MAT 542 Assignment No. 3 Due: February 12, 2003

1) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be power series with $R = 1$. A point z_0 on the unit circle is called regular if there exists a disk $D(\rho, r)$ with the center at some point ρ, $|\rho| < 1$, and radius $r > 0$ such that $z_0 \in D(\rho, r)$ and corresponding power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\rho)}{n!}(z-\rho)^n$$

absolutely converges on $D(\rho, r)$. In particular, $\lim_{z \to z_0^-} f(z)$ exists. Otherwise, z_0 is called singular point.

(a) Prove that if all coefficients a_n are real and the series $\sum_{n=0}^{\infty} a_n$ diverges to infinity, then $z = 1$ is a singular point for f.

(b) Show by example that weaker condition $|a_0 + \cdots + a_n| \to \infty$ does not guarantee that $z = 1$ is a singular point.

(c) Prove that if $a_n \geq 0$ for all n then $z = 1$ is a singular point.

2) Prove that for function $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ every point on the unit circle is singular.

3) Suppose that $f(z)$ is analytic and satisfies the condition $|f^2(z) - 1| < 1$ in the region Ω. Prove that either $\Re f(z) > 0$ or $\Re f(z) < 0$ throughout Ω.

4) Let $S(z) = \frac{az+b}{cz+d}$ be a Möbius transformation with real coefficients normalized such that $ad - bc = 1$. Prove that if S not the identity, then

(a) S is elliptic iff $-2 < a + d < 2$

(b) S is parabolic iff $a + d = \pm 2$

(c) S is hyperbolic iff $(a+d)^2 > 4$.

5) Express cross ratios corresponding to all 24 permutations of four points z_1, z_2, z_3, z_4 in terms of $\lambda = (z_1, z_2, z_3, z_4)$.

6) Let z_1, z_2, z_3, z_4 be consecutive points on the circle. Prove that

$$|z_1 - z_3||z_2 - z_4| = |z_1 - z_2||z_3 - z_4| + |z_2 - z_3||z_1 - z_4|$$

and interpret the result geometrically.

7) Find the most general Möbius transformation of the circle $|z| = R$ into itself.

8) Map the region between $|z| = 1$ and $|z - 1/2| = 1/2$ conformally onto the half plane.

9) Map conformally the complement of the arc $|z| = 1$, $\Im z \geq 0$ on the outside of the unit circle so that the points ∞ correspond to each other.