MAT 534: HOMEWORK 8

DUE THU OCT 22

Problems marked by asterisk (*) are optional.

1. Prove that $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain.

2. Determine the greatest common divisor in $\mathbb{Q}[x]$ of the polynomials $a(x) = x^3 + 4x^2 + x - 6$ and $b(x) = x^5 - 6x + 5$ and write it as a linear combination of $a(x)$ and $b(x)$.

3. Let $p \in \mathbb{Z}$ be a prime number of the form $p = 4k + 1$,and let $p = \pi\bar{\pi}$ be its factorization into primes in $\mathbb{Z}[\sqrt{-1}]$.

(a) Prove that $\mathbb{Z}[\sqrt{-1}]/(p)$ is a finite ring, with $|\mathbb{Z}[\sqrt{-1}]/(p)| = p^2$.

(b) Use Chinese Remainder Theorem to prove that $\mathbb{Z}[\sqrt{-1}]/(p) \simeq \mathbb{Z}_p \times \mathbb{Z}_p$.

4. Consider the ring $R = \mathbb{Z}[\sqrt{-5}]$.

(a) Prove that elements $2, 3, 1 \pm \sqrt{-5}$ are irreducible in R.

(b) Show that R is not U.F.D. because

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}).$$

(c) Define the ideals

$$I = (2, 1 + \sqrt{-5}),$$
$$J = (3, 2 + \sqrt{-5}),$$
$$J' = (3, 2 - \sqrt{-5}).$$

Prove that these ideals are prime (see hint in in Exercise 8, p. 293 in the book).

(d) Prove that $(2) = I^2$, $(3) = JJ'$, $(1 - \sqrt{-5}) = IJ$, $(1 + \sqrt{-5}) = IJ'$. Deduce from this that both factorizations $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ give the same presentation for (6) as a product of prime ideals: $(6) = I^2JJ'$.

5. Dummit and Foote, problems 10 and 13 on p. 257.

6. Dummit and Foote, problem 5 on p. 267.

***7.** Dummit and Foote, problems 10 and 11 on p. 269.