_i w) \varepsilon(x_i - 1) + \phi(\partial_i w) u(x_i - 1)) \\ &= \sum_{i=1}^n \phi(\partial_i w) u(x_i) \end{aligned}$$

for any $w \in \mathbb{Z}\Pi$. Conversely for any n -tuple (u_1, \dots, u_n) of elements of V , the map $u: \Pi \rightarrow V$ defined by

$$u(w) = \sum \phi(\partial_i w) u_i \quad (3.2)$$

is a cocycle. Thus (3.2) defines an isomorphism of the vector space $Z^1(\Pi; V_\phi)$ of V_ϕ -valued cocycles on Π with the vector space V^n .

3.5. We may use (3.2) to compute the differential of a word map. Let $w \in \Pi$ be a word $w(x_1, \dots, x_n)$ in variables x_1, \dots, x_n and let G be a Lie group. Then w defines an analytic map $w: G^n \rightarrow G$ by substitution. Let us identify the tangent spaces to G with the Lie algebra \mathfrak{G} by regarding \mathfrak{G} as right-invariant vector fields and extending each tangent vector to G at a given

$g \in G$ to a unique right-invariant vector field. Then the differential dw of the word map $w: G^n \rightarrow G$ at $(g_1, \dots, g_n) \in G^n$ is a linear map $dw: \mathfrak{G}^n \rightarrow \mathfrak{G}$ which may vary with (g_1, \dots, g_n) .

Each point (g_1, \dots, g_n) of G^n determines a unique homomorphism $\phi \in \text{Hom}(\Pi, G)$. By the discussion in Section 1, tangent vectors to $\text{Hom}(\Pi, G)$ correspond to cocycles $\Pi \rightarrow \mathfrak{G}_{\text{Ad}\phi}$. Thus for each $(u_1, \dots, u_n) \in \mathfrak{G}^n$, the map $w \rightarrow dw(u_1, \dots, u_n)$ is the cocycle $\Pi \rightarrow \mathfrak{G}_{\text{Ad}\phi}$ which associates to each x_i the vector $u_i \in \mathfrak{G}$. In particular $dw(u_1, \dots, u_n) = \sum_{i=1}^n \phi(\partial_i w) u_i$.

We may rewrite this in a more suggestive form. For each $i = 1, \dots, n$ let dx_i be the projection $\mathfrak{G}^n \rightarrow \mathfrak{G}$ onto the i th factor. It is also the differential at the word map x_i , which is just projection $G^n \rightarrow G$ onto the i th factor. Then as a mapping $\mathfrak{G}^n \rightarrow \mathfrak{G}$ the differential of the word map $w: G^n \rightarrow G$ is the total differential

$$dw = \sum_{i=1}^n \partial_i w dx_i. \tag{3.3}$$

3.6. Now suppose that π is a group with n generators; thus π admits a presentation Π/\mathcal{R} where \mathcal{R} is a normal subgroup of Π , consisting of relations between the generators of π . Representations $\pi \rightarrow G$ correspond bijectively to representations $\Pi \rightarrow G$ which map \mathcal{R} to the identity. Similarly if V is a π -module (and hence a Π -module) cocycles $\pi \rightarrow V$ correspond bijectively to cocycles $\Pi \rightarrow V$ which are zero on \mathcal{R} . It follows that $\text{Hom}(\pi, G)$ corresponds to the subset of $\phi \in \text{Hom}(\Pi, G) = G^n$ with $\phi(R) = 1$ for all $R \in \mathcal{R}$. If G is an algebraic Lie group then it follows from 3.3 that the Zariski tangent space to $\text{Hom}(\pi, G) \subset G^n$ at $\phi \in \text{Hom}(\pi, G)$ is the subspace

$$\begin{aligned} Z^1(\pi; \mathfrak{G}_{\text{Ad}\phi}) = & \left\{ (u_1, \dots, u_n) \in \mathfrak{G}^n \mid \sum_{i=1}^n \text{Ad } \phi(\partial_i R) u_i = 0 \right. \\ & \left. \text{for all } R \in \mathcal{R} \right\}. \end{aligned}$$

3.7. EXAMPLE: SURFACE GROUPS. Suppose that π is the fundamental group of a closed orientable surface of genus p . Then π is generated by $2p$ generators $A_1, B_1, \dots, A_p, B_p$ subject to the relation $R = \prod_{i=1}^p [A_i, B_i]$ where $[A, B] = ABA^{-1}B^{-1}$, i.e., $\pi = \Pi/\mathcal{R}$ where Π is the free group on $A_1, B_1, \dots, A_p, B_p$ and \mathcal{R} is the normal subgroup generated by R .

Let G be a Lie group with a fixed Ad-invariant nondegenerate symmetric bilinear form on its Lie algebra \mathfrak{G} . We now prove the following result, which readily implies Proposition 1.2:

PROPOSITION. *The rank of $dR: \mathfrak{G}^{2p} \rightarrow \mathfrak{G}$ at a point $(A_1, B_1, \dots, A_p, B_p) \in G^{2p}$ equals the codimension of the centralizer of $\{A_1, B_1, \dots, A_p, B_p\}$.*

Proof. By 3.3 the image $dR(\mathfrak{G}^{2p})$ is a sum $\sum_{j=1}^p (\text{Ad}(\partial R/\partial A_j)(\mathfrak{G}) + \text{Ad}(\partial R/\partial B_j)(\mathfrak{G}))$ so its orthogonal complement $dR(\mathfrak{G}^{2p})^\perp$ is an intersection

$$\bigcap_{j=1}^p (\text{Ad}(\partial R/\partial A_j)(\mathfrak{G})^\perp \cap \text{Ad}(\partial R/\partial B_j)(\mathfrak{G})^\perp).$$

Here $\partial R/\partial A_j = C_{j-1}(I - A_j B_j A_j^{-1})$ and $\partial R/\partial B_j = C_{j-1}(A_j - A_j B_j A_j^{-1} B_j^{-1})$ where $C_j = \prod_{i=1}^j [A_i, B_i]$.

If T is an orthogonal transformation of an inner product space, then $(\text{image}(I - T))^\perp = \text{Ker}(I - T)$. Moreover if S is another orthogonal transformation then $(\text{image } S(I - T))^\perp = S(\text{image}(I - T))^\perp = S(\text{Ker}(I - T)) = \text{Ker}(I - STS^{-1})$.

Applying these facts we find that $(\text{Ad}(\partial R/\partial A_j)(\mathfrak{G}^{2p}))^\perp = \text{Ker Ad}(I - C_{j-1}^{-1} A_j B_j A_j^{-1} C_{j-1})$ equals the Lie subalgebra $\zeta(C_{j-1}^{-1} A_j B_j A_j^{-1} C_{j-1})$ centralized by $C_{j-1}^{-1} A_j B_j A_j C_{j-1}$. Similarly $(\text{Ad}(\partial R/\partial B_j)(\mathfrak{G}^{2p}))^\perp = \zeta(C_{j-1}^{-1} A_j B_j A_j^{-1} B_j^{-1} A_j^{-1} C_{j-1}^{-1})$. Since

$$\{C_{j-1}^{-1} A_j B_j A_j^{-1} C_{j-1}, C_{j-1}^{-1} A_j B_j A_j^{-1} B_j^{-1} A_j^{-1} C_{j-1}\}$$

generates the same group as does $\{C_{j-1}^{-1} A_j C_{j-1}, C_{j-1}^{-1} B_j C_{j-1}\}$ we find that $(\text{Ad}(\partial R/\partial A_j)(\mathfrak{G}))^\perp \cap (\text{Ad}(\partial R/\partial B_j)(\mathfrak{G}))^\perp = \zeta(C_{j-1}^{-1} A_j C_{j-1}) \cap \zeta(C_{j-1}^{-1} B_j C_{j-1})$. Finally since $\{A_1, B_1, C_1^{-1} A_2 C_1, C_1^{-1} B_2 C_1, \dots, C_{p-1}^{-1} A_p C_{p-1}, C_{p-1}^{-1} B_p C_{p-1}\}$ generates the same group as does $\{A_1, B_1, A_2, B_2, \dots, A_p, B_p\}$ we find that $dR(\mathfrak{G}^{2p})^\perp$ equals the common centralizer ζ of all the A_i and B_i . Thus the image of the differential dR and the centralizer ζ have complementary dimension. The proof of Proposition 3.7, and hence Proposition 1.2, is complete. Q.E.D.

3.8. The Fundamental Cycle of a Surface Group

Now we derive a closed expression for a 2-dimensional cycle in group cohomology, based on the free differential calculus.² We use this formula to write the symplectic structure $\omega^{(B)}$ on $\text{Hom}(\pi, G)/G$ in terms of the restriction to $\text{Hom}(\pi, G)$ of an algebraic tensor field on G^n (where n is the number of generators in a fixed presentation for π).

Recall that an Eilenberg–MacLane 1-chain (resp. 2-chain) on a group π is a \mathbb{Z} -linear combination of elements of π (resp. $\pi \times \pi$). Thus we will systematically confuse 1-chains and 2-chains on π with elements of the group rings $\mathbb{Z}\pi$ and $\mathbb{Z}\pi \times \pi$. The boundary of a 2-chain $z = \sum_{i=1}^n n_i(a_i, b_i)$, $n_i \in \mathbb{Z}$, $a_i, b_i \in \pi$, is $\partial z = \sum_{i=1}^n n_i(a_i - a_i b_i + b_i)$. (More generally we will write as 2-chains expressions of the form $z = \sum n_i(a_i, b_i)$ where a_i, b_i are now elements of $\mathbb{Z}\pi$, under the natural convention that the operation (a, b) is \mathbb{Z} -bilinear in a and b . Then the boundary of a 2-chain $z = (a, b)$ is given by the formula $\partial z = a\varepsilon(b) - ab + \varepsilon(a)b$.)

² This formula may also be found in Brown [B].

The classical formula of Hopf [HO] (compare also Lyndon [L]) expresses the 2-dimensional homology group $H_2(\pi)$ of a group π in terms of a free presentation $\pi = \Pi/\mathcal{R}$. Hopf's formula states $H_2(\pi) \cong ([\Pi, \Pi] \cap \mathcal{R})/[\Pi, \mathcal{R}]$ where the symbol $[\Pi, \mathcal{R}]$ means the subgroup of Π generated by all commutators $[A, B]$ when $A \in \Pi$ and $B \in \mathcal{R}$, etc. Thus to every relation R which is a product of commutators in the generators there is an associated homology class in $H_2(\pi)$. Geometrically this is just the statement that 2-dimensional homology classes in a space X (here $X \simeq \mathcal{R}(\pi, 1)$) are represented by a map from a surface into X .

Here is the recipe. Let $R \in [\Pi, \Pi] \cap \mathcal{R}$. Consider the 2-chain $z_R = \sum_{i=1}^n (\partial_i R, x_i)$ on Π . Its boundary is

$$\begin{aligned} \partial z_R &= \sum_{i=1}^n (\varepsilon(x_i) \partial_i R - x_i \partial_i R + x_i \varepsilon(\partial_i R)) \\ &= \sum_{i=1}^n (\partial_i R)(1 - x_i) = 1 - R \end{aligned}$$

by 3.1 and the fact that $\varepsilon(\partial_i R) = \text{total exponent sum of } x_i \text{ in } R = 0$ since $R \in [\Pi, \Pi]$. Thus the image of z_R in the space of 1-chains on π is a cycle.

3.9. PROPOSITION. *Let R_p be the canonical relation $\prod_{i=1}^p [A_i, B_i]$ for the fundamental group π of a surface of genus p . Let z_p be the 2-cycle on π corresponding to the chain $\sum_{i=1}^p ((\partial R_p/\partial A_i, A_i) + (\partial R_p/\partial B_i, B_i))$. Then its homology class $[z_p]$ generates $H_2(\pi)$.*

Proof. Consider first the case $p = 1$. Then $R_j = ABA^{-1}B^{-1}$ and $z_{R_1} = (I - ABA^{-1}, A) + (A - ABA^{-1}B^{-1}, B)$ projects to the 2-cycle on $\pi z_1 = (I, A) + (A, B) - (B, A) - (I, B)$ which is homologous to $(A, B) - (B, A)$. Let α, β be cohomology classes lying in $H^1(\pi)$ which are dual to the homology classes corresponding to A and B . Then $(\alpha \cup \beta)[z_1] = 1$ and since $\alpha \cup \beta$ generates $H^2(\pi)$, so does $[z_1]$ generate $H_2(\pi)$.

For general p , consider a surface S of genus p and the degree one map $f: S \rightarrow T^2$ which collapses to a point all but one handle. In the canonical presentations, such a map induces the homomorphism $f_* \pi_1(S) \rightarrow \pi_1(T^2)$ defined by, say, $f_*(A_1) = A$, $f_*(B_1) = B$ and $f_*(A_j) = f_*(B_j) = 1$ ($j > 1$). In the CW decompositions corresponding to the presentations, the free group Π appears as $\pi_1(S^{(1)})$ where $S^{(1)}$ is the 1-skeleton and it is easy to see (e.g., by making f cellular) that $f_*^{(1)}(R_p) = R_1$ and $f_*(z_p) = z_1$. Since $f_*: H_2(S) \rightarrow H_2(T^2)$ is an isomorphism and $[z_1]$ generates $H_2(T^2)$, it follows that $[z_p]$ generates $H_2(S) = H_2(\pi)$. Q.E.D.

Remark. A very similar, but not quite as explicit, formula has been given by Lyndon [L].

3.10. Using the free differential calculus, it is easy to express $\omega^{(B)}$ as an algebraic 2-tensor on $\text{Hom}(\pi, G)$ invariant under G . (Here we use the

term “algebraic” in its usual sense if G is an algebraic group so that $\text{Hom}(\pi, G)$ is an algebraic variety; otherwise we simply mean that $\omega^{(B)}$ is expressed in terms of words in π , the group multiplication in G , the adjoint representation and the bilinear form B .)

Let u, v be two tangent vectors to $\text{Hom}(\pi, G)$; that is, u, v , are cocycles in $Z^1(\pi; \mathfrak{G}_{\text{Ad}\phi})$. (In terms of the generators $u_i = u(x_i), v_i = v(x_i)$, this simply means $\sum_{i=1}^n \partial_i R u_i = \sum_{i=1}^n \partial_i R v_i = 0$.) Their symplectic product $\omega^{(B)}([u], [v])$ equals the 2-cocycle $B_*(u \cup v)$ evaluated on the fundamental cycle of π . Now $B_*(u \cup v)$ is the 2-cocycle on π which associates to $(x, y) \in \pi \times \pi$ the number $B(u(x), xv(y)) = B(x^{-1}u(x), v(y)) = B(u(-x^{-1}), v(y))$. Thus if $\sum n_i(a_i, b_i)$ is a fundamental 2-cycle for π , then

$$\omega^{(B)}([u], [v]) = \sum n_i B(u(-a_i^{-1}), v(b_i)).$$

Let $\#$ denote the antiautomorphism of $\mathbb{Z}\pi$ defined by $\#(\sum n_i a_i) = \sum n_i a_i^{-1}$. Then applying the formula above to the canonical cycle $\sum_{i=1}^n (\partial_i R, x_i)$ we obtain

$$\begin{aligned} \omega^{(B)}([u], [v]) &= - \sum_{i=1}^n B(u(\#\partial_i R), v(x_i)) \\ &= - \sum_{i,j=1}^n B(\partial_i \# \partial_j R(u_i), v_j). \end{aligned} \tag{3.4}$$

This expression is an algebraic 2-tensor field on G^n in the above sense. Restricting u and v to be cocycles, the tensor defined by 3.4 is invariant under G and skew-symmetric; however, neither property holds in general for the tensor on G^n .

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REFERENCES

[AH1] L. AHLFORS, Some remarks on Teichmüller’s space of Riemann surfaces, *Ann. of Math.* **74** (1961), 171–191.
 [AH2] L. AHLFORS, “Lectures on Quasiconformal Mappings,” Van Nostrand, New York, 1966.
 [AB] M. ATIYAH AND R. BOTT, The Yang–Mills equations over Riemann surfaces, *Philos. Trans. Roy. Soc. London A* **308** (1982), 523–615.
 [B] K. BROWN, “Cohomology of groups,” Graduate Texts in Mathematics No. 87, Springer–Verlag, New York, 1982.
 [BE] L. BERS, Uniformization, moduli and Kleinian groups, *Bull. London Math. Soc.* **4** (1972), 257–300.
 [BI] J. BIRMAN, Braids, links and mapping class groups, *Ann. of Math. Stud.* **82** (1974).
 [E] C. EARLE, Teichmüller theory, in “Discrete Groups and Automorphism Functions” (W. Harvey, Ed.), pp. 143–162, Academic Press, New York, 1977.

- [F] R. FOX, The free differential calculus. I. Derivations in the group ring, *Ann. of Math.* **57** (1953), 547–560.
- [G1] W. GOLDMAN, “Discontinuous Groups and the Euler Class,” Thesis, University of California, Berkeley, 1980.
- [G2] W. GOLDMAN, Characteristic classes and representations of discrete subgroups of Lie groups, *Bull. Amer. Math. Soc. (N. S.)* **6** (1982), 91–94.
- [G3] W. GOLDMAN, Representations of fundamental groups of surfaces, in “Proceedings of the Special Year in Topology, Maryland 1983–1984,” Lecture Notes in Mathematics, Springer-Verlag, New York, to appear.
- [G4] W. GOLDMAN, Invariant functions on Lie groups and Hamiltonian flows of surfaces, submitted.
- [G5] W. GOLDMAN, An ergodic action of the mapping class group, in preparation.
- [GU] R. C. GUNNING, “Lectures on Vector Bundles over Riemann Surfaces,” Princeton Univ. Press, Princeton, N.J., 1967.
- [H] D. HEJHAL, Monodromy groups and Poincaré series, *Bull. Amer. Math. Soc.* **84** (1978), 339–376.
- [HO] H. HOPF, Fundamentalgruppe und zweite Bettische gruppe, *Comm. Math. Helv.* **14** (1942), 257–309.
- [JM] D. JOHNSON AND J. MILLSON, Deformation spaces of compact hyperbolic manifolds, in “Discrete Groups in Geometry and Analysis,” Proceedings of a Conference Held at Yale University in Honor of G. D. Mostow on his Sixtieth Birthday, to appear.
- [KS] K. KODAIRA AND D. C. SPENCER, On deformations of complex analytic structures, I, **67** (1958), 328–401, 403–466; **71** (1960), 43–76.
- [K] I. KRA, “Automorphic Forms and Kleinian Groups,” Benjamin, Reading, Mass. 1972.
- [L] R. LYNDON, Cohomology theory of groups with one defining relation, *Ann. of Math.* **52** (1950), 650–665.
- [MW] J. MARSDEN AND A. WEINSTEIN, Reduction of symplectic manifolds with symmetry, *Rep. Math. Phys.* **5** (1974), 121–130.
- [M] J. MILNOR, Construction of universal bundles, I, *Ann. of Math.* **63** (1958), 272–284.
- [N] M. S. NARASIMHAN, Geometry of moduli of vector bundles over an algebraic curve, in “Actes Int. Congress de Math., Nice 1970,” Tome 2, pp. 199–201.
- [NS1] M. S. NARASIMHAN AND C. S. SESHADRI, Holomorphic vector bundles on a compact Riemann surface, *Math. Ann.* **155** (1964), 69–80.
- [NS2] M. S. NARASIMHAN AND C. S. SESHADRI, Stable and unitary vector bundles on a compact Riemann surface, *Ann. of Math.* **82** (1965), 540–567.
- [NR] A. NIJENHUIS AND R. RICHARDSON, Deformations of homomorphisms of Lie groups and Lie algebras, *Bull. Amer. Math. Soc.* **73** (1967), 175.
- [R] M. S. RAGHUNATHAN, “Discrete Subgroups of Lie Groups,” Springer-Verlag, New York, 1972.
- [SH] G. SHIMURA, Sur les intégrales attachées aux formes automorphes, *J. Math. Soc. Japan* **11** (1959), 291–311.
- [ST] N. STEENROD, “The Topology of Fiber Bundles,” Princeton Univ. Press, Princeton, N.J., 1951.
- [WE] A. WEINSTEIN, “Lectures on Symplectic Manifolds,” C.B.M.S. 29, Amer. Math. Soc., Providence, R.I., 1977.
- [WO1] S. WOLPERT, An elementary formula for the Fenchel–Nielsen twist, *Comm. Math. Helv.* **56** (1981), 132–135.
- [WO2] S. WOLPERT, The Fenchel–Nielsen deformation, *Ann. of Math.* **115** (1982), 501–528.
- [WO3] S. WOLPERT, On the symplectic geometry of deformations of a hyperbolic surface, *Ann of Math.* **117** (1983), 207–234.