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GROUP COHOMOLOGY AND THE SYMPLECTIC STRUCTURE ON THE MODULI SPACE OF REPRESENTATIONS

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0. Introduction. The space of equivalence classes of irreducible representations of the fundamental group of a compact-oriented surface of genus at least 2 in a Lie group has a natural symplectic form. In [AB], Atiyah and Bott described this symplectic structure using methods from gauge theory. Goldman [G] constructed the skew-symmetric pairing algebraically using methods from group cohomology. Using Poincaré duality, Goldman showed that the pairing is nondegenerate and identified it with the symplectic structure given by gauge theory.

For a punctured surface, the space of equivalence classes of irreducible representations also admits a symplectic structure, provided some boundary conditions are imposed (namely, fixing the conjugacy classes of the holonomies around the punctures). The symplectic structure on the moduli space was constructed analytically using gauge theory in [BG1] and [BG2]. An algebraic description of the symplectic form using parabolic cohomology is given in [GHJW]. The required nondegeneracy is proved using the cohomology of *group systems*; this can be thought of as an analogue of compactly supported cohomology.

This paper can be considered a companion to [GHJW]. The principal aim of this paper is to prove, using ideas from the Hodge theory of forms and the explicit use of Fox calculus, the nondegeneracy of the skew-symmetric pairings which then give the above symplectic structures.

In the process of the proof, we introduce a Riemannian metric on the above spaces of equivalence classes of representations of the fundamental group. The choice of this metric was motivated and in some sense dictated by the explicit description of the symplectic structure by Goldman [G] in the compact case. We came up with this metric in an attempt to mimic a harmonic theory of forms in the cohomology of groups. This metric, we believe, is interesting in its own right and needs further study. The striking feature of this metric on the moduli space of representations is that it can be given explicitly once a presentation of the fundamental group of the surface is chosen and does not apparently decode any complex structure or metric on the surface itself.

In the first section we describe the metric on the moduli space in the compact surface case and prove the nondegeneracy of the skew-symmetric pairing, which

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then defines the symplectic structure. In the next sections we briefly describe the parabolic moduli space and present the proof of nondegeneracy in the general n-puncture case and for a general reductive Lie group.

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1. Metric on the moduli space (compact case). Let S be a compact-orientable surface of genus-$g \geqslant 2$, and let $\pi = \pi_1(S, s_0)$ be the fundamental group based at $s_0 \in S$. π has the familiar presentation

$$\pi = \left\langle A_1, B_1, \ldots, A_g, B_g \;\middle|\; \prod_{i=1}^{g} [A_i, B_i] = I \right\rangle,$$

where I is the identity element. Let $C_j = \prod_{i=1}^{j} [A_i, B_i]$ and $R = C_g$ so that $R = I$ is the unique relation among the generators of π.

The integral group ring $\mathbb{Z}[\pi]$ of π has a natural anti-automorphism $\#$ defined by

$$\#\left(\sum n_i a_i\right) = \sum n_i a_i^{-1} \quad \text{for } n_i \in \mathbb{Z} \text{ and } a_i \in \pi.$$

Using the notation from the Fox calculus (see [G, §3]), we know the following:

$$\frac{\partial R}{\partial A_i} = C_{i-1}(I - A_i B_i A_i^{-1}) = C_{i-1} - C_i B_i, \tag{1}$$

$$\frac{\partial R}{\partial B_i} = C_{i-1}(A_i - A_i B_i A_i^{-1} B_i^{-1}) = C_{i-1} A_i - C_i, \tag{2}$$

so that

$$\# \frac{\partial R}{\partial A_i} = C_{i-1}^{-1} - B_i^{-1} C_i^{-1}, \tag{3}$$

$$\# \frac{\partial R}{\partial B_i} = A_i^{-1} C_{i-1}^{-1} - C_i^{-1}. \tag{4}$$

Let G be a compact Lie group with Lie algebra $\mathfrak{G}$, and let $\langle\ ,\ \rangle$ be a G-invariant inner product on $\mathfrak{G}$. The moduli space $\mathscr{M} = \mathrm{Hom}(\pi, G)/G$ of conjugacy classes of representations of π into G admits a natural skew-symmetric pairing. More precisely if $\rho : \pi \to G$ is a representation, the composition $\pi \xrightarrow{\rho} G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{G})$ makes the Lie algebra $\mathfrak{G}$ into a π-module and the tangent space $T_\rho(\mathrm{Hom}(\pi, G))$ can be identified with the space $Z^1(\pi, \mathfrak{G}_\rho)$ of 1-cocycles of π with values in this π-module $\mathfrak{G}_\rho$. That is,

$$Z^1(\pi, \mathfrak{G}_\rho) = \{u : \pi \to \mathfrak{G}_\rho \mid u(xy) = u(x) + \mathrm{Ad}\,\rho(x)u(y) \text{ for } x, y \in \pi\}.$$

At the equivalence class $[\rho]$, the tangent space $T_{[\rho]}(\mathscr{M})$ can then be identified with the first group cohomology $H^1(\pi, \mathfrak{G}_\rho)$. The following composition of the cup product and $\langle\ ,\ \rangle$

$$Z^1(\pi, \mathfrak{G}_\rho) \times Z^1(\pi, \mathfrak{G}_\rho) \xrightarrow{\cup} Z^2(\pi, \mathfrak{G}_\rho \otimes \mathfrak{G}_\rho) \xrightarrow{\langle,\rangle} Z^2(\pi, \mathbb{R})$$

induces a skew-symmetric pairing on cohomology

$$H^1(\pi, \mathfrak{G}_\rho) \times H^1(\pi, \mathfrak{G}_\rho) \to H^2(\pi, \mathbb{R}) \simeq \mathbb{R}.$$

From now on we will assume that ρ is irreducible. For simplicity, we write $x \cdot u(y)$ for $\mathrm{Ad}\,\rho(x)(u(y))$. We know from [G, (3.4)] that the formula

$$w(u, v) = -\sum_{i=1}^{g} \left\{ \left\langle u\left(\# \frac{\partial R}{\partial A_i}\right), v(A_i) \right\rangle + \left\langle u\left(\# \frac{\partial R}{\partial B_i}\right), v(B_i) \right\rangle \right\} \tag{5}$$

for $u, v \in Z^1(\pi, \mathfrak{G}_\rho)$ gives the symplectic form on $H^1(\pi, \mathfrak{G}_\rho)$. The main aim of this section is to prove the following.

THEOREM 1. *The skew-symmetric pairing w is nondegenerate on $H^1(\pi, \mathfrak{G}_\rho)$.*

Poincaré duality is invoked in [G] to prove nondegeneracy of the above pairing, which defines the symplectic structure on the moduli space $\mathscr{M}$. We prove nondegeneracy of the above pairing without apparent use of Poincaré duality by introducing a metric (positive definite inner product) on $Z^1(\pi, \mathfrak{G}_\rho)$. Our heuristic is to look for a dual 1-cocycle v for a given 1-cocycle u such that $w(u, v)$ is positive in analogy with the Hodge theory of forms. By closely examining the above expression for $w(u, v)$ and the special case of a genus-1 surface, we "naively" define the dual 1-cocycle v of u by setting

$$v(A_i) = u\left(\# \frac{\partial R}{\partial A_i}\right),$$

$$v(B_i) = u\left(\# \frac{\partial R}{\partial B_i}\right).$$

However if we consider the analogous situation in the Hodge theory of forms, the dual of a closed form need not be closed. The dual of a closed form is closed if and only if the form is harmonic. We also know from [G, 3.6] that v is a 1-cocycle if

$$\sum_{i=1}^{g}\left\{\frac{\partial R}{\partial A_i}v(A_i)+\frac{\partial R}{\partial B_i}v(B_i)\right\}=0.$$

That is,

$$\sum_{i=1}^{g}\left\{\frac{\partial R}{\partial A_i}u\left(\#\frac{\partial R}{\partial A_i}\right)+\frac{\partial R}{\partial B_i}u\left(\#\frac{\partial R}{\partial B_i}\right)\right\}=0.$$

We reinterpret the above condition as the characterising condition for harmonicity. But the space of *harmonic cocycles* should also be characterised as being the orthogonal complement to the space of 1-coboundaries with respect to a suitable metric on the space of 1-coycles. In our situation the space of 1-coboundaries can be easily described. Morever we expect the dual map ϕ to lead to an almost complex structure, and $w(u, \phi(v))$ to possibly define an inner product on the first cohomology. This leads us to define a metric $\langle\ ,\ \rangle_\rho$ on $Z^1(\pi, \mathfrak{G}_\rho)$, which we use to define the space of harmonic cocycles and the *dual* of a harmonic cocycle. This will enable us to prove the nondegeneracy of the skew-symmetric pairing w.

We now define a nonnegative symmetric pairing $\langle\ ,\ \rangle_\rho$ on $Z^1(\pi, \mathfrak{G}_\rho)$ as

$$\langle u,v\rangle_\rho=\sum_{i=1}^{g}\left\{\left\langle u\left(\#\frac{\partial R}{\partial A_i}\right),v\left(\#\frac{\partial R}{\partial A_i}\right)\right\rangle+\left\langle u\left(\#\frac{\partial R}{\partial B_i}\right),v\left(\#\frac{\partial R}{\partial B_i}\right)\right\rangle\right\}. \tag{6}$$

We claim that this pairing actually defines an inner product on $Z^1(\pi, \mathfrak{G}_\rho)$.

PROPOSITION 1. *The pairing $\langle\ ,\ \rangle_\rho$ on $Z^1(\pi, \mathfrak{G}_\rho)$ is positive definite.*

Proof. We need to prove that $\langle u,u\rangle_\rho = 0 \Rightarrow u = 0$ for $u \in Z^1(\pi, \mathfrak{G}_\rho)$. Since u is a 1-cocycle, it suffices to prove that $u(A_i) = u(B_i) = 0$ for all $i = 1, \ldots, g$. Since $\langle\ ,\ \rangle$ is an inner product on $\mathfrak{G}$,

$$\langle u,u\rangle_\rho=0\Rightarrow u\left(\#\frac{\partial R}{\partial A_i}\right)=u\left(\#\frac{\partial R}{\partial B_i}\right)=0 \qquad \forall i. \tag{7}$$

It follows from (3) and (4) that

$$u(C_{i-1}^{-1})=u(B_i^{-1}C_i^{-1}), \tag{8}$$

$$u(C_i^{-1})=u(A_i^{-1}C_{i-1}^{-1}). \tag{9}$$

Assume now by induction that $u(A_j) = u(B_j) = u(C_j) = 0$ for $j < i$, where we set $A_0 = B_0 = I$. Note that $C_0 = I$, so our induction hypothesis is satisfied for $i = 1$. Equation (8) and the induction hypothesis imply

$$u(C_iB_i) = 0.$$

Since $C_iB_i = C_{i-1}A_iB_iA_i^{-1}$, it follows by our induction hypothesis that

$$u(A_iB_iA_i^{-1}) = 0. \tag{10}$$

Morever $0 = u(C_iB_i) = u(C_i) + C_iu(B_i)$, which implies

$$u(B_i) = -C_i^{-1}u(C_i) = u(C_i^{-1}).$$

Equation (9) and the induction hypothesis imply

$$u(C_i^{-1}) = u(A_i^{-1}).$$

Hence $u(B_i) = u(A_i^{-1})$. It follows then that

$$\begin{aligned} u(A_iB_i) &= u(A_i) + A_iu(B_i) \\ &= u(A_i) + A_iu(A_i^{-1}) \\ &= u(A_i) - u(A_i) \\ &= 0. \end{aligned}$$

Hence it follows from (10) that

$$u(A_i^{-1}) = 0.$$

Thus we obtain $0 = u(A_i^{-1}) = u(B_i) = u(C_i)$ and our induction argument can be continued. Thus, for all $i = 1, \dots, g$, $u(A_i) = u(B_i) = 0$; that is, u vanishes on all generators $\Rightarrow u = 0$. Therefore $\langle\ ,\ \rangle_\rho$ is a metric on $Z^1(\pi, \mathfrak{G}_\rho)$. □

Our aim now is to define the analogue of the Hodge $*$-operator on $Z^1(\pi, \mathfrak{G}_\rho)$ using the metric $\langle\ ,\ \rangle_\rho$ on $Z^1(\pi, \mathfrak{G}_\rho)$. However, the Hodge $*$-operator in general does not send closed forms to closed forms, unless the forms are harmonic as well. Hence our heuristic now is to describe the orthogonal complement $B^1(\pi, \mathfrak{G}_\rho)^\perp$ to the space of coboundaries $B^1(\pi, \mathfrak{G}_\rho)$ in $Z^1(\pi, \mathfrak{G}_\rho)$. We think of $B^1(\pi, \mathfrak{G}_\rho)^\perp$ as the representative for the space of harmonic forms and call $B^1(\pi, \mathfrak{G}_\rho)^\perp$ the space of harmonic cocycles. Clearly $B^1(\pi, \mathfrak{G}_\rho)^\perp$ is isomorphic to

$H^1(\pi, \mathfrak{G}_\rho)$. We have

$$B^1(\pi, \mathfrak{G}_\rho)^\perp = \{u \in Z^1(\pi, \mathfrak{G}_\rho) \mid \langle u, d\mu \rangle_\rho = 0 \ \ \forall \mu \in \mathfrak{G}_\rho\}.$$

The coboundary $d\mu$ for $\mu \in \mathfrak{G}_\rho$ is defined by

$$d\mu(x) = x \cdot \mu - \mu \qquad \forall x \in \pi.$$

Clearly

$$\begin{aligned} d\mu\left(\# \frac{\partial R}{\partial A_i}\right) &= d\mu(C_{i-1}^{-1}) - d\mu(B_i^{-1} C_i^{-1}) \\ &= (C_{i-1}^{-1} \cdot \mu - \mu) - (B_i^{-1} C_i^{-1} \cdot \mu - \mu) \\ &= C_{i-1}^{-1} \cdot \mu - B_i^{-1} C_i^{-1} \cdot \mu = \left(\# \frac{\partial R}{\partial A_i}\right) \cdot \mu. \end{aligned}$$

Similarly $d\mu\left(\# \frac{\partial R}{\partial B_i}\right) = \left(\# \frac{\partial R}{\partial B_i}\right) \cdot \mu$. It is easily checked that

$$\left\langle u\left(\# \frac{\partial R}{\partial A_i}\right), d\mu\left(\# \frac{\partial R}{\partial A_i}\right) \right\rangle = \left\langle \frac{\partial R}{\partial A_i} u\left(\# \frac{\partial R}{\partial A_i}\right), \mu \right\rangle, \tag{11}$$

and similarly for B_i. Hence

$$\langle u, d\mu \rangle_\rho = \left\langle \sum_{i=1}^{g} \left(\frac{\partial R}{\partial A_i} u\left(\# \frac{\partial R}{\partial A_i}\right) + \frac{\partial R}{\partial B_i} u\left(\# \frac{\partial R}{\partial B_i}\right) \right), \mu \right\rangle.$$

Consequently we have (since $\langle \ , \ \rangle$ is an inner product) the characterisation of harmonic cocycles

$$B^1(\pi, \mathfrak{G}_\rho)^\perp = \left\{ u \in Z^1(\pi, \mathfrak{G}_\rho) \,\middle|\, \sum_{i=1}^{g} \left(\frac{\partial R}{\partial A_i} u\left(\# \frac{\partial R}{\partial A_i}\right) + \frac{\partial R}{\partial B_i} u\left(\# \frac{\partial R}{\partial B_i}\right) \right) = 0 \right\},$$

$$Z^1(\pi, \mathfrak{G}_\rho) = B^1(\pi, \mathfrak{G}_\rho) \oplus B^1(\pi, \mathfrak{G}_\rho)^\perp.$$

We now are all set to define an analogue of the Hodge $*$-map ϕ: $B^1(\pi, \mathfrak{G}_\rho)^\perp \to Z^1(\pi, \mathfrak{G}_\rho)$. For $u \in B^1(\pi, \mathfrak{G}_\rho)^\perp$, define $\phi(u)$ as follows on the generators

$$\phi(u)(A_i) = u\left(\# \frac{\partial R}{\partial A_i}\right),$$

$$\phi(u)(B_i) = u\left(\# \frac{\partial R}{\partial B_i}\right).$$

We now need to check that $\phi(u)$ is a cocycle. $\phi(u)$ can be extended as a 1-cocyle to the free group defined on the generators $A_1, B_1, \ldots, A_g, B_g$. We have then the following identity (see [G, 3.6]):

$$\phi(u)(R) = \sum_{i=1}^{g} \left\{ \frac{\partial R}{\partial A_i} \phi(u)(A_i) + \frac{\partial R}{\partial B_i} \phi(u)(B_i) \right\}.$$

Thus from [G, 3.6], we know that a function $v \colon \pi \to \mathfrak{G}_\rho$ is a 1-cocycle if and only if it satisfies the relation

$$v(R) = \sum_{i=1}^{g} \left\{ \frac{\partial R}{\partial A_i} v(A_i) + \frac{\partial R}{\partial B_i} v(B_i) \right\} = 0.$$

Clearly

$$\sum_{i=1}^{g} \left\{ \frac{\partial R}{\partial A_i} \phi(u)(A_i) + \frac{\partial R}{\partial B_i} \phi(u)(B_i) \right\} = \sum_{i=1}^{g} \left\{ \frac{\partial R}{\partial A_i} u\left(\# \frac{\partial R}{\partial A_i}\right) + \frac{\partial R}{\partial B_i} u\left(\# \frac{\partial R}{\partial B_i}\right) \right\}$$
$$= 0,$$

since u is a harmonic cocycle. Therefore $\phi(u)$ is a 1-cocycle.

The nondegeneracy of the skew-symmetric pairing on $T_{[\rho]}(\mathcal{M}) \approx H^1(\pi, \mathfrak{G}_\rho)$ now follows easily from the Hodge theory developed above. For $u \in B^1(\pi, \mathfrak{G}_\rho)^\perp \approx H^1(\pi, \mathfrak{G}_\rho)$ we show that there exists an element $v \in B^1(\pi, \mathfrak{G}_\rho)^\perp$ such that $w(u, v) \neq 0$. We simply set $v = \phi(u)$. It follows from (5) that

$$\begin{aligned} w(u, \phi(u)) &= -\sum_{i=1}^{g} \left\{ \left\langle u\left(\# \frac{\partial R}{\partial A_i}\right), \phi(u)(A_i) \right\rangle + \left\langle u\left(\# \frac{\partial R}{\partial B_i}\right), \phi(u)(B_i) \right\rangle \right\} \\ &= -\sum_{i=1}^{g} \left\{ \left\langle u\left(\# \frac{\partial R}{\partial A_i}\right), u\left(\# \frac{\partial R}{\partial A_i}\right) \right\rangle + \left\langle u\left(\# \frac{\partial R}{\partial B_i}\right), u\left(\# \frac{\partial R}{\partial B_i}\right) \right\rangle \right\} \\ &= -\langle u, u \rangle_\rho \neq 0 \quad \text{if } u \neq 0 \end{aligned}$$

by the proposition. Therefore the form w on $T_{[\rho]}(\mathcal{M})$ is nondegenerate.

2. Metric on the moduli space on an n-punctured surface. We briefly describe the moduli space on the punctured surface (see [BG1] and [GHJW]). Let $M = S \backslash \{p_1, \ldots, p_n\}$ be the surface of genus-$g \geqslant 2$ punctured at finitely many points $p_1, \ldots, p_n$. Let $C_1, \ldots, C_n$ be fixed conjugacy classes in G. The fundamental group $\tilde{\pi} := \pi_1(M, p_0)$ based at $p_0 \in M$ admits the following familiar presentation:

$$\tilde{\pi} = \left\langle A_1, B_1, \ldots, A_g, B_g, \alpha_1, \ldots, \alpha_n \,\middle|\, \left(\prod_{i=1}^{g} [A_i, B_i]\right) \alpha_1 \cdots \alpha_n = I \right\rangle.$$

Let $\mathscr{R} = \{\rho \in \operatorname{Hom}(\tilde{\pi}, G) \mid \rho(\alpha_i) \in C_i \text{ for } i = 1, \ldots, n\}$. G acts on $\mathscr{R}$ by conjugate action, and the quotient $\mathscr{R}/G$ is called the parabolic moduli space and denoted by $\mathscr{P}$.

The space $Z^1_{\text{par}}(\tilde{\pi}, \mathfrak{G}_\rho)$ of parabolic cocycles is defined by

$$Z^1_{\text{par}}(\tilde{\pi}, \mathfrak{G}_\rho) = \{u \in Z^1(\tilde{\pi}, \mathfrak{G}_\rho) \mid u(\alpha_j) \text{ is a coboundary}\}.$$

Saying $u(\alpha_j)$ is a coboundary means $u(\alpha_j) = \alpha_j \cdot \mu_j - \mu_j$ for some $\mu_j \in \mathfrak{G}_\rho$. $T_\rho(\mathscr{R})$ can be identified with $Z^1_{\text{par}}(\tilde{\pi}, \mathfrak{G}_\rho)$, and $T_{[\rho]}(\mathscr{P})$ with the first parabolic cohomology group $H^1_{\text{par}}(\tilde{\pi}, \mathfrak{G}_\rho) := Z^1_{\text{par}}(\tilde{\pi}, \mathfrak{G}_\rho)/B^1(\tilde{\pi}, \mathfrak{G}_\rho)$.

Notation. ρ is always assumed to be an irreducible representation. A product of elements in a group indexed by the empty set denotes the identity element of the group. We remark that a cocycle vanishes on the identity element of a group.

We know from [GHJW, §9] that the following formula gives a skew-symmetric pairing on $Z^1_{\text{par}}(\tilde{\pi}, \mathfrak{G}_\rho)$:

$$w(u, v) = \sum_{i=1}^{g} \left\{ \left\langle u\left(\# \frac{\partial R}{\partial A_i}\right), v(A_i) \right\rangle + \left\langle u\left(\# \frac{\partial R}{\partial B_i}\right), v(B_i) \right\rangle \right\}$$

$$- \sum_{j=1}^{n} \langle u(R\alpha_1 \cdots \alpha_{i-1}), (R\alpha_1 \cdots \alpha_{i-1}) \cdot v(\alpha_i) \rangle + \sum_{j=1}^{n} \langle \mu_i, v(\alpha_i) \rangle, \qquad (12)$$

where $u, v \in Z^1_{\text{par}}(\tilde{\pi}, \mathfrak{G}_\rho)$ and $u(\alpha_i) = \alpha_i \cdot \mu_i - \mu_i$ for some $\mu_i \in \mathfrak{G}$, $i = 1, \ldots, n$. Our aim now is to show that this pairing is nondegenerate on $H^1_{\text{par}}(\tilde{\pi}, \mathfrak{G}_\rho)$.

On $Z^1_{\text{par}}(\tilde{\pi}, \mathfrak{G}_\rho)$ we define an inner product. For $u, v \in Z^1_{\text{par}}(\pi, \mathfrak{G}_\rho)$ so that

$$u(\alpha_j) = \alpha_i \cdot \mu_j - \mu_j,$$

$$v(\alpha_j) = \alpha_j \cdot v_j - v_j$$

for $i = 1, \ldots, n$, and $\mu_i, v_i \in \mathfrak{G}_\rho$, we set

$$\langle u, v \rangle_\rho = \sum_{i=1}^{g} \left\{ \left\langle u\left(\# \frac{\partial R}{\partial A_i}\right), v\left(\# \frac{\partial R}{\partial A_i}\right) \right\rangle + \left\langle u\left(\# \frac{\partial R}{\partial B_i}\right), v\left(\# \frac{\partial R}{\partial B_i}\right) \right\rangle \right\}$$

$$+ \sum_{j=1}^{n} \langle u(\alpha_j \cdots \alpha_n) - u(\alpha_{j+1} \cdots \alpha_n), v(\alpha_j \cdots \alpha_n) - u(\alpha_{j+1} \cdots \alpha_n) \rangle. \qquad (13)$$

Proposition 2. *$\langle\ ,\ \rangle_\rho$ is an inner product on $Z^1_{\text{par}}(\tilde{\pi}, \mathfrak{G}_\rho)$.*

Proof. If $\langle u, u\rangle_\rho = 0$, then for $i = 1, \dots, g$, $j = 1, \dots, n$,

$$\begin{aligned}0 &= \left\langle u\left(\# \frac{\partial R}{\partial A_i}\right), u\left(\# \frac{\partial R}{\partial A_i}\right)\right\rangle \\ &= \left\langle u\left(\# \frac{\partial R}{\partial B_i}\right), u\left(\# \frac{\partial R}{\partial B_i}\right)\right\rangle \\ &= \langle u(\alpha_j \cdots \alpha_n) - u(\alpha_{j+1} \cdots \alpha_n), u(\alpha_j \cdots \alpha_n) - u(\alpha_{j+1} \cdots \alpha_n)\rangle \\ &\Rightarrow u\left(\# \frac{\partial R}{\partial A_i}\right) = u\left(\# \frac{\partial R}{\partial B_i}\right) = u(\alpha_j \cdots \alpha_n) - u(\alpha_{j+1} \cdots \alpha_n) = 0. \qquad (14)\end{aligned}$$

By the proof of Proposition 1, $u(A_i) = u(B_i) = 0$. By decreasing induction assume that $u(\alpha_j) = 0$ for $j > j_0$. Upon substituting $j = n$ in (14), we see that $u(\alpha_n) = 0$. Since $u(\alpha_j \cdots \alpha_n) = u(\alpha_j) + \alpha_j u(\alpha_{j+1} \cdots \alpha_n)$, the induction hypothesis and (14) together imply that $u(\alpha_{j_0}) = 0$. Hence u vanishes on all the generators A_i, B_i, and α_j, and so $u = 0$. Therefore $\langle\ ,\ \rangle_\rho$ is a positive definite inner product on $Z^1_{\text{par}}(\tilde{\pi}, \mathfrak{G}_\rho)$. □

The space of *harmonic* parabolic cocycles is given by

$$B^1(\pi, \mathfrak{G}_\rho)^\perp = \{u \in Z^1_{\text{par}}(\tilde{\pi}, \mathfrak{G}_\rho), \langle u, d\mu\rangle_\rho = 0 \ \forall v \in \mathfrak{G}_\rho\}$$

$$\begin{aligned}\langle u, d\mu\rangle_\rho &= \left\{\sum_{i=1}^{g}\left\{\left\langle u\left(\# \frac{\partial R}{\partial A_i}\right), d\mu\left(\# \frac{\partial R}{\partial A_i}\right)\right\rangle + \left\langle u\left(\# \frac{\partial R}{\partial B_i}\right), d\mu\left(\# \frac{\partial R}{\partial B_i}\right)\right\rangle\right\}\right\} \\ &\quad + \langle u(\alpha_j \cdots \alpha_n) - u(\alpha_{j+1} \cdots \alpha_n), d\mu(\alpha_j \cdots \alpha_n) - d\mu(\alpha_{j+1} \cdots \alpha_n)\rangle \\ &= \left\{\sum_{i=1}^{g}\left\{\left\langle \frac{\partial R}{\partial A_i} u\left(\# \frac{\partial R}{\partial A_i}\right), v\right\rangle + \left\langle \frac{\partial R}{\partial B_i} u\left(\# \frac{\partial R}{\partial B_i}\right), v\right\rangle\right\}\right\} \qquad \text{by (11)} \\ &\quad + \langle u(\alpha_j \cdots \alpha_n) - u(\alpha_{j+1} \cdots \alpha_n), d\mu(\alpha_j \cdots \alpha_n) - d\mu(\alpha_{j+1} \cdots \alpha_n)\rangle.\end{aligned}$$

But

$$\begin{aligned}&\langle u(\alpha_j \cdots \alpha_n) - u(\alpha_{j+1} \cdots \alpha_n), d\mu(\alpha_j \cdots \alpha_n) - d\mu(\alpha_{j+1} \cdots \alpha_n)\rangle \\ &= \langle u(\alpha_j \cdots \alpha_n) - u(\alpha_j \cdots \alpha_n), (\alpha_{j+1} \cdots \alpha_n)v - (\alpha_{j+1} \cdots \alpha_n)v\rangle \\ &= \langle (\alpha_j \cdots \alpha_n)^{-1} u(\alpha_j \cdots \alpha_n) - (\alpha_j \cdots \alpha_n)^{-1} u(\alpha_{j+1} \cdots \alpha_n) \\ &\quad - (\alpha_{j+1} \cdots \alpha_n)^{-1} u(\alpha_j \cdots \alpha_n) + (\alpha_{j+1} \cdots \alpha_n)^{-1} u(\alpha_{j+1} \cdots \alpha_n), v\rangle.\end{aligned}$$

Since $(\alpha_j \cdots \alpha_n)^{-1} = R\alpha_1 \cdots \alpha_{j-1}$, we have the characterisation

$$B^1(\pi, \mathfrak{G}_\rho)^\perp = \left\{ u \in Z^1_{\mathrm{par}}(\pi, \mathfrak{G}_\rho) \;\middle|\; \sum_{i=1}^{n} \left\{ \frac{\partial R}{\partial A_i} u\left(\# \frac{\partial R}{\partial A_i}\right) + \frac{\partial R}{\partial B_i} u\left(\# \frac{\partial R}{\partial B_i}\right) \right\} \right.$$
$$+ \sum_{j=1}^{n} \{ R\alpha_1 \cdots \alpha_{j-1} u(\alpha_j \cdots \alpha_n) - R\alpha_1 \cdots \alpha_{j-1} u(\alpha_{j+1} \cdots \alpha_n)$$
$$\left. - R\alpha_1 \cdots \alpha_j u(\alpha_j \cdots \alpha_n) + R\alpha_1 \cdots \alpha_j u(\alpha_{j+1} \cdots \alpha_n) \} = 0 \right\}.$$

For $u \in B^1(\tilde{\pi}, \mathfrak{G}_\rho)^\perp$ we define $\phi(u)$ on the generators

$$\phi(u)(A_i) = u\left(\# \frac{\partial R}{\partial A_i}\right),$$

$$\phi(u)(B_i) = u\left(\# \frac{\partial R}{\partial B_i}\right),$$

$$\phi(u)(\alpha_j) = u(\alpha_j \cdots \alpha_n) - u(\alpha_{j+1} \cdots \alpha_n) - \alpha_j(u(\alpha_j \cdots \alpha_n) - u(\alpha_{j+1} \cdots \alpha_n)).$$

$\phi(u)$ can be thought of as a 1-cocyle on the free group generated by the $(2g+n)$ generators A_i, B_i, and α_j. $\tilde{\pi}$ is the group defined by the $(2g+n)$ generators A_i, B_i, and α_j, together with the relation $R\alpha_1 \cdots \alpha_n = 1$. For $\phi(u)$ to define a 1-cocycle on $\tilde{\pi}$, we need to check that $\phi(u)(R\alpha_1 \cdots \alpha_n) = \phi(u)(1) = 0$. We have

$$\phi(u)(R\alpha_1 \cdots \alpha_n)$$
$$= \phi(u)(R) + \sum_{j=1}^{n} R\alpha_1 \cdots \alpha_{j-1} \phi(u)(\alpha_j)$$
$$= \left\{ \sum_{i=1}^{g} \left(\frac{\partial R}{\partial A_i} \phi(u)(A_i) + \frac{\partial R}{\partial B_i} \phi(u)(B_i) \right) \right\} \qquad \text{by [G, 3.6]}$$
$$+ \sum_{j=1}^{n} R\alpha_1 \cdots \alpha_{j-1} \{ u(\alpha_j \cdots \alpha_n) - u(\alpha_{j+1} \cdots \alpha_n) - \alpha_j(u(\alpha_j \cdots \alpha_n) - u(\alpha_{j+1} \cdots \alpha_n)) \}$$
$$= \left\{ \sum_{i=1}^{g} \left(\frac{\partial R}{\partial A_i} \phi(u)(A_i) + \frac{\partial R}{\partial B_i} \phi(u)(B_i) \right) \right\} + \sum_{j=1}^{n} \{ R\alpha_1 \cdots \alpha_{j-1} u(\alpha_j \cdots \alpha_n)$$
$$- R\alpha_1 \cdots \alpha_{j-1} u(\alpha_{j+1} \cdots \alpha_n) - R\alpha_1 \cdots \alpha_j u(\alpha_j \cdots \alpha_n) + R\alpha_1 \cdots \alpha_j u(\alpha_{j+1} \cdots \alpha_n) \}$$
$$= 0$$

since $u \in B^1(\tilde{\pi}, \mathfrak{G}_\rho)^\perp$. Therefore $\phi(u)$ is a cocycle. Moreover, by its very definition $\phi(u)$ is parabolic. Hence we obtain a map $\phi\colon H^1_{\mathrm{par}}(\pi, \mathfrak{G}_\rho) \approx B^1(\tilde{\pi}, \mathfrak{G}_\rho)^\perp \to Z^1_{\mathrm{par}}(\tilde{\pi}, \mathfrak{G}_\rho)$. We have the following proposition.

PROPOSITION 3. *For* $u \in B^1(\pi, \mathfrak{G}_\rho)^\perp$,

$$w(u, \phi(u)) = \langle u, u \rangle_\rho.$$

Proof. We have

$$w(u,\phi(u)) = \sum_{i=1}^{g}\left\{\left\langle u\left(\# \frac{\partial R}{\partial A_i}\right), u\left(\# \frac{\partial R}{\partial A_i}\right)\right\rangle + \left\langle u\left(\# \frac{\partial R}{\partial B_i}\right), u\left(\# \frac{\partial R}{\partial B_i}\right)\right\rangle\right\}$$
$$- \sum_{j=1}^{n} \langle u(R\alpha_1 \cdots \alpha_{j-1}), R\alpha_1 \cdots \alpha_{j-1}\phi(u)(\alpha_j)\rangle + \sum_{j=1}^{n}\langle \mu_j, \phi(u)(\alpha_j)\rangle. \tag{15}$$

Now

$$-\langle u(R\alpha_1 \cdots \alpha_{j-1}), R\alpha_1 \cdots \alpha_{j-1}\phi(u)(\alpha_j)\rangle + \langle \mu_j, \phi(u)(\alpha_j)\rangle$$
$$= \langle R\alpha_1 \cdots \alpha_{j-1}u(\alpha_j \cdots \alpha_n), R\alpha_1 \cdots \alpha_{j-1}\phi(u)(\alpha_j)\rangle + \langle \mu_j, \phi(u)(\alpha_j)\rangle$$
$$(\text{since } u(R\alpha_1 \cdots \alpha_{j-1}) = -R\alpha_1 \cdots \alpha_{j-1}u(\alpha_j \cdots \alpha_n))$$
$$= \langle u(\alpha_j \cdots \alpha_n) + \mu_j, \phi(u)(\alpha_j)\rangle$$
$$= \langle \alpha_j\mu_j + \alpha_j u(\alpha_{j+1} \cdots \alpha_n), \phi(u)(\alpha_j)\rangle$$
$$= \langle \alpha_j\mu_j + \alpha_j u(\alpha_{j+1} \cdots \alpha_n), u(\alpha_j \cdots \alpha_n) - u(\alpha_{j+1} \cdots \alpha_n)\rangle$$
$$- \langle \alpha_j\mu_j + \alpha_j u(\alpha_{j+1} \cdots \alpha_n), \alpha_j(u(\alpha_j \cdots \alpha_n) - u(\alpha_{j+1} \cdots \alpha_n))\rangle$$
$$= \langle \alpha_j\mu_j + \alpha_j u(\alpha_{j+1} \cdots \alpha_n) - \mu_j - u(\alpha_{j+1} \cdots \alpha_n), u(\alpha_j \cdots \alpha_n) - u(\alpha_{j+1} \cdots \alpha_n)\rangle$$
$$= \langle u(\alpha_j \cdots \alpha_n) - u(\alpha_{j+1} \cdots \alpha_n), u(\alpha_j \cdots \alpha_n) - u(\alpha_{j+1} \cdots \alpha_n)\rangle.$$

Hence it follows from (13) and (15) that $w(u, \phi(u)) = \langle u, u \rangle_\rho$. □

We obtain the following theorem as a corollary.

THEOREM 2. *The skew-symmetric pairing* w *defined by (12) is nondegenerate on* $T_{[\rho]}(\mathscr{P}) \approx H^1_{\text{par}}(\tilde{\pi}, \mathfrak{G}_\rho)$.

3. The case of the general reductive Lie group. We now indicate how the constructions of the previous sections can be extended to a noncompact reductive Lie group. Let G be a connected reductive real Lie group with Lie algebra $\mathfrak{G}$. Let Θ be a Cartan involution on G. Θ is an involutive automorphism of G, and the set of elements fixed under Θ is a maximal compact subgroup of G. In particular, if G is compact, then Θ is the identity on G. Let θ be the corresponding automorphism at the Lie algebra level. If $G \subset GL(n, \mathbf{C})$ is linear, then Θ can be taken to be the map that sends an invertible matrix to its conjugate transpose inverse, and then $\theta(X) = -\overline{X}^t$. Let Ad denote the adjoint representation of G on its Lie algebra.

Let B denote the Killing form on $\mathfrak{G}$. Let X, Y denote general elements in $\mathfrak{G}$, and let g be an element of G. Denote by $\langle\ ,\ \rangle$ the form

$$\langle X, Y \rangle = -\text{Real part of } \mathrm{B}(X, \theta(Y)).$$

It is known then that $\langle\ ,\ \rangle$ is a positive-definite, symmetric, bilinear form on $\mathfrak{G}$. We have the following:

$$\langle Ad(g)X, Y \rangle = \langle X, Ad(\Theta(g^{-1}))Y \rangle,$$

$$Ad(\Theta(g))(X) = \theta(Ad(g)(\theta(X))).$$

We can now easily carry over our earlier arguments to a connected, reductive Lie group G. We define the inner product $\langle\ ,\ \rangle_\rho$ on 1-cocycles as before, using now the modified inner product $\langle\ ,\ \rangle$ defined as above on $\mathfrak{G}$. This allows us to define the space of "harmonic" 1-cocycles as the orthogonal complement to the space of 1-coboundaries. It is easy to see, using the above formulas, that if v is a 1-cocycle, then it is orthogonal to the coboundaries if and only if $\theta(v)$ satisfies the same equations as before. For a harmonic 1-cocyle u, the formula for the "dual" cocycle $\phi(u)$ is given by applying θ to the old formulas. That is,

$$\phi(u)(A_i) = \theta\left(u\left(\# \frac{\partial R}{\partial A_i}\right)\right),$$

and so on. Finally it is clear from the formulas for the skew-symmetric pairing w, and the same arguments as before, that for a harmonic 1-cocycle u, $w(u, \phi(u)) = \langle u, u \rangle_\rho$. Consequently we can conclude that $w(u, \phi(u))$ is nonzero, if u is nonzero.

REFERENCES

[AB] M. F. ATIYAH AND R. BOTT, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. Lond. Ser. A **308** (1983), 523–615.

[BG1] I. BISWAS AND K. GURUPRASAD, *Principal bundles on open surfaces and invariant functions on Lie groups*, Internat J. Math. **4** (1993), 535–544.

[BG2] ———, *On some geometric invariants associated to the space of flat connections on an open surface*, Canad. Math. Bull. **39** (1996), 169–177.

[G] W. M. GOLDMAN, *The symplectic nature of fundamental groups of surfaces*, Adv. Math. **54** (1984), 200–225.

[Gu] K. GURUPRASAD, *Flat connections, geometric invariants and the symplectic nature of the fundamental group of surfaces*, Pacific J. Math. **162** (1994), 45–55.

[GHJW] K. GURUPRASAD, J. HUEBSCHMANN, L. JEFFREY, AND A. WEINSTEIN, *Group systems, groupoids, and the moduli spaces of parabolic bundles*, Duke Math. J. **89** (1997), 377–412.

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