MAT 542 Assignment No 1 Due: January 29, 2003

COMPLEX ANALYSIS

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(1) a) Let $\{z_n\}$ be a sequence of complex numbers. Assume

$$|z_n - z_m| < \frac{1}{1+|n-m|},$$

for all n and m. Evaluate

$$\lim_{n\to\infty} z_n.$$

What more can you say about this sequence?

b) Let $\{z_n\}$ be a sequence with

$$\lim_{n\to\infty} z_n = 0.$$

Let $\{w_n\}$ be a bounded sequence. Show that

$$\lim_{n\to\infty} w_n z_n = 0.$$

(c) Let $\{z_n\}$ be a sequence with

$$\lim_{n\to\infty} z_n = A.$$

Prove that

$$\lim_{n\to\infty} \frac{z_1 + \cdots + z_n}{n} = A.$$

(2) Verify the Cauchy-Riemann equations for the function $f(z) = z^3$ by splitting f into its real and imaginary parts.

(3) Let $x = r\cos\theta,\ y = r\sin\theta$. Show that the Cauchy-Riemann equations in polar coordinates for $F = U + \imath V$ are:

$$r\frac{\partial U}{\partial r} = \frac{\partial V}{\partial \theta}$$

$$r\frac{\partial V}{\partial r} = -\frac{\partial U}{\partial \theta}.$$

(4) Suppose $z = x + \imath y$. Define

$$f(z) = \frac{xy^2 (x + \imath y)}{x^2 + y^4},$$

for $z \neq 0$, and $f(0) = 0$. Show that

$$\lim \frac{f(z) - f(0)}{z} = 0$$

as $z \to 0$ along any straight line. Show that as $z \to 0$ along the curve $x = y^2$, the limit of the difference ratio is $\frac{1}{2}$, thus showing that $f'(0)$ does not exist.

(5) Determine the radius of convergence of each of the following series:

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \sum_{n=1}^{\infty} \frac{z^n}{n}, \quad \sum_{n=0}^{\infty} n! z^n.$$

(6) Prove that if $|a_n| \leq M$ for $n \geq 0$, then the power series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $\rho \geq 1$.

(7) Does there exist a holomorphic function f on $\mathbb{C}$ whose real part is (here $z = x + \imath y$)
   a) $u(x, y) = e^x$
   b) $u(x, y) = e^x [x \cos y - y \sin y]$.

   If yes, exhibit the holomorphic function, if not, prove it.

(8) Prove the fundamental theorem of algebra: Let $a_0, \ldots, a_{n-1}$ be complex numbers ($n \geq 1$). Let $p(z) = z^n + a_{n-1} z^{n-1} + \ldots + a_0$. Then there exists a number $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.
   Hint:
   a) Show there is an $M > 0$ and an $R > 0$ so that $|f(z)| \geq M$ for $|z| \geq R$.
   b) Show next that there is a $z_0 \in \mathbb{C}$ such that $|f(z_0)| = \min \{|f(z)|; z \in \mathbb{C}\}$.
   c) By the change of variable, $f(z + z_0) = g(z)$, it suffices to show that $g(0) = 0$.
   d) Write $g(z) = \alpha + \beta z^m + \ldots + C_n z^n$ with $\beta \neq 0$. Choose ω such that

   $$\omega^m = -\frac{\alpha}{\beta}.$$

   If $\alpha \neq 0$, obtain the contradiction $|g(\omega z)| < |\alpha|$ for some z.

   P.S. We will later have a simpler proof of this theorem.

(9) Let p be a complex valued polynomial of two real variables:

$$p(z) = \sum a_{ij} x^i y^j.$$

Write

$$p(z) = \sum P_j(z)\overline{z}^j,$$

Where each P_j is of the form $P_j(z) = \sum b_{ij} z^i$. Prove that p is an entire function if and only if

$$0 = P_1 = P_2 = \dots.$$

(10) Show that the positive integers cannot be partitioned nontrivially into a finite set of arithmetic progressions with no common differences.
Hint: You should make a good use of the geometric series.