MAT 542 Assignment No. 2 Due: February 5, 2003

1) Let y_n be increasing real sequence such that $y_n \to \infty$. Prove that (Stolz theorem)

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}}$$

if the limit in the right hand side exists (or equal to $\pm\infty$). Show how the problem 1 part (c) from HW 1 (due to Cauchy) immediately follows from Stolz theorem.

2) Let $\{a_n\}$ and $\{b_n\}$ be positive sequences. Prove that
   (a)
   $$\overline{\lim}_n a_n b_n \le \overline{\lim}_n a_n \overline{\lim}_n b_n,$$
   provided the right hand side is not of the indeterminate form $0 \times \infty$. Give an example when strict inequality holds.
   (b) If $\lim_n a_n$ exists, show then (a) is equality provided the right side is not indeterminate, i.e.,
   $$\overline{\lim}_n a_n b_n = \lim_n a_n \overline{\lim}_n b_n.$$

3) Find all the roots of $\cos z = 2$.

4) (a) Find: $\Re(\sin z)\,, \Im(\sin z)\,, \Re(\cos z)\,, \Im(\cos z)\,.$
   (b) Prove that
   $$|\sin z|^2 = \sin^2 x + \sinh^2 y\; (z = x + \imath y)\;, |\cos z|^2 = \cos^2 x + \sinh^2 y.$$
   (c) Derive addition formulas for $\cosh(a+b)$ and $\sinh(a+b)$.
   (d) Evaluate $D\sinh z, D\cosh z$, and $\cosh^2 z - \sinh^2 z$.

5) Prove, using power series, that $e^{-z} = 1/e^z$.

6) Is it always true that $\mathrm{Log}(e^z) = z$? Support your answer with either a proof or a counterexample.

7) Let $\{k_n\}$ be a sequence in which every positive integer appears once and only once. Let $\sum_n a_n$ be a series, putting $a'_n = a_{k_n}$,we say that $\sum_n a'_n$ is a rearrangement of $\sum_n a_n$.
   (a) Let $a_n \in \mathbb{R}$. Assume that $\sum_n a_n$ is convergent but not absolutely convergent. Let $a \in \mathbb{R}$. Show that there is a rearrangement $\sum_n a'_n$ of $\sum_n a_n$ such that $a = \sum_n a'_n$ (Riemann theorem).
   b) Show that $\sum_n a_n$ converges absolutely if and only if every rearrangement converges to the same sum.

   Hint: for part (a), write $\sum_n a_n = \sum_n p_n - \sum_n q_n$ with positive p_n and q_n and use that if convergent series is not absolutely convergent, then series $\sum_n p_n$ and $\sum_n q_n$ are divergent (it is easy to prove this).

8) Let $\{a_n\}$ be a real sequence. Show that

$$\overline{\lim}_n a_n = \sup\left\{\alpha; \alpha = \lim_n b_n\right\},$$

with$\{b_n\}$ a convergent subsequence of a_n

$$\underline{\lim}_n a_n = \inf\left\{\alpha; \alpha = \lim_n b_n\right\},$$

with $\{b_n\}$ as above.
In this exercise a sequence $\{b_n\}$ with $\lim b_n = +\infty$ (similarly $-\infty$) is considered a convergent sequence.

9) (a) Given two points z_1, z_2, $|z_1| < 1$, $|z_2| < 1$, show that for every point $z \neq 1$ in the closed triangle with vertices z_1, z_2 and 1,

$$\frac{|1-z|}{1-|z|} \leq K,$$

where K is a constant that depends only on z_1 and z_2.

(b) Determine the smallest value of K for $z_1 = \frac{1+\imath}{2}$ and $z_2 = \frac{1-\imath}{2}$.