MAT 542 Assignment No. 6 Due: April 7, 2003

(1) Let f be holomorphic in the disc $|z| < R$ and let $M(r) = \sup \{|f(z)|; |z| = r\}$ for $0 \leq r < R$. Prove:
   a) $M(r)$ is a continuous, monotone non-decreasing function of r for $0 \leq r < R$.
   b) If f is not constant, $M(r)$ is strictly increasing.

(2) Let f be holomorphic in a bounded region Ω, continuous in the closure $\bar{\Omega}$. Suppose that $N > 0$ is the minimum value of $|f(z)|$ in $\bar{\Omega}$. Prove that $|f(z)| > N$ for z in Ω unless f is a constant. Show by example that $|f(z)|$ may attain its minimum value at a point of Ω if $N = 0$, without being constant in Ω.

(3) If $f(z)$ is holomorphic in an open set containing the closed disc $|z| \leq 1$ and if $|f(z)| < 1$ for $|z| = 1$, then the equation $f(z) = z^n$ has exactly n solutions in $|z| < 1$ for any integer $n \geq 0$.

(4) Suppose f is holomorphic on an open set Ω containing the closed Jordan path γ and its interior. If f is real at all points of γ, show that f must be constant inside γ. (*Hint:* Use the argument principle to find the number of solutions inside γ of the equation $f(z) = a$, where a is not real.)

(5) Use Roche's theorem to prove the fundamental theorem of algebra.

(6) Problem No. 1 on p. 154 in Ahlfors.

(7) Problem No. 3 on p. 154 in Ahlfors.

(8) Problem No. 1 on p. 161 in Ahlfors.

(9) Evaluate the following integrals.
   (a)
$$\int_0^\infty \frac{dx}{(a + bx^2)^n}; \ n \in \mathbb{Z}, \, n, a, b > 0.$$
   (b)
$$\int_0^\infty \frac{dx}{1 + x^n}; \ n \in \mathbb{Z}, \, n \geq 2.$$
   (c) Problem No. 3 (h) on p. 161 in Ahlfors.
   (d) Problem No. 3 (i) on p. 161 in Ahlfors.
   (e) Problem No. 4 on p. 161 in Ahlfors.