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Scientific heritage of L. D. Faddeev. Survey of papers

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Abstract. This survey was written by students of L. D. Faddeev under the editorship of L. A. Takhtajan. Sections 1.1, 1.2, 2–4, and 6 were written by Takhtajan, §§ 1.3 and 1.4 by F. A. Smirnov, §§ 5.1 and 5.2 by E. K. Sklyanin, §§ 5.3–5.6 by Sklyanin, Smirnov, and Takhtajan, § 7.1 by M. A. Semenov-Tian-Shansky, §§ 7.2–7.6 by Takhtajan and S. L. Shatashvili, § 7.7 by A. Yu. Alekseev and Shatashvili, and § 8 by I. Ya. Aref'eva.

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Contents

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1. Scattering theory

In 1951–1956, when L. D. Faddeev was studying at university, quantum mechanics was still regarded as a new field. Indeed, only 25 years had passed since the Schrödinger equation was published in 1926! The remarkable achievements of quantum mechanics and quantum field theory (its unification with the special theory of relativity) attracted both theoretical physicists and mathematicians. A fundamental role in the description of quantum phenomena is played by the scattering theory, which studies changes in the quantum particle state as it passes through a potential barrier. The mathematical formalism uses a Hamiltonian operator (a Hamiltonian) acting in the Hilbert space $L^2(\mathbb{R}^3)$,

$$H = H_0 + V, \tag{1.1}$$

where $H_0 = -\Delta$ is the Hamiltonian of a free particle,[1] Δ is the Laplace operator in $\mathbb{R}^3$, and V is the operator of multiplication in $L^2(\mathbb{R}^3)$ by a measurable function $v(\boldsymbol{x})$ which decays appropriately as $|\boldsymbol{x}| \to \infty$. For example, it suffices to require that $v(\boldsymbol{x})$ be bounded and

$$v(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-3-\varepsilon}) \quad \text{as } |\boldsymbol{x}| \to \infty. \tag{1.2}$$

Specifically, the evolution of the particle is described by the Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = H\psi \tag{1.3}$$

for the wave function $\psi(t) \in L^2(\mathbb{R}^3)$. If (1.2) holds, then there are solutions

$$\psi_\pm(t) = e^{-itH_0}\psi_\pm$$

of the free Schrödinger equation such that

$$\|\psi(t) - \psi_\pm(t)\| \to 0 \quad \text{as } t \to \pm\infty.$$

The passage from a free motion as $t \to -\infty$ to a free motion as $t \to +\infty$ is given by the scattering operator

$$\psi_+ = S\psi_-.$$

The operator S is unitary on $L^2(\mathbb{R}^3)$ and commutes with a free-particle Hamiltonian H_0. More precisely,

$$S = U_+^* U_-,$$

where the *wave operators* $U_\pm$ are defined by

$$U_\pm = \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}. \tag{1.4}$$

The limits exist in the strong operator topology. The wave operators intertwine the free Hamiltonian and the perturbed Hamiltonian:

$$HU_\pm = U_\pm H_0. \tag{1.5}$$

In the so-called stationary approach to the scattering theory, S is defined as an integral operator with kernel obtained from a solution of the stationary Schrödinger equation

$$-\Delta\psi(\boldsymbol{x}) + v(\boldsymbol{x})\psi(\boldsymbol{x}) = k^2\psi(\boldsymbol{x}). \tag{1.6}$$

Namely, for all $\boldsymbol{k} \in \mathbb{R}^3$, the equation (1.6) with $k = |\boldsymbol{k}|$ has solutions $u^{(\pm)}(\boldsymbol{x}, \boldsymbol{k})$ satisfying the so-called *radiation conditions* as $r = |\boldsymbol{x}| \to \infty$:

$$u^{(\pm)}(\boldsymbol{x}, \boldsymbol{k}) = e^{i(\boldsymbol{k},\boldsymbol{x})} + f^{(\pm)}(k, \boldsymbol{\omega}, \boldsymbol{n})\frac{e^{\pm ikr}}{r} + o\left(\frac{1}{r}\right).$$

[1] For convenience we let the Planck constant $\hbar$ be equal to 1 and the mass of a particle to 1/2.

Here $k=|\boldsymbol{k}|$, $\boldsymbol{k}=k\boldsymbol{\omega}$, and $\boldsymbol{x}=r\boldsymbol{n}$, where $\boldsymbol{\omega}$ and $\boldsymbol{n}$ are in S^2 (the two-dimensional sphere in $\mathbb{R}^3$) and $(\,\cdot\,,\cdot\,)$ is the standard scalar product in $\mathbb{R}^3$. We write $\mathscr{F}$ for the Fourier transform operator on $L^2(\mathbb{R}^3)$,

$$\widehat{\psi}(\boldsymbol{k})=\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}e^{-i(\boldsymbol{k},\boldsymbol{x})}\psi(\boldsymbol{x})\,d^3\boldsymbol{x},$$

which switches between the position and momentum representations. In the momentum representation, the scattering operator takes the form $\widehat{S}=\mathscr{F}^{-1}S\mathscr{F}$ and is given explicitly by

$$(\widehat{S}\psi)(\boldsymbol{k})=\psi(\boldsymbol{k})+\frac{i}{2\pi}\int_{\mathbb{R}^3}\delta(\boldsymbol{k}^2-\boldsymbol{l}^2)f(\boldsymbol{k},\boldsymbol{l})\psi(\boldsymbol{l})\,d^3\boldsymbol{l},$$

where

$$\delta(\boldsymbol{k}^2-\boldsymbol{l}^2)=\frac{\delta(k-l)}{2k}\,,\quad f(\boldsymbol{k},\boldsymbol{l})=f^{(+)}(k,\boldsymbol{\omega},\boldsymbol{\omega}'),\quad \boldsymbol{k}=k\boldsymbol{\omega},\quad \boldsymbol{l}=k\boldsymbol{\omega}'.$$

In physics, the operator $\widehat{S}$ is referred to as the *S-matrix*, and the function $f(\boldsymbol{k},\boldsymbol{l})$ (defined for $|\boldsymbol{k}|=|\boldsymbol{l}|$) as the *scattering amplitude*. The S-matrix is a fundamental object in quantum mechanics and quantum field theory.

1.1. First papers. Scattering theory was Faddeev's first love. It all began at a student seminar organized by O. A. Ladyzhenskaya and devoted to the monograph *Mathematical aspects of the quantum theory of fields* [96] by Friedrichs. Faddeev was the main speaker and dreamed of being able in the future to get a serious hold on quantum field theory, which describes quantum systems with infinitely many degrees of freedom. For its study one first needs to understand systems with finitely many degrees of freedom, that is, quantum mechanics. In his first paper [1],[2] published in 1956, Faddeev proved that the Fourier transform $\widehat{v}(\boldsymbol{p})$ of a potential $v(\boldsymbol{x})$ is a limit of the scattering amplitude $f(k,\boldsymbol{\omega},\boldsymbol{\omega}')$ as $k\to\infty$ at fixed $k(\boldsymbol{\omega}-\boldsymbol{\omega}')=\boldsymbol{p}$. Hence, in this case the potential $v(\boldsymbol{x})$ is uniquely determined by its S-matrix.

In the 1950s, the mathematical theory of operators arising in quantum mechanics was an inspiration and a source of important problems for the spectral theory of differential operators. Achievements of the Soviet mathematical school, which was brilliantly represented by M. Sh. Birman, I. M. Gelfand, M. G. Krein, B. M. Levitan, V. A. Marchenko, A. Ya. Povzner, and others, were second to none. At that time one major and difficult problem was development of the spectral theory for a multidimensional Schrödinger operator with decaying potential that describes the scattering of a quantum particle on a potential centre. The problem of two interacting particles reduces to this case after separation of the centre of mass. The spectral theory of the Schrödinger operator for three interacting particles was at that time an insoluble problem.

In 1955, Povzner was the first to prove the eigenfunction expansion theorem for a three-dimensional Schrödinger operator[3] with compactly supported potential.

[2]Refereed by Norman Levinson in *Mathematical Reviews*.

[3]Multidimensional Schrödinger operators can be studied in the same way.

In his paper [2] of 1957,[4] Faddeev substantially developed Povzner's method and proved the eigenfunction expansion theorem in the case when the potential $v(\boldsymbol{x})$ and its gradient $\nabla v(\boldsymbol{x})$ satisfy (1.2). He also proved an equiconvergence theorem: the eigenfunction expansion converges at a point $\boldsymbol{x} \in \mathbb{R}^3$ if the Fourier integral expansion converges at this point.

In [4], published in a physics journal, Faddeev proposed an elegant and rigorous proof of the analytic properties of the zero-angle scattering amplitude[5] $f(k, \boldsymbol{\omega}, \boldsymbol{\omega})$ as a function of the energy $E = k^2$. Namely, when the potential $v(\boldsymbol{x})$ satisfies (1.2), he proved that the function

$$f(E) = f(\sqrt{E}\,, \boldsymbol{\omega}, \boldsymbol{\omega})$$

with fixed $\boldsymbol{\omega} \in S^2$ extends analytically to the complex plane $\mathbb{C}$ slit along the positive real semiaxis, except for finitely many points E_l on the negative real semiaxis, where it can have simple poles with real residues d_l. This gives a rigorous mathematical proof of the so-called *dispersion relation*

$$f(E) = f(\infty) + \frac{1}{\pi} \int_0^\infty \frac{\operatorname{Im} f(E')}{E - E'}\, dE' + \sum_{l=1}^{n} \frac{d_l}{E - E_l}\,, \tag{1.7}$$

where $E \in \mathbb{C} \setminus [0, \infty)$ and $f(\infty) = -\dfrac{1}{4\pi} \displaystyle\int_{\mathbb{R}^3} v(\boldsymbol{x})\, d^3\boldsymbol{x}$. This elegant argument was reproduced in the famous textbook *Quantum mechanics*: *non-relativistic theory* by Landau and Lifshitz [109] (Chap. XVII, § 130) with a footnote: "The idea of this proof is due to *L. D. Faddeev* (1958)."

1.2. The Friedrichs model. In [6], Ladyzhenskaya and Faddeev studied the Friedrichs model in the perturbation theory of the continuous spectrum. The method developed in [6] and improved in [15] was successfully used by Faddeev in various problems of scattering theory, from three-particle scattering theory [13] to the spectral theory of automorphic functions [19]!

Thus, let $\mathfrak{H} = L^2(I, \mathfrak{h})$ be the space of functions $f(x)$ on an interval $I \subseteq \mathbb{R}$ with values in an auxiliary Hilbert space $\mathfrak{h}$ such that $\|f(x)\|_{\mathfrak{h}}$ is square-integrable on I, and let H_0 be the operator of multiplication by x.[6] Write V for an integral operator whose kernel $v(x, y)$ is compact in $\mathfrak{h}$ and satisfies the realness condition

$$v(x, y) = v^*(y, x) \quad \text{for all } x, y \in I$$

and certain boundedness and Hölder conditions, which take the following form in the case when $I = \mathbb{R}$.

Condition A_{θ_0} (boundedness).

$$\|v(x, y)\|_{\mathfrak{h}} \leqslant K(1 + |x| + |y|)^{-\theta_0}, \qquad \theta_0 > \frac{1}{2}.$$

[4]Refereed by the famous analyst Lars Gårding in *Mathematical Reviews*!
[5]Also known as the *forward scattering amplitude*.
[6]Here we use the notation of [15].

Condition B_{μ_0} (smoothness).

$$\|v(x+h,y+k)-v(x,y)\|_{\mathfrak{h}} \leqslant K(1+|x|+|y|)^{-\theta_0}(|h|^{\mu_0}+|k|^{\mu_0}), \qquad \mu_0 > \frac{1}{2}.$$

Faddeev developed the following method for studying the perturbed operator[7]

$$H = H_0 + V.$$

Consider the second Hilbert identity

$$R(\lambda) - R_0(\lambda) = R_0(\lambda)VR(\lambda), \qquad \lambda \in \mathbb{C} \setminus I, \tag{1.8}$$

for the resolvents

$$R(\lambda) = (H-\lambda I)^{-1} \quad \text{and} \quad R_0(\lambda) = (H_0 - \lambda I)^{-1}$$

of the self-adjoint operators H and H_0. This equation is unusable for a study of the resolvent $R(\lambda)$ since the operator $R_0(\lambda)$ is non-compact. But if we put

$$T(\lambda) = V - VR(\lambda)V, \tag{1.9}$$

then

$$R(\lambda) = R_0(\lambda) - R_0(\lambda)T(\lambda)R_0(\lambda) \tag{1.10}$$

and the operator $T(\lambda)$ satisfies the equation

$$T(\lambda) = V - VR_0(\lambda)T(\lambda), \qquad \lambda \in \mathbb{C} \setminus I. \tag{1.11}$$

In [15] this equation was proved to be Fredholm in the Banach space $\mathfrak{B}(\theta,\mu)$ of Hölder functions $f(x)$ with the norm

$$\|f\|_{\theta,\mu} = \sup_{x\in I,\ |h|\leqslant 1} (1+|x|)^{\theta}\left[|f(x)| + \frac{|f(x+h)-f(x)|}{|h|^{\mu}}\right], \qquad \mu,\theta > 0,$$

for all $\theta < \theta_0$ and $\mu < \mu_0$. This remarkable result enables one to establish that H has finitely many eigenvalues of finite multiplicity, to describe its eigenfunctions of the continuous spectrum, and to prove the corresponding expansion theorem. The proof of the compactness of the operator $VR_0(\lambda)$ uses the following beautiful line of argument.[8] One first proves that this operator is bounded and acts from $\mathfrak{B}(\theta,\mu)$ to $\mathfrak{B}(\theta',\mu')$, where $\theta' > \theta$ and $\mu' > \mu$. The space $\mathfrak{B}(\theta',\mu')$ is naturally embedded in $\mathfrak{B}(\theta,\mu)$, and this embedding maps weakly convergent sequences to strongly convergent ones.

[7] An example of the operator H is the Schrödinger operator (1.6) in the momentum representation. The variable x in the Friedrichs model corresponds to the momentum variable $\boldsymbol{k}$ in the other sections.

[8] Faddeev said: "In this paper, O. A. (Ladyzhenskaya) helped me by stating a splendid compactness criterion in the space of Hölder functions".

1.3. Potentials of zero radius. In the early 1960s Faddeev became seriously interested in quantum field theory (QFT), which was in a somewhat chaotic state after the discovery of the 'zero-charge' paradox by Landau and Pomeranchuk.

Both the beauty and the difficulty of QFT stem from the fact that it deals with local interactions. Faddeev always tried to find a simple model for understanding complicated phenomena. In the joint paper [11] with Berezin, he chose a model given by the Schrödinger equation (1.6) in $\mathbb{R}^3$ with Dirac's δ-function as the potential:

$$-\Delta\psi(\boldsymbol{x}) + \varepsilon\delta(\boldsymbol{x})\psi = E\psi(\boldsymbol{x}). \tag{1.12}$$

Clearly, the potential $\delta(\boldsymbol{x})$ does not belong to $L^2(\mathbb{R}^3)$. Hence, the suggestion in [11] was to regularize the problem by studying an equation

$$-\Delta\psi(\boldsymbol{x}) + \varepsilon(N)\int_{\mathbb{R}^3} u_N(\boldsymbol{x},\boldsymbol{y})\psi(\boldsymbol{y})\,d^3\boldsymbol{y} = E\psi(\boldsymbol{x}), \tag{1.13}$$

where

$$u_N(\boldsymbol{x},\boldsymbol{y}) \to \delta(\boldsymbol{x})\delta(\boldsymbol{y}) \quad \text{as } N\to\infty$$

and $\varepsilon(N)$ is allowed to depend on N for self-consistency, as will be seen below. In the language of QFT, the parameter N is called the ultraviolet cut-off, and $\varepsilon(N)$ is called the bare coupling constant. The equation (1.13) determines a well-defined self-adjoint operator in $L^2(\mathbb{R}^3)$.

Passing to the Fourier transform and choosing a piecewise-constant approximation for the kernel $u_N(\boldsymbol{x},\boldsymbol{y})$, one can solve the spectral problem (1.13) quite easily using the methods of the Friedrichs model (see § 1.2). This yields the following remarkable result. For the wave function to have a non-trivial limit as $N\to\infty$, one should take

$$\varepsilon(N) = \frac{8\pi^3\alpha}{1-4\pi\alpha N}\,, \tag{1.14}$$

where α is an arbitrary constant, which in QFT would be called the physical or renormalized coupling constant. We shall return to the physics interpretation of this formula soon, but first we explain its mathematical meaning.

For a mathematician, it is natural to begin the study of (1.12) by considering the closed symmetric operator $-\Delta$ in $L^2(\mathbb{R}^3)$, defined on smooth compactly supported functions on $\mathbb{R}^3$ vanishing at the point $\boldsymbol{x}=0$. This operator has deficiency indices $(1,1)$ and, therefore, admits a one-parameter family of self-adjoint extensions. The parameter α labels this family of operators H_α, and the kernel of the resolvent $(H_\alpha - zI)^{-1}$ takes the following form in the momentum representation:

$$R(\boldsymbol{p},\boldsymbol{q},z) = \frac{\delta(\boldsymbol{p}-\boldsymbol{q})}{\boldsymbol{p}^2-z} - \frac{\alpha}{1+2\pi^2 i\alpha\sqrt{z}\,\operatorname{sign}\operatorname{Im}\sqrt{z}}\,\frac{1}{(\boldsymbol{p}^2-z)(\boldsymbol{q}^2-z)}\,. \tag{1.15}$$

In the case of a two-dimensional Schrödinger operator, reconsidered by Faddeev after many years [76], the renormalized coupling constant depends logarithmically on the cut-off parameter N, while in the one-dimensional case, the Schrödinger equation (1.12) describes an infinite potential barrier and the corresponding Schrödinger operator is self-adjoint. In dimension $n\geqslant 4$ the operator $-\Delta$ in $L^2(\mathbb{R}^n)$, defined on smooth compactly supported functions on $\mathbb{R}^n$ vanishing at the point $\boldsymbol{x}=0$, is essentially self-adjoint. This corresponds to $\varepsilon=0$ in (1.12) and is related to Sobolev's embedding theorems.

We now return to the interpretation of (1.14) by physicists. To obtain an extension with a given coupling constant α, one should let N tend to infinity and $\varepsilon(N)$ to zero in a compatible way. This suggests another approach to this problem using perturbation theory with respect to ε. Such an approach is indeed possible and (1.14) is then obtained by summing a geometric series. Remarkably, this is in accordance with the removal of the ultraviolet cut-off and letting $N\to\infty$.

In the modern language, a field theory is said to be asymptotically free if it admits a self-consistent procedure similar to that described above. Examples of asymptotically free field theories are non-Abelian gauge theories which will be discussed later. Their ultraviolet behaviour is diametrically opposite to the paradoxical behaviour discovered by Landau and Pomeranchuk in the Abelian case (of quantum electrodynamics).

1.4. Three-body problem. Faddeev's best-known result in scattering theory is his solution of the quantum-mechanical scattering problem for three particles. It was announced in [8], [10], and [12] and discussed in detail in [13]. Consider the Schrödinger operator for three pairwise-interacting particles:

$$H_3=H_0+V_{12}+V_{23}+V_{13}$$

in the Hilbert space $L^2(\mathbb{R}^9)$, where

$$H_0=-\frac{1}{2m_1}\Delta_1-\frac{1}{2m_2}\Delta_2-\frac{1}{2m_3}\Delta_3$$

is the free Hamiltonian and the V_{ij} are the operators of multiplication by potentials $v_{ij}(\boldsymbol{x}_i-\boldsymbol{x}_j)$, which are assumed to be smooth and sufficiently rapidly decaying at infinity. It is also assumed that the corresponding two-particle operators

$$H_2=-\frac{1}{2m_1}\Delta_1-\frac{1}{2m_2}\Delta_2+V_{12}\quad\text{and so on}$$

in $L^2(\mathbb{R}^6)$ have only one discrete-spectrum point and no virtual levels at the lower edge of the continuous spectrum.[9]

The three-body problem is difficult for many reasons, of which we mention the two most significant. First, the total potential does not decrease along some directions in the nine-dimensional configuration space $\mathbb{R}^9$ (for example, $\boldsymbol{x}_1-\boldsymbol{x}_2=\text{const}$). Second, two of the three particles can form a bound state in the process of scattering (conservation of energy prohibits this in the two-particle case). In mathematical terms, this means that the continuous spectrum of H_3 differs substantially from that of the free three-particle Hamiltonian. Clearly, these two difficulties are closely related. Overcoming them is the main content of Faddeev's work.

As in the Friedrichs model [4], [15], Faddeev uses the Fourier transform (or the momentum representation), where the free Hamiltonians of particles are the operators of multiplication by $\boldsymbol{k}_i^2/(2m_i)$, and the potentials V_{ij} are given by integral

[9] The first restriction is inessential and can easily be omitted.

operators whose kernels, to be described below, contain δ-functions in some variables. The two-particle problem is reduced by separation of the centre of mass to a study of the operator

$$\boldsymbol{h} = \boldsymbol{h}_0 + \boldsymbol{v}, \qquad \boldsymbol{h}_0 = \frac{\boldsymbol{k}^2}{2m},$$

in $L^2(\mathbb{R}^3)$ (here and in what follows we use the notation from [13]).

In the three-particle case, Faddeev separates the centre of mass and considers the coordinates conjugate to the familiar Jacobi coordinates in the three-body problem. This yields three pairs of variables:

$$\boldsymbol{k}_{23}, \boldsymbol{p}_1, \quad \boldsymbol{k}_{31}, \boldsymbol{p}_2, \quad \text{and} \quad \boldsymbol{k}_{12}, \boldsymbol{p}_3,$$

each of which can be used independently. It is convenient to use the notation

$$\boldsymbol{k}_\alpha, \boldsymbol{p}_\alpha, \quad \text{where } \alpha = 23, 31, 12, \text{ and } \quad \boldsymbol{p}_{23} = \boldsymbol{p}_1,\ \boldsymbol{p}_{31} = \boldsymbol{p}_2,\ \boldsymbol{p}_{12} = \boldsymbol{p}_3.$$

The three-particle Hamiltonian H_3 is reduced to the following operator acting in $L^2(\mathbb{R}^6)$:

$$\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{V}_{23} + \boldsymbol{V}_{31} + \boldsymbol{V}_{12},$$

where for each $\alpha = 23, 31, 12$ we have

$$\boldsymbol{H}_0 = \frac{\boldsymbol{k}_\alpha^2}{2m_\alpha} + \frac{\boldsymbol{p}_\alpha^2}{2n_\alpha}$$

and m_α and n_α are simply expressed in terms of the masses m_1, m_2, and m_3 of the individual particles. The operators $\boldsymbol{V}_\alpha$ are integral operators with the kernels

$$V_\alpha(\boldsymbol{k}_\alpha, \boldsymbol{p}_\alpha; \boldsymbol{k}'_\alpha, \boldsymbol{p}'_\alpha) = v_\alpha(\boldsymbol{k}_\alpha - \boldsymbol{k}'_\alpha)\delta(\boldsymbol{p}_\alpha - \boldsymbol{p}'_\alpha), \qquad \alpha = 23, 31, 12.$$

The presence of the δ-function of $\boldsymbol{p}_\alpha$ reflects a difficulty (already mentioned above) of a three-particle problem.

As in the Friedrichs model, the functions $v_\alpha(\boldsymbol{k})$ defining the difference-type kernels V_α are subject to the realness condition

$$\overline{v_\alpha(-\boldsymbol{k})} = v_\alpha(\boldsymbol{k})$$

and the decay and smoothness conditions A_{θ_0} and B_{μ_0}. When $\theta_0 > 1/2$, these conditions coincide with the hypotheses of the well-known Kato theorem, and therefore the operators $\boldsymbol{h}$ and $\boldsymbol{H}$ are self-adjoint and their resolvents

$$\boldsymbol{r}(z) = (\boldsymbol{h} - zI)^{-1} \quad \text{and} \quad \boldsymbol{R}(z) = (\boldsymbol{H} - zI)^{-1}$$

are bounded operators for $z \in \mathbb{C} \setminus \mathbb{R}$. Faddeev's main tool is a study of the resolvent $\boldsymbol{R}(z)$, whose behaviour near $\mathbb{R}$ determines the spectral properties of the operator.

The two-particle resolvent $\boldsymbol{r}(z)$ is studied following the scheme of the Friedrichs model given above and using an operator $\boldsymbol{t}(z)$ which is defined by (1.9) and satisfies (1.11).[10] We now pass to the three-particle problem and consider the Hilbert identity

$$\boldsymbol{R}(z) - \boldsymbol{R}_0(z) = \boldsymbol{R}_0(z)\boldsymbol{V}\boldsymbol{R}(z), \tag{1.16}$$

[10]The improvements in [15] in comparison with [4] stem precisely from Faddeev's work in [13]!

where
$$\boldsymbol{V} = \boldsymbol{V}_{23} + \boldsymbol{V}_{13} + \boldsymbol{V}_{12} \quad \text{and} \quad \boldsymbol{R}_0(z) = (\boldsymbol{H}_0 - zI)^{-1}.$$
As in § 1.2, the operator
$$\boldsymbol{T}(z) = \boldsymbol{V} - \boldsymbol{V}\boldsymbol{R}(z)\boldsymbol{V}$$
satisfies the equation
$$\boldsymbol{T}(z) = \boldsymbol{V} - \boldsymbol{V}\boldsymbol{R}_0(z)\boldsymbol{T}(z) \tag{1.17}$$
and we have
$$\boldsymbol{R}(z) = \boldsymbol{R}_0(z) - \boldsymbol{R}_0(z)\boldsymbol{T}(z)\boldsymbol{R}_0(z). \tag{1.18}$$
However, in contrast to (1.11), this equation cannot be used to investigate the operator $\boldsymbol{T}(z)$. The thing is that, for example, the kernel of the operator $\boldsymbol{V}_{23}\boldsymbol{R}_0(z)$ contains a δ-function and this singularity persists under iterations. At the same time, the singularity disappears in the kernel of the operator product $\boldsymbol{V}_{23}\boldsymbol{R}_0(z)\boldsymbol{V}_{13}\boldsymbol{R}_0(z)$. This observation led Faddeev to his discovery of the class of integral equations bearing his name!

Namely, we put
$$\boldsymbol{M}_{\alpha\beta}(z) = \delta_{\alpha\beta}\boldsymbol{V}_\alpha - \boldsymbol{V}_\alpha\boldsymbol{R}(z)\boldsymbol{V}_\beta,$$
so that
$$\boldsymbol{T}(z) = \sum_{\alpha,\beta} \boldsymbol{M}_{\alpha\beta}(z).$$
The following system of equations for the operators $\boldsymbol{M}_{\alpha\beta}(z)$ is obtained from (1.16):
$$\boldsymbol{M}_{\alpha\beta}(z) = \delta_{\alpha,\beta}\boldsymbol{V}_\alpha - \boldsymbol{V}_\alpha\boldsymbol{R}_0(z)\sum_{\gamma}\boldsymbol{M}_{\gamma\beta}(z). \tag{1.19}$$
The system (1.19) is no better than (1.11): its iterations still give rise to kernels containing δ-functions. To remedy this, Faddeev proposes an approach that should be learned by everybody working seriously in quantum mechanics! Namely, he suggests using the knowledge already obtained about the two-particle problem and summing up these unpleasant terms, which are completely of two-particle origin. More precisely, we transfer the terms containing $\boldsymbol{M}_{\alpha\beta}$ from the right-hand side of (1.19) to the left-hand side:
$$(I + \boldsymbol{V}_\alpha\boldsymbol{R}_0(z))\boldsymbol{M}_{\alpha\beta}(z) = \delta_{\alpha\beta}\boldsymbol{V}_\alpha - \boldsymbol{V}_\alpha\boldsymbol{R}_0(z)\sum_{\gamma\neq\alpha}\boldsymbol{M}_{\gamma\beta}(z) \tag{1.20}$$
and invert the operators $(I + \boldsymbol{V}_\alpha\boldsymbol{R}_0(z))$. To do this, we introduce operators $\boldsymbol{T}_\alpha(z)$ with kernels
$$t_\alpha\left(\boldsymbol{k}_\alpha, \boldsymbol{k}'_\alpha, z - \frac{\boldsymbol{p}_\alpha^2}{2n_\alpha}\right)\delta(\boldsymbol{p}_\alpha - \boldsymbol{p}'_\alpha),$$
where $t_\alpha(\boldsymbol{k}_\alpha, \boldsymbol{k}'_\alpha, z)$ is the kernel of the operator of the two-particle problem with potential $v_\alpha(\boldsymbol{k})$ and mass m_α. The operators $\boldsymbol{T}_\alpha(z)$ satisfy the equations
$$\boldsymbol{T}_\alpha(z) = \boldsymbol{V}_\alpha - \boldsymbol{V}_\alpha\boldsymbol{R}_0(z)\boldsymbol{T}_\alpha(z),$$

whence $\boldsymbol{T}_\alpha(z) = (I + \boldsymbol{V}_\alpha \boldsymbol{R}_0(z))^{-1}\boldsymbol{V}_\alpha$ and we obtain

$$\boldsymbol{M}_{\alpha\beta}(z) = \delta_{\alpha\beta}\boldsymbol{T}_\alpha(z) - \boldsymbol{T}_\alpha(z)\boldsymbol{R}_0(z)\sum_{\gamma\neq\alpha}\boldsymbol{M}_{\gamma\beta}(z). \tag{1.21}$$

These are the famous Faddeev equations for the quantum three-particle system!

The equations (1.21) are the main tool for studying the resolvent $\boldsymbol{R}(z)$, which is constructed from their solutions by the simple formulae (1.17) and (1.18). Since the kernels of the constant terms in (1.21) contain δ-functions, one should consider the operators

$$\boldsymbol{W}_{\alpha\beta}(z) = \boldsymbol{M}_{\alpha\beta}(z) - \delta_{\alpha\beta}\boldsymbol{T}_\alpha(z),$$

which satisfy the same equations with other constant terms:

$$\boldsymbol{W}_{\alpha\beta}(z) = \boldsymbol{W}^{(0)}_{\alpha\beta}(z) - \boldsymbol{T}_\alpha(z)\boldsymbol{R}_0(z)\sum_{\gamma\neq\alpha}\boldsymbol{W}_{\gamma\beta}(z), \tag{1.22}$$

where

$$\boldsymbol{W}^{(0)}_{\alpha\alpha}(z) = 0, \quad \boldsymbol{W}^{(0)}_{\alpha\beta}(z) = -\boldsymbol{T}_\alpha(z)\boldsymbol{R}_0(z)\boldsymbol{T}_\beta(z). \tag{1.23}$$

Further analysis of these equations is incredibly complicated. We can only admire Faddeev's technical power revealed in his sophisticated calculations and clever estimations of various singular integrals.

Thus, define a family of operators $\widetilde{\boldsymbol{A}}(z)$ acting on triples of functions $\chi(z) = \{\chi_{23}(\boldsymbol{k},\boldsymbol{p};z), \chi_{31}(\boldsymbol{k},\boldsymbol{p};z), \chi_{12}(\boldsymbol{k},\boldsymbol{p};z)\}$ by the formula

$$(\widetilde{\boldsymbol{A}}(z)\chi(z))_\alpha = -\boldsymbol{T}_\alpha(z)\boldsymbol{R}_0(z)\sum_{\gamma\neq\alpha}\chi_\gamma(z).$$

Separate the contribution of the discrete spectrum to the kernels $t_\alpha(\boldsymbol{k},\boldsymbol{k}',z)$:

$$t_\alpha(\boldsymbol{k},\boldsymbol{k}',z) = \frac{\varphi_\alpha(\boldsymbol{k})\overline{\varphi_\alpha(\boldsymbol{k}')}}{z+\varkappa_\alpha^2} + \widehat{t}_\alpha(\boldsymbol{k},\boldsymbol{k}',z),$$

where the $\varphi_\alpha(\boldsymbol{k}_\alpha)$ are the normalized eigenfunctions of the corresponding two-particle operators with eigenvalues $-\varkappa_\alpha^2$. The kernels $\widehat{t}_\alpha(\boldsymbol{k},\boldsymbol{k}',z)$ are analytic in the whole of the z-plane slit along the positive semiaxis and have continuous boundary values on the edges of the slit. This representation suggests that we should express the functions $\chi_\alpha(\boldsymbol{k},\boldsymbol{p};z)$ in the form

$$\chi_\alpha(\boldsymbol{k},\boldsymbol{p};z) = \rho_\alpha(\boldsymbol{k},\boldsymbol{p};z) + \frac{\varphi_\alpha(\boldsymbol{k}_\alpha)\sigma_\alpha(\boldsymbol{p}_\alpha)}{z+\varkappa_\alpha^2 - \boldsymbol{p}_\alpha^2/(2n_\alpha)} \tag{1.24}$$

and thus obtain an operator $\boldsymbol{A}(z)$ acting on vector-valued functions

$$\omega(\boldsymbol{k},\boldsymbol{p};z) = \bigl\{\rho_{23}(\boldsymbol{k},\boldsymbol{p};z), \rho_{31}(\boldsymbol{k},\boldsymbol{p};z), \rho_{12}(\boldsymbol{k},\boldsymbol{p};z), \sigma_{23}(\boldsymbol{p}_{23}), \sigma_{31}(\boldsymbol{p}_{31}), \sigma_{12}(\boldsymbol{p}_{12})\bigr\}.$$

However, $\boldsymbol{A}(z)$ cannot be applied directly to the analysis of (1.22) and (1.23) since it maps the smooth functions $\rho_\alpha(\boldsymbol{k},\boldsymbol{p};z)$ to the functions $\rho'_\alpha(\boldsymbol{k},\boldsymbol{p};z)$ that are determined by $\omega'(z) = \boldsymbol{A}(z)\omega(z)$ and can have singularities induced by the singularity of the free resolvent $\boldsymbol{R}_0(z)$. Faddeev's remarkable discovery was that these singularities become milder under iterations and $\boldsymbol{A}(z)^n$ with $n > 4$ does not generate them at all. Moreover, endowing the set of Hölder vector-valued functions ω with the structure of a Banach space similar to $\mathfrak{B}(\theta,\mu)$, he proved that the operators $\boldsymbol{A}(z)^n$ with $n > 4$ are compact. Using Nikolskii's well-known theorem on operators with compact powers on a Banach space, he proved that the inhomogeneous equation with operator $\boldsymbol{A}$ has a unique solution if and only if the homogeneous equation

$$\boldsymbol{A}(z)\omega = 0 \tag{1.25}$$

has only a zero solution, which is the case for non-real z. It is particularly difficult to prove that the operator $\boldsymbol{H}$ has no singular spectrum, that is, the set of points z for which the homogeneous equation (1.25) has a non-trivial solution is a countable closed subset of a finite interval of the real axis which has no limit points other than $-\varkappa_\alpha^2$, $\alpha = 23, 31, 12$. All such points z (except possibly the limit points) belong to the discrete spectrum[11] of $\boldsymbol{H}$.

In contrast to the two-particle case, the absolutely continuous spectrum of $\boldsymbol{H}$ in the three-particle problem does not coincide with the spectrum of the free Hamiltonian $\boldsymbol{H}_0$. The eigenfunction expansion theorem for the operator $\boldsymbol{H}$ was proved in [13] and says that the projection of $\boldsymbol{H}$ onto the subspace of $L^2(\mathbb{R}^6)$ corresponding to the absolutely continuous spectrum is unitarily equivalent to the operator

$$\widehat{\boldsymbol{H}} = \widetilde{\boldsymbol{H}}_0 \oplus \widetilde{\boldsymbol{H}}_{23} \oplus \widetilde{\boldsymbol{H}}_{31} \oplus \widetilde{\boldsymbol{H}}_{12}$$

in the Hilbert space

$$\begin{gathered}\widehat{\mathfrak{H}} = \mathfrak{H}_0 \oplus \mathfrak{H}_{23} \oplus \mathfrak{H}_{31} \oplus \mathfrak{H}_{12}, \\ \mathfrak{H}_0 = L^2(\mathbb{R}^6), \qquad \mathfrak{H}_\alpha = L^2(\mathbb{R}^3), \quad \alpha = 23, 31, 12,\end{gathered} \tag{1.26}$$

given explicitly by the formulae

$$\widetilde{\boldsymbol{H}}_0 = \frac{\boldsymbol{k}_\alpha^2}{2m_\alpha} + \frac{\boldsymbol{p}_\alpha^2}{2n_\alpha}\,, \qquad \widetilde{\boldsymbol{H}}_\alpha = \frac{\boldsymbol{p}_\alpha^2}{2m_\alpha} - \varkappa_\alpha^2,$$

where the first expression is independent of α.

Similarly to the two-particle case, but in a much more complicated and involved way, it is proved in [13] that there are wave operators $\boldsymbol{U}^\pm\colon \widehat{\mathfrak{H}} \to L^2(\mathbb{R}^6)$ such that

$$\boldsymbol{U}^*\boldsymbol{U} = I, \quad \boldsymbol{U}\boldsymbol{U}^* = I - \boldsymbol{P}_d\,, \quad \boldsymbol{H}\boldsymbol{U} = \boldsymbol{U}\widehat{\boldsymbol{H}}, \tag{1.27}$$

where $\boldsymbol{U} = \boldsymbol{U}^\pm$ and $\boldsymbol{P}_d$ is the projection onto the subspace of $L^2(\mathbb{R}^6)$ corresponding to the discrete spectrum of $\boldsymbol{H}$. In the stationary approach, the wave operators

$$\boldsymbol{U}^\pm = \boldsymbol{U}_0^\pm \oplus \boldsymbol{U}_{23}^\pm \oplus \boldsymbol{U}_{31}^\pm \oplus \boldsymbol{U}_{12}^\pm \tag{1.28}$$

are defined in terms of the solution of the scattering problem for $\boldsymbol{H}$, and in the non-stationary approach they are defined similarly to (1.4):

$$\boldsymbol{U}_0^\pm = \lim_{t\to\pm\infty} e^{it\boldsymbol{H}}\boldsymbol{J}_0 e^{-it\widetilde{\boldsymbol{H}}_0}, \qquad \boldsymbol{U}_\alpha^\pm = \lim_{t\to\pm\infty} e^{it\boldsymbol{H}}\boldsymbol{J}_\alpha e^{-it\widetilde{\boldsymbol{H}}_\alpha}.$$

[11] An inaccuracy in the proof of Lemma 7.11 in [13] was corrected by Yafaev [127].

Here $\boldsymbol{J}_0$ is the operator of identification of $\mathfrak{H}_0$ with $L^2(\mathbb{R}^6)$, and the operators $\boldsymbol{J}_\alpha$ embed $\mathfrak{H}_\alpha$ isometrically in $L^2(\mathbb{R}^6)$:

$$(\boldsymbol{J}_\alpha f)(\boldsymbol{k}_\alpha, \boldsymbol{p}_\alpha) = \varphi_\alpha(\boldsymbol{k}_\alpha) f(\boldsymbol{p}_\alpha).$$

The scattering matrix

$$\boldsymbol{S} = \boldsymbol{U}^{(+)*}\boldsymbol{U}^{(-)}$$

is a unitary operator on $\widehat{\mathfrak{H}}$, which has a block structure in accordance with the decompositions (1.26) and (1.28). The blocks of the S-matrix describe different physical processes: scattering of three particles into three particles, or into one particle and a bound state of the other two, and an inverse process of scattering of one particle and a bound state of the other two into an analogous state with or without transition. This reflects the more versatile nature of three-particle scattering compared to scattering by a central potential. The first is an example of the so-called *multi-channel scattering* while the second is a *one-channel scattering*.

Faddeev's equations are a powerful mathematical tool for studying the quantum three-body problem. They were used in numerical calculations for concrete processes of nuclear physics in the 1970s. These equations give incomparably better results than naïve attempts to solve the Schrödinger equation numerically. Nowadays an excellent accuracy in the calculation of scattering processes can be obtained by solving Faddeev's equations numerically using a PC. The iterations converge very quickly. We conclude by expressing our general impression from this work of Ludwig Dmitrievich Faddeev. His technical arsenal and virtuosity were second to none, his ingenuity was boundless, and his diligence must serve as an example for new generations of Russian scientists.

Faddeev's ideas and methods have been successfully developed by his students. The many-body problem was studied by O. A. Yakubovskii. The proof of the compactness of integral operators in this case becomes incredibly complicated even compared with [13]. The second direction of considerable interest for Faddeev himself is a generalization to slowly decaying potentials, with Coulomb interaction as the most interesting case. This problem was studied by S. P. Merkur'ev. The main difficulty arises because the trajectories of particles in a slowly decaying potential field are not asymptotically linear, whence the Fourier transform cannot be used and one must work in the position representation, which is very difficult. These and other results are described in the resulting monograph [50] by Faddeev and Merkur'ev.

2. Quantum inverse scattering problem

Another important problem of the spectral theory of differential operators is the so-called inverse problem. It has two aspects.

(a) The spectral aspect involves recovering a differential operator from its spectral function.

(b) The quantum-mechanical aspect involves finding the analytic properties of the scattering amplitude $f(\boldsymbol{k}, \boldsymbol{l})$ for the Schrödinger operator (1.6) and recovering the potential $v(\boldsymbol{x})$ from a given scattering amplitude. Faddeev made fundamental contributions to the solution of the inverse scattering problem and its applications to the theory of integrable non-linear evolution equations.

2.1. Radial Schrödinger equation. Faddeev's survey [7] (1959), devoted to the inverse problem for the radial Schrödinger equation, plays a special role. This equation arises from the separation of variables in (1.6) for a spherically symmetric potential

$$v(\boldsymbol{x}) = v(r), \qquad r = |\boldsymbol{x}|.$$

As a result, we obtain the equation[12]

$$-\frac{d^2\psi}{dr^2} + v(r)\psi(r) = k^2\psi(r), \qquad r > 0, \tag{2.1}$$

with the boundary condition $\psi(0) = 0$. An analogue of (1.2) is

$$\int_0^\infty r|v(r)|\,dr < \infty, \tag{2.2}$$

and in this case the solution $\varphi(r,k)$ of (2.1) with the initial data

$$\varphi(0,k) = 0 \quad \text{and} \quad \varphi'(0,k) = 1$$

has the following asymptotics for real k as $r \to \infty$:

$$\varphi(r,k) = \frac{1}{2ik}\big(\overline{M(k)}e^{ikr} - M(k)e^{-ikr}\big). \tag{2.3}$$

The function $M(k)$ satisfies $M(-k) = \overline{M(k)}$ and extends analytically to the upper half-plane of the variable k. We have

$$M(k) = 1 + o(1) \quad \text{as } k \to \infty,\ \operatorname{Im} k > 0 \tag{2.4}$$

and $M(k)$ has a finite number N of simple zeros $k = i\varkappa_l$, which correspond to the discrete eigenvalues $-k_l^2$ of the radial Schrödinger operator. The functions

$$A(k) = |M(k)| \quad \text{and} \quad \eta(k) = \arg M(k)$$

are referred to as the *asymptotic amplitude* and the *asymptotic phase*. The corresponding S-matrix is the operator of multiplication by the function

$$S(k) = \frac{M(-k)}{M(k)} = e^{-2i\eta(k)}$$

in $L^2(0,\infty)$, and we have $\eta(k)\big|_{-\infty}^{\infty} = 2\pi N$.

Before Faddeev's survey [7], which has become classical, there were several approaches to the inverse problem. They were proposed by Marchenko, Krein, and Gelfand and Levitan. Marchenko's approach enables one to deduce necessary and sufficient conditions for the S-matrix: the function $1 - S(k)$ must be the Fourier transform of an absolutely continuous function $F(x) \in L^1(0,\infty)$ such that

$$\int_0^\infty x|F'(x)|\,dx < \infty.$$

[12] In the case when $l = 0$. The eigenvalues of the Laplacian on the two-dimensional sphere S^2 in $\mathbb{R}^3$ are $l(l+1)$.

The corresponding potential $v(r)$ is constructed from the solution of an integral equation (Marchenko's equation) and satisfies (2.2). Krein's approach uses integral equations for Krein's canonical system. The approach of Gelfand and Levitan recovers a differential operator from its spectral function using another integral equation (the Gelfand–Levitan equation).

With an ingenious use of the method of transformation operators (operators U that satisfy (1.5) and are different from the wave operators $U_{\pm}$), which goes back to Friedrichs, Faddeev proved in [7] that these approaches are equivalent and established relations between the corresponding integral equations. In particular, the spectral function $\rho(k)$ in the Gelfand–Levitan approach is nothing other than $1/|M(k)|^2$ in the approaches of Marchenko and Krein.[13] The survey [7] and its more recent continuation [28], to be discussed below, became reference books for several generations of experts in scattering theory and mathematical physics, both in the USSR and elsewhere.

We cite from [7]: "It is interesting to note that in the USSR the inverse problem has been studied on the whole by mathematicians, whereas abroad it has been studied almost exclusively by physicists". Faddeev's students inherited his esteem for the pioneering papers by the Soviet mathematicians Z. S. Agranovich, Birman, Gelfand, M. G. Krein, Levitan, Marchenko, Povzner, and others. He kept offprints of Krein's notes in *Doklady Akademii Nauk SSSR* with special esteem and admired the depth of Krein's results and the conciseness of their statements. The origin of [7] is also interesting. In 1958, the academician N. N. Bogolyubov invited Faddeev to give a talk on the inverse problem for the radial Schrödinger equation at a conference organized by the Dubna Laboratory of Theoretical Physics. Gelfand, Krein, Levitan, and Marchenko were among his audience. The talk was based on a paper which Faddeev had prepared for his postgraduate examination, and it resulted in an invitation to the 25-year old author to write a survey in *Uspekhi Mat. Nauk*!

2.2. Trace identities. Another important problem in the spectral theory of differential operators involves finding the so-called *trace identities*, which express the (appropriately regularized) spectral trace of a differential operator in terms of its coefficients. These identities may be regarded as far-reaching generalizations of the equality between the matrix trace and the sum of eigenvalues in the finite-dimensional case.[14] The first important result of this kind was obtained in 1953 by Gelfand and Levitan in the simplest case of a regular Sturm–Liouville operator

$$L=-\frac{d^2}{dx^2}+v(x)$$

on the closed interval $[0,\pi]$ with zero boundary conditions. In $L^2(0,\pi)$ the operator L has a simple discrete spectrum of eigenvalues λ_n with a limit point at infinity. If $v(x)\in C^2(0,\pi)$, then

$$\lambda_n=n^2+c+O\left(\frac{1}{n^2}\right)\quad \text{as } n\to\infty,$$

[13] A similar relation between the spectral function and the S-matrix plays an important role in the representation theory of semisimple Lie groups.

[14] The equality of the matrix trace and the spectral trace for trace-class integral operators on $L^2(0,1)$ with continuous kernels is a well-known theorem of B. L. Lidskii.

where

$$c = \frac{1}{\pi}\int_0^\pi v(x)\,dx.$$

The equality

$$\sum_{n=1}^\infty(\lambda_n - c - n^2) = \frac{v(0)+v(\pi)}{4}$$

is the Gelfand–Levitan trace formula, which expresses the trace of the difference $L - L_0$ of two regular Sturm–Liouville operators, where $L_0 = -d^2/dx^2$. Another much more sophisticated example of a trace identity is Selberg's famous trace formula for the Laplace–Beltrami operator on a fundamental domain of a Fuchsian group of the first kind on the Lobachevskii plane.

Faddeev's papers [3] and [9] made a fundamental contribution to the trace identities for the Schrödinger equation. In [3] he considered a singular Sturm–Liouville operator: the radial Schrödinger operator

$$H = -\frac{d^2}{dr^2} + v(r)$$

with zero boundary conditions and a potential $v(r)$ satisfying (2.2). The operator H has a simple absolutely continuous spectrum filling $[0,\infty)$ and a finite number of negative eigenvalues. We define the trace of the difference $H_1 - H_2$ of two such operators:

$$\mathrm{Tr}(H_1 - H_2) = \lim_{R\to\infty}\int_{-\infty}^R \lambda\,d\,\mathrm{Tr}(E^1_\lambda - E^2_\lambda),$$

where E_λ is the resolution of the identity for H in the von Neumann spectral theorem (the distribution function of the corresponding projection-valued measure on $\mathbb{R}$). Using scattering theory, Faddeev proved that if

$$\int_0^\infty (v_1(x) - v_2(x))\,dx = 0,$$

then the limit of the integral as $R\to\infty$ exists and is equal to $-(v_1(0)-v_2(0))/4$. This is the trace identity in the singular case.

This topic was further developed in the joint paper [9] with V. S. Buslaev, Faddeev's first student,[15] who considered the difference $R_\lambda - R^0_\lambda$ between the resolvents of H and the free operator H_0. Here

$$R_\lambda = (H - \lambda I)^{-1}$$

for λ in the complement to the spectrum of H in $\mathbb{C}$, I is the identity operator on $L^2(\mathbb{R})$, and similarly for H_0. The remarkable result is that for such λ the operator $R_\lambda - R^0_\lambda$ is trace-class and

$$\mathrm{Tr}(R_\lambda - R^0_\lambda) = -\frac{d}{d\lambda}\log M(\sqrt{\lambda}), \qquad 0 < \arg\sqrt{\lambda} < \pi, \tag{2.5}$$

[15] Buslaev had Ladyzhenskaya and Faddeev as scientific advisors.

where $M(k)$ is the function introduced in (2.3). Since VR_λ^0 is trace-class, it follows from (2.5) that

$$M(\sqrt{\lambda}\,) = \det(I + VR_\lambda^0),$$

where det is the Fredholm determinant. This gives an expression for the regularized determinant of $H - \lambda I$. Moreover, when the potential $v(r)$ is smooth and decays rapidly with all derivatives as $r \to \infty$, both sides of (2.5) admit an asymptotic expansion in inverse powers of $k = \sqrt{\lambda}$ as $k \to \infty$ in the upper half-plane, and this expansion extends by smoothness to the real axis. The expansion for $\log M(k)$ is obtained from a dispersion relation of type (1.7), which follows from the analytic properties of $M(k)$ described above. The expansion of the left-hand side of (2.5) is obtained by reducing the Schrödinger equation to a Riccati equation. Its coefficients are given by the integrals of certain recursively defined polynomials in the function $v(r)$ and its derivatives over the positive semiaxis as well as the values of these derivatives at $r = 0$. These identities will play a fundamental role in the proof of the complete integrability of the Korteweg–de Vries equation as an infinite-dimensional Hamiltonian system!

2.3. One-dimensional Schrödinger equation. The one-dimensional Schrödinger equation

$$-\psi''(x) + v(x)\psi(x) = k^2\psi(x), \qquad -\infty < x < \infty, \tag{2.6}$$

is intermediate between the three-dimensional (1.6) and radial (2.1) Schrödinger equations. An analogue of (2.2) is

$$\int_{-\infty}^{\infty} (1 + |x|)|v(x)|\,dx < \infty. \tag{2.7}$$

Under this condition, H has a two-fold absolutely continuous spectrum filling $[0, \infty)$ and a finite number of negative eigenvalues. A complete study of the direct and inverse problems for the equation (2.6) with condition (2.7) was undertaken by Faddeev in [5] and constituted his Ph.D. thesis "Properties of the S-matrix for scattering by a local potential" (1959), which was published as [16]. In this case, the S-matrix is the 2×2 matrix $S(k)$ which is defined for $k \neq 0$ in terms of solutions of the scattering problem.

Namely, let $f_1(x, k)$ and $f_2(x, k)$ be the Jost solutions. For real k they are uniquely determined by the asymptotic conditions

$$\begin{aligned} f_1(x, k) &= e^{ikx} + o(1) \qquad \text{as } x \to \infty, \\ f_2(x, k) &= e^{-ikx} + o(1) \quad \text{as } x \to -\infty \end{aligned}$$

and extend analytically to the upper half-plane of the variable k for every fixed x. The transition coefficients $a(k)$ and $b(k)$ are defined for $k \neq 0$ by the formula

$$f_2(x, k) = a(k) f_1(x, -k) + b(k) f_1(x, k)$$

and satisfy the symmetry conditions $\overline{a(k)} = a(-k)$, $\overline{b(k)} = b(-k)$ and the normalization condition

$$|a(k)|^2 = 1 + |b(k)|^2.$$

The solutions $u_{1,2}(x,k) = f_{1,2}(x,k)/a(k)$ satisfy the one-dimensional analogue of the radiation conditions, and the S-matrix takes the form

$$S(k) = \begin{pmatrix} s_{11}(k) & s_{12}(k) \\ s_{21}(k) & s_{22}(k) \end{pmatrix},$$

where

$$s_{11}(k) = s_{22}(k) = \frac{1}{a(k)}, \quad s_{12}(k) = \frac{b(k)}{a(k)}, \quad s_{21} = \frac{\overline{b(k)}}{a(k)}.$$

The S-matrix is unitary:

$$S^*(k)S(k) = I.$$

It satisfies the realness condition $\overline{S(k)} = S(-k)$ and

$$S(k) = I + O(|k|^{-1}) \quad \text{as } |k| \to \infty.$$

In quantum mechanics, the function $t(k) = s_{11}(k)$ is called the transmission coefficient, and $r(k) = s_{12}(k)$ is called the reflection coefficient.

By using Marchenko's integral equations it was proved in [6] and [16] that under the condition (2.7) the Fourier transforms $F_1(x)$ and $F_2(x)$ of the functions $s_{12}(k)$ and $s_{21}(k)$ are absolutely continuous and, for every a,

$$\int_a^\infty (1+|x|)|F_1'(x)|\,dx < \infty \quad \text{and} \quad \int_{-\infty}^a (1+|x|)|F_2'(x)|\,dx < \infty.$$

The entries of the S-matrix satisfy the conditions[16]

$$\lim_{k\to 0} \frac{k(s_{12}(k)+1)}{s_{11}(k)} = \lim_{k\to 0} \frac{k(s_{21}(k)+1)}{s_{11}(k)} = 0$$

and

$$|s_{12}(k)| = |s_{21}(k)| \leqslant 1 - \frac{Ck^2}{1+k^2} \quad \text{as } k \to 0.$$

Similarly to the function $M(k)$ in the radial case, $a(k)$ extends analytically to the upper half-plane of the variable k with asymptotic behaviour (2.4) and can have only a finite number N of simple zeros $i\varkappa_l$ there. The following relation is an analogue of (2.5):

$$\operatorname{Tr}(R_\lambda - R_\lambda^0) = -\frac{d}{d\lambda} \log a(\sqrt{\lambda}\,), \qquad 0 < \arg\sqrt{\lambda} < \pi. \tag{2.8}$$

Here

$$f_2(x, i\varkappa_l) = c_l f_1(x, i\varkappa_l), \qquad c_l \neq 0,$$

whence the $-\varkappa_l^2$ are the negative eigenvalues of H. The normalization constants m_l are given by

$$m_l^{-1} = \int_{-\infty}^\infty |f_1(x, i\varkappa_l)|^2\,dx = \frac{i\dot{a}(i\varkappa_l)}{c_l}.$$

[16]These conditions, which improve the relations (2.8) in [16], were stated in Marchenko's monograph [116] (see Chap. 3, § 5). We also note that the American mathematicians Deift and Trubowitz [93] suggested a stronger condition $\int_{-\infty}^\infty (1+x^2)|v(x)|\,dx < \infty$ instead of (2.7) for solving the inverse problem. Marchenko's analysis shows that this condition is not necessary.

Using the condition of analyticity of $1/s_{11}(k)$, the relation

$$|s_{11}(k)|^2 = 1 - |s_{12}(k)|^2,$$

and the above formulae for the entries of $S(k)$, we get that the S-matrix is completely determined by specifying one of the entries $s_{12}(k)$ or $s_{21}(k)$ and the poles $i\varkappa_l$ of $s_{11}(k)$. Thus, this enables us to recover the function $\log s_{11}(k)$ in terms of $\log|s_{11}(k)|^2 = \log(1-|s_{12}(k)|^2)$ using the Poisson–Schwarz formula, an analogue of the dispersion relation (1.7):

$$s_{11}(k) = \exp\biggl\{-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log(1-|s_{12}(p)|^2)}{k-p}\,dp\biggr\}\prod_{l=1}^{N}\frac{k+i\varkappa_l}{k-i\varkappa_l}\,, \qquad \operatorname{Im} k > 0, \quad (2.9)$$

and

$$s_{11}(k) = \lim_{\varepsilon\to 0} s_{11}(k+i\varepsilon) \quad \text{for } k \in \mathbb{R}.$$

Since $s_{12}(-k) = \overline{s_{12}(k)}$, the whole matrix $S(k)$ is determined by a single complex-valued function $s_{12}(k)$ on the positive semiaxis $k>0$ and a finite set of $i\varkappa_l$. This corresponds to the single real-valued function $v(x)$ on the whole real axis $-\infty < x < \infty$, thus giving a correct calculation of functional parameters.

The main result of [6] and [16] (Faddeev's theorem) says that all these conditions on the matrix $S(k)$ and the constants $m_l > 0$ are also sufficient for $S(k)$ to be the S-matrix of the one-dimensional Schrödinger equation with a potential $v(x)$ satisfying (2.7), eigenvalues $-\varkappa_l^2$, and normalization constants m_l. In other words, the potential $v(x)$ is uniquely recovered from the set $s = (r(k), \varkappa_l, m_l)$ of the so-called *scattering data*. Thus, by analysing Marchenko's equation on the right-hand endpoint, one proves the existence of a potential $v_1(x)$ satisfying the same bound as for $F_1'(x)$, and Marchenko's equation on the left-hand endpoint gives rise to a potential $v_2(x)$ satisfying the same bound as for $F_2'(x)$. Finally, using the formula

$$s_{21}(k) = \frac{s_{12}(-k)s_{11}(k)}{s_{11}(-k)}\,,$$

the unitarity of the S-matrix, and Marchenko's equations, one proves that

$$v_1(x) = v_2(x) = v(x).$$

2.4. Three-dimensional Schrödinger equation. The inverse problem for the three-dimensional Schrödinger operator is much more complicated than the analogous problem for the radial and one-dimensional Schrödinger equations. The main difference from the one-dimensional case is that, at first sight, the S-matrix, which is uniquely determined by the potential $v(\boldsymbol{x})$, depends on a larger number of functional parameters than the potential itself. Namely, the scattering amplitude $f(\boldsymbol{k},\boldsymbol{l})$ is a complex-valued function of the energy $E = \boldsymbol{k}^2 = \boldsymbol{l}^2$, $0 \leqslant E < \infty$, and two vectors in the unit sphere S^2, while $v(\boldsymbol{x})$ is a real-valued function on $\mathbb{R}^3$ and thus depends on the radius r, $0 \leqslant r < \infty$, and a vector in S^2. The symmetry property $f(\boldsymbol{k},\boldsymbol{l}) = f(-\boldsymbol{l},-\boldsymbol{k})$ and the unitarity of the S-matrix reduce the scattering amplitude to a real-valued symmetric function of energy and two vectors in S^2, which is insufficient. Thus, the problem is to find all properties of the S-matrix uniquely determined by the potential $v(\boldsymbol{x})$.

This important and very hard problem was solved in Faddeev's papers [17], [18], and [25], with a detailed exposition in the survey [28]. Faddeev himself regarded these as his technically most advanced papers and was proud of them. The main idea is to define and study a family of transformation operators $\{U_{\boldsymbol{\gamma}}\}_{\boldsymbol{\gamma}\in S^2}$ in $L^2(\mathbb{R}^3)$, $HU_{\boldsymbol{\gamma}} = U_{\boldsymbol{\gamma}}H_0$, which are Volterra operators in the direction $\boldsymbol{\gamma}$:

$$(U_{\boldsymbol{\gamma}}\psi)(\boldsymbol{x}) = \psi(\boldsymbol{x}) + \int_{(\boldsymbol{x}-\boldsymbol{y},\boldsymbol{\gamma})>0} A_{\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{y})\psi(\boldsymbol{y})\,d^3\boldsymbol{y}.$$

To prove the existence of the operators $U_{\boldsymbol{\gamma}}$ it suffices to show that for every $\boldsymbol{\gamma}\in S^2$ the Schrödinger equation (1.6) has a solution $f_{\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{k})$ which extends analytically to the upper half-plane of the variable $s = (\boldsymbol{k},\boldsymbol{\gamma})$ for all fixed values of $\boldsymbol{x}$ and $\boldsymbol{k}_\perp = \boldsymbol{k} - (\boldsymbol{k},\boldsymbol{\gamma})\boldsymbol{\gamma}$, admits there the bound

$$|f_{\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{k})e^{-is(\boldsymbol{x},\boldsymbol{\gamma})}| \leqslant C,$$

and has the following asymptotic behaviour for large s:

$$f_{\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{k})e^{-is(\boldsymbol{x},\boldsymbol{\gamma})} = e^{i(\boldsymbol{k}_\perp,\boldsymbol{x})} + o(1).$$

These solutions are multidimensional analogues of Jost solutions.

In the one-dimensional case the existence of Jost solutions was proved using the Volterra-type Green functions $G_1(x-y,k)$ and $G_2(x-y,k)$ of the operator $H_0-\lambda I$:

$$G_1(x,k) = -\theta(-x)\,\frac{\sin kx}{k} \quad \text{and} \quad G_2(x,k) = \theta(x)\,\frac{\sin kx}{k}\,,$$

where $k=\sqrt{\lambda}$ and $\theta(x)$ is the Heaviside function:

$$\theta(x) = 1 \quad \text{for } x>0, \qquad \theta(x) = 0 \quad \text{for } x<0.$$

We recall that the ordinary Green function $G(x-y,k)$ (the kernel of the resolvent R^0_λ of the operator H_0 in $L^2(\mathbb{R})$) is

$$G(x,k) = -\frac{e^{ik|x|}}{2ik}\,.$$

In the three-dimensional case, the ordinary Green function $G(\boldsymbol{x}-\boldsymbol{y},\boldsymbol{k})$ (the kernel of the resolvent R^0_λ of the operator H_0 in $L^2(\mathbb{R}^3)$) is given by the classical formula

$$G(\boldsymbol{x},\boldsymbol{k}) = -\frac{1}{4\pi}\,\frac{e^{i|\boldsymbol{k}|\,|\boldsymbol{x}|}}{|\boldsymbol{x}|}\,, \qquad \lambda = \boldsymbol{k}^2.$$

Remarkably, the existence of the functions $f_{\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{k})$ is proved using the multidimensional generalization of Volterra-type Green functions which was discovered by Faddeev in [17] (now referred to as Faddeev–Green functions). These functions form a family $G_{\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{k})$, where $\boldsymbol{\gamma}\in S^2$, and are given by the formula

$$G_{\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{k}) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{e^{i(\boldsymbol{l},\boldsymbol{x})}}{\boldsymbol{k}^2-\boldsymbol{l}^2+i0(\boldsymbol{k}-\boldsymbol{l},\boldsymbol{\gamma})}\,d^3\boldsymbol{l},$$

where the distribution $(x+i0a)^{-1}$ is understood as

$$(x+i0)^{-1} \quad \text{for } a>0 \quad \text{and} \quad (x-i0)^{-1} \quad \text{for } a<0.$$

The Faddeev–Green function $G_{\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{k})$ has an important analyticity property: for all fixed $\boldsymbol{\gamma}$, $\boldsymbol{x}$, and $\boldsymbol{k}_\perp$ it extends analytically to the upper half-plane of the variable $s=(\boldsymbol{k},\boldsymbol{\gamma})$, where it satisfies the estimate

$$|G_{\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{k})e^{-is(\boldsymbol{x},\boldsymbol{\gamma})}| \leqslant \frac{C}{|\boldsymbol{x}|}\,.$$

The integral equation

$$u_{\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{k}) = e^{i(\boldsymbol{k},\boldsymbol{x})} + \int_{\mathbb{R}^3} G_{\boldsymbol{\gamma}}(\boldsymbol{x}-\boldsymbol{y},\boldsymbol{k})v(\boldsymbol{y})u_{\boldsymbol{\gamma}}(\boldsymbol{y},\boldsymbol{k})\,d^3\boldsymbol{y}, \qquad \boldsymbol{\gamma}\in S^2, \tag{2.10}$$

is a multidimensional analogue of the integral equation for the Jost functions. However, in contrast to the one-dimensional case, (2.10) is not a Volterra-type equation, but just a Fredholm equation. Thus, the solution $u_{\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{k})$ is bounded only when the homogeneous equation with a given s has no non-trivial bounded solutions; otherwise the solution $u_{\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{k})$ has poles at points s with $\operatorname{Im} s>0$. It was shown in [28] that such singular values of s always exist if the Schrödinger operator H has discrete eigenvalues. Further use of Faddeev's method requires an analogue of Kato's theorem asserting that the homogeneous equation (2.10) has no bounded solutions for real s. This is the case for small potentials, but in general one must *assume* that the potential $v(\boldsymbol{x})$ satisfies this condition. In [28] it is called *Condition* C. If it holds, then

$$u_{\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{k}) = \overline{u_{\boldsymbol{\gamma}}(\boldsymbol{x},-\boldsymbol{k})}$$

and the analogues $f_{\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{k})$ of the Jost solutions are given by

$$f_{\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{k}) = u_{\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{k})\Delta_{\boldsymbol{\gamma}}(\boldsymbol{k}),$$

where $\Delta_{\boldsymbol{\gamma}}(\boldsymbol{k})$ is the regularized Fredholm determinant of the equation (2.10). Using integral equations and differentiation with respect to $\boldsymbol{\gamma}$ (the Lie derivatives of the action of $\mathrm{SO}(3)$ on S^2) with ingenuity, Faddeev proved in [28] that the regularized determinants $\Delta_{\boldsymbol{\gamma}}(\boldsymbol{k})$ can be expressed in terms of the scattering amplitude $f(\boldsymbol{k},\boldsymbol{l})$.

The analytic properties of the S-matrix can be completely described in terms of the solutions $u_{\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{k})$. Thus, we put[17]

$$h_{\boldsymbol{\gamma}}(\boldsymbol{k},\boldsymbol{l}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} e^{-i(\boldsymbol{l},\boldsymbol{x})}v(\boldsymbol{x})u_{\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{k})\,d^3\boldsymbol{x}$$

and define the operators $Q^{(\pm)}_{\boldsymbol{\gamma}}$ by

$$(Q^{(\pm)}_{\boldsymbol{\gamma}}\psi)(\boldsymbol{k}) = \psi(\boldsymbol{k}) \pm \frac{1}{2\pi i}\int_{\mathbb{R}^3} \delta(\boldsymbol{k}^2-\boldsymbol{l}^2)h_{\boldsymbol{\gamma}}(\boldsymbol{k},\boldsymbol{l})\theta(\pm(\boldsymbol{k}-\boldsymbol{l},\boldsymbol{\gamma}))\psi(\boldsymbol{l})\,d^3\boldsymbol{l}.$$

[17] Note that the amplitude $f(\boldsymbol{k},\boldsymbol{l})$ in [28] is our $f(\boldsymbol{k},\boldsymbol{l})$ divided by $-2\pi^2$. Therefore, our function $h_{\boldsymbol{\gamma}}(\boldsymbol{k},\boldsymbol{l})$ is $h_{\boldsymbol{\gamma}}(\boldsymbol{k},\boldsymbol{l})$ in [28] multiplied by $-2\pi^2$.

Remarkably, for any $\boldsymbol{\gamma} \in S^2$ the operators $Q_{\boldsymbol{\gamma}}^{(\pm)}$ determine a factorization of the scattering operator:

$$\widehat{S} = (Q_{\boldsymbol{\gamma}}^{(+)})^{-1} Q_{\boldsymbol{\gamma}}^{(-)},$$

where the function $h_{\boldsymbol{\gamma}}(\boldsymbol{k}, \boldsymbol{l})$ is uniquely determined by the scattering amplitude via the integral equation

$$h_{\boldsymbol{\gamma}}(\boldsymbol{k}, \boldsymbol{l}) = f(\boldsymbol{k}, \boldsymbol{l}) + \frac{1}{2\pi i} \int_{\mathbb{R}^3} \delta(\boldsymbol{k}^2 - \boldsymbol{m}^2) h_{\boldsymbol{\gamma}}(\boldsymbol{k}, \boldsymbol{m}) \theta((\boldsymbol{m} - \boldsymbol{k}, \boldsymbol{\gamma})) f(\boldsymbol{m}, \boldsymbol{l})\, d^3\boldsymbol{m}. \quad (2.11)$$

Condition C is equivalent to the unique solubility of (2.11).

A key point is the following analyticity property of the functions $h_{\boldsymbol{\gamma}}(\boldsymbol{k}, \boldsymbol{l})$, which Faddeev discovered. For $(\boldsymbol{k}, \boldsymbol{\gamma}) = (\boldsymbol{l}, \boldsymbol{\gamma}) = s$ and any fixed $\boldsymbol{k}_\perp$, $\boldsymbol{l}_\perp$ and $\boldsymbol{\gamma}$, the function $h_{\boldsymbol{\gamma}}(\boldsymbol{k}, \boldsymbol{l})$ extends analytically to the upper half-plane of the variable s with finite-order poles at the singular values of s. We stress that the proof of analyticity uses the locality of the potential $v(\boldsymbol{x})$ (that is, the fact that V is the operator of multiplication by $v(\boldsymbol{x})$ in $L^2(\mathbb{R}^3)$) in a crucial way. This property of $h_{\boldsymbol{\gamma}}(\boldsymbol{k}, \boldsymbol{l})$ discovered by Faddeev is a far-reaching generalization of the analyticity of the forward scattering amplitude.

To sum up, we get that the scattering amplitude $f(\boldsymbol{k}, \boldsymbol{l})$ of a smooth potential $v(\boldsymbol{x})$ decaying at infinity and satisfying Condition C, possesses the following properties.

I. Equation (2.11) is uniquely soluble for all $\boldsymbol{\gamma} \in S^2$. This gives a family of solutions $h_{\boldsymbol{\gamma}}(\boldsymbol{k}, \boldsymbol{l})$.

II. The function $\Delta_{\boldsymbol{\gamma}}(\boldsymbol{k})$ constructed from $h_{\boldsymbol{\gamma}}(\boldsymbol{k}, \boldsymbol{l})$ has a bounded analytic continuation to the upper half-plane of the variable $s = (\boldsymbol{k}, \boldsymbol{\gamma})$.

III. For $(\boldsymbol{k}, \boldsymbol{\gamma}) = (\boldsymbol{l}, \boldsymbol{\gamma})$ and fixed $\boldsymbol{k}_\perp$ and $\boldsymbol{l}_\perp$ the functions $h_{\boldsymbol{\gamma}}(\boldsymbol{k}, \boldsymbol{l})\Delta_{\boldsymbol{\gamma}}(\boldsymbol{k})$ also extend analytically to the upper half-plane of the variable $s = (\boldsymbol{k}, \boldsymbol{\gamma})$.

A fundamental result due to Faddeev [28] says that the properties I–III of the scattering amplitude $f(\boldsymbol{k}, \boldsymbol{l})$ are also sufficient! Thus, if these conditions hold, then there is a local potential $v(\boldsymbol{x})$ (whose uniqueness was proved in [1]) with scattering amplitude $f(\boldsymbol{k}, \boldsymbol{l})$. Thus, the potential is constructed in terms of solutions of Gelfand–Levitan type equations for some functions $A_{\boldsymbol{\gamma}}(\boldsymbol{x}, \boldsymbol{y})$, which in turn determine the solutions $f_{\boldsymbol{\gamma}}(\boldsymbol{x}, \boldsymbol{k})$. The Gelfand–Levitan equations form a family (parametrized by $\boldsymbol{\gamma} \in S^2$) of integral equations with positive kernels expressed in terms of the scattering amplitude. For every $\boldsymbol{\gamma} \in S^2$ the solution $A_{\boldsymbol{\gamma}}(\boldsymbol{x}, \boldsymbol{y})$ of the Gelfand–Levitan equation gives rise to an integral operator $V_{\boldsymbol{\gamma}}$ with kernel $V_{\boldsymbol{\gamma}}(\boldsymbol{x}, \boldsymbol{y})$ which is local in the direction $\boldsymbol{\gamma}$. Using the analyticity properties of the scattering amplitude (described above) one can establish relations between the $V_{\boldsymbol{\gamma}}$ with distinct $\boldsymbol{\gamma}$ and prove that

$$V_\gamma = V \quad \text{for all } \boldsymbol{\gamma} \in S^2$$

and $V(\boldsymbol{x}, \boldsymbol{y}) = \delta(\boldsymbol{x} - \boldsymbol{y}) v(\boldsymbol{x})$!

We leave it to the reader to work out all these details following [28]. Of course, [28] gives only a general way to solve the inverse problem for the three-dimensional Schrödinger operator. The task of proving the main assumption (Condition C) and establishing correspondences between function classes of potentials and scattering amplitudes similarly to the one-dimensional case (considered above) still awaits a solution. Perhaps some readers will be able to bring this important and difficult problem to completion.

The scattering matrix is the most important object in quantum theory. Solution of the inverse scattering problem in various cases tells us whether it contains full information about the system. This question has special importance in quantum field theory. Integrable models of quantum field theory, to be described in § 5, are the only cases in which an affirmative answer is known. Namely, it was shown by F. A. Smirnov, a student of Faddeev, that the knowledge of the factorized scattering matrix determines the local observables completely by means of a certain system of equations. Here, as in the cases considered above, a decisive role is played by analytic properties of the scattering matrix and the matrix entries of local observables.

3. Spectral theory of automorphic functions

In the 1950s the classical theory of automorphic forms and functions was on the rise in connection with Selberg's famous paper [119], translated into Russian in 1957. At the same time Gelfand, Graev, Pyatetskii-Shapiro, and Fomin established a connection between Selberg's approach and the theory of infinite-dimensional representations of semisimple Lie groups. Namely, let G be a real semisimple Lie group, K a maximal compact subgroup of G, and Γ a discrete subgroup of G such that the volume[18] of $\Gamma\backslash G$ is finite. The space $\Gamma\backslash G/K$ is acted on by a representation of the commutative algebra $\mathscr{D}$ of invariant differential operators (the Laplace operators), and the main task is to derive an expansion in eigenfunctions of operators in $\mathscr{D}$ on $L^2(\Gamma\backslash G/K)$. In the case when $\Gamma\backslash G$ is compact this problem was solved in the monograph *Representation theory and automorphic functions* [97] (issue 6 of the series *Generalized functions*) by Gelfand, Graev, and Pyatetskii-Shapiro using the methods of representation theory. Namely, the representation of G induced by a finite-dimensional unitary representation of Γ decomposes into a countable direct sum of irreducible unitary representations with finite multiplicities, which proves the eigenfunction expansion theorem in this case.

3.1. Expansion in eigenfunctions of the Laplacian. Especially interesting is the case of rank 1 when

$$G=\mathrm{SL}(2,\mathbb{R}), \qquad K=\mathrm{SO}(2),$$

the homogeneous space G/K is the Lobachevskii plane realized as the upper half-plane

$$\mathbb{H}=\{z=x+iy\colon y>0\},$$

and Γ is a Fuchsian group of the first kind which acts on $\mathbb{H}$ by fractional-linear transformations. Then the main task is reduced to studying the Laplace–Beltrami operator A of the Poincaré metric,

$$Af=-y^2\biggl(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\biggr), \tag{3.1}$$

[18] With respect to the Haar measure on G.

and to prove the theorem on expansion in eigenfunctions of the operator A in the Hilbert space $\mathscr{H} = L^2(\Gamma\backslash\mathbb{H})$. We already mentioned that when the closure $\overline{F}$ of a fundamental domain[19] is compact, this problem can be solved completely using the methods of representation theory. But when $\overline{F}$ is non-compact and has finite area in the Lobachevskii geometry, the problem becomes much more difficult. The result obtained in [97] is an expansion of the Hilbert space $\mathscr{H}$ into an orthogonal sum of two A-invariant subspaces. The first subspace consists of functions having zero integrals over all horocycles in $\Gamma\backslash\mathbb{H}$, and A has a discrete (possibly finite) spectrum on this subspace. On the second subspace, it was written in [97] (see Chap. 1, § 6) that: "It can be shown that the discrete spectrum of the second subspace has only a finite number of points, and that the spectrum of its remaining part is continuous of finite multiplicity. This multiplicity of the continuous spectrum is equal to the minimum number[20] of cusps of a fundamental domain of Γ. The proof is based on perturbation theory for differential operators. To avoid overloading the book with special problems in the theory of differential operators, we give an account of this proof elsewhere".

This is the problem solved in Faddeev's paper [19], written in 1966! Thus, the following assertions were proved in [19].

(a) The spectrum of the Laplace operator consists of an n-fold absolutely continuous spectrum filling the semiaxis $1/4 \leqslant \lambda < \infty$ and a discrete spectrum of finite multiplicity lying on the semiaxis $0 \leqslant \lambda < \infty$ and having no limit points on any finite interval.

(b) The so-called Eisenstein–Maass series (over the cosets of Γ), which converge absolutely for $\operatorname{Re} s > 1$, admit a meromorphic continuation to the whole complex s-plane with poles for $\operatorname{Re} s < 1/2$.

(c) The system of eigenfunctions of the continuous spectrum for A is given by the analytic continuations of the Eisenstein–Maass series to the line $\operatorname{Re} s = 1/2$.

(d) The theorem on expansion in eigenfunctions of A holds in $\mathscr{H}$.

Although these results were well known to experts,[21] and Selberg himself used them explicitly to derive the trace formula, in the general case their complete proofs had not been published. Thus, [19], written 10 years after Selberg's famous paper, was the first to give rigorous proofs of all these results. Although Faddeev understood the importance of these results and the methods used to obtain them, he modestly wrote in the introduction to [19] that "this paper may have only a methodological value". The fundamental role of [19] was mentioned by Lang in his monograph [110], whose second part is devoted to an exposition of this paper of Faddeev. Thus, Lang writes in the Introduction: "The Faddeev paper on the spectral decomposition of the Laplace operator on the upper half-plane is an exceedingly good introduction to analysis, placing the latter in a nice geometric framework. Any good senior undergraduate or first year graduate student should be able to read most of it, and I have reproduced it (with the addition of many details left out for more expert readers by Faddeev) as Chap. XIV. Faddeev's method comes from

[19] Here F is a fundamental domain of Γ in $\mathbb{H}$, that is, an open subset of $\mathbb{H}$ such that $\gamma_1 F \cap \gamma_2 \overline{F} = \varnothing$ for $\gamma_1 \neq \gamma_2$ and the union $\bigcup \gamma \overline{F}$ over all $\gamma \in \Gamma$ is equal to $\mathbb{H}$.

[20] More precisely, to the number n of inequivalent cusps of Γ. – *Note by L. A. T.*

[21] An approach using potential theory was given in Selberg's lectures at Göttingen University in 1954, still unpublished at the time.

perturbation theory and scattering theory, and as such is interesting for its own sake, as well as to analysts who may know the analytic part and may want to see how it applies in the group-theoretic context".

In concrete terms, Faddeev's method involves using perturbation theory for continuous spectra which he developed in [15] for the Friedrichs model (see § 1.2). He first considers the self-adjoint operator A_0 in $L^2(\mathbb{H})$ (the Laplacian on the Lobachevskii plane defined by the differential expression (3.1)). As in the case of the Schrödinger equation it is convenient to use the parametrization

$$\lambda = s(1-s),$$

where $\lambda \in \mathbb{C} \setminus [1/4, \infty)$ and s satisfies the condition $\operatorname{Re} s > 1/2$. For $\operatorname{Re} s > 1$, the resolvent

$$R_0(s) = (A_0 - s(1-s)I)^{-1}$$

of A_0 is an integral operator whose kernel $k(z, z'; s)$ depends on the invariant distance $\rho(z, z')$ on $\mathbb{H}$ and is given by a simple definite integral, easily expressed in terms of a hypergeometric function. It can easily be proved that the differential expression (3.1) determines a unique self-adjoint operator A acting in $\mathscr{H} = L^2(\Gamma\backslash\mathbb{H})$, and its resolvent

$$R(s) = (A - s(1-s)I)^{-1}$$

for $\operatorname{Re} s > 2$ is an integral operator whose kernel $r(z, z'; s)$ is obtained by the method of images:

$$r(z, z'; s) = \sum_{\gamma \in \Gamma} k(z, \gamma z'; s), \tag{3.2}$$

where the series converges absolutely for $\operatorname{Re} s > 1$. The resolvent $R(s)$ satisfies Hilbert's first identity

$$R(s) - R(s') = (s(1-s) - s'(1-s'))R(s')R(s).$$

Putting $s' = \varkappa$, where $\varkappa > 0$ is sufficiently large, and writing

$$R = R(\varkappa), \quad \omega(s) = s(1-s) - \varkappa(1-\varkappa),$$

we obtain the equation

$$R(s) = R + \omega(s) R R(s). \tag{3.3}$$

Like (1.8), the equation (3.3) is unusable for the study of $R(s)$ since the operator R is non-compact and has continuous spectrum. However, one can distinguish and explicitly invert the principal part that generates this spectrum. Consider for simplicity the case of one cusp[22] at $i\infty$ and choose the fundamental domain F of Γ in the form

$$F = F_0 \cup F_1,$$

where $\overline{F}_0$ is compact and $\overline{F}_1$ is the strip

$$\{z = x + iy \colon 0 \leqslant x \leqslant 1,\ y \geqslant a\} \quad \text{for some } a > 0.$$

[22]The case of several cusps is studied in a similar way.

Write P_0 and $P_1 = I - P_0$ for the orthogonal projection operators on $\mathscr{H}$ corresponding to multiplication by the characteristic functions of F_0 and F_1. Using (3.2), we easily prove that for $\varkappa > 2$ the operators

$$R_{00} = P_0 R P_0, \quad R_{01} = P_0 R P_1, \quad \text{and} \quad R_{10} = P_1 R P_0$$

are compact. To study the cusp part of R, that is, the operator $R_{11} = P_1 R P_1$, we define an orthogonal projection P of the subspace $P_1\mathscr{H}$ onto $L^2([a,\infty); y^{-2}\,dy)$ using integration over horocycles of F_1:

$$f(z) \mapsto P(f)(y) = \int_0^1 (x+iy)\,dx, \qquad y \geqslant a.$$

It follows from (3.2) that R_{11} is an integral operator with kernel

$$R_{11} = \sum_{\gamma\in\Gamma_\infty} k(z, \gamma z'; \varkappa),$$

where

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},\ n \in \mathbb{Z} \right\}.$$

Hence we get that $R_{11} = PR_{11}P + R'_{11}$, where $T = PR_{11}P$ is an integral operator acting in $L^2([a,\infty); y^{-2}\,dy)$ with kernel $t(y, y'; \varkappa)$,

$$t(y, y'; \varkappa) = \frac{1}{2\varkappa - 1} \begin{cases} y^{\varkappa} y'^{1-\varkappa}, & y < y', \\ y^{1-\varkappa} y'^{\varkappa}, & y > y', \end{cases}$$

and R_{11} is a compact operator. The operator T is the value at $s = \varkappa$ of the resolvent

$$R_0(s) = (B - s(1-s)I)^{-1}$$

of the self-adjoint operator B in $L^2([a,\infty); y^{-2}\,dy)$ that is given by the differential expression $-y^2\, d^2\varphi/dy^2$ with the boundary condition

$$\varphi(a) = \varkappa a \varphi'(a).$$

A remarkable observation by Faddeev, which is necessary for application of the method developed in [15], says that A may be regarded as a perturbation of the operator B with the same absolutely continuous spectrum!

More precisely, writing $R = T + V$, where V is a compact operator, and using again the Hilbert identity for $R_0(s)$,

$$(I - \omega(s)T)^{-1} = I + \omega(s)R_0(s),$$

we rewrite (3.3) in the following way:

$$R(s) = R_0(s) + (I + \omega(s)R_0(s))V + \omega(s)(I + \omega(s)R_0(s))VR(s).$$

Finally, putting

$$R(s) = R_0(s) + (I + \omega(s)R_0(s))B(s)(I + \omega(s)R_0(s)),$$

we obtain an integral equation

$$B(s) = V + H(s)B(s) \tag{3.4}$$

for the resulting operator $B(s)$, where

$$H(s) = V(I + \omega(s)R_0(s)).$$

Thus $B(s)$ is an analogue of the operator $T(\lambda)$ given by (1.9) in the Friedrichs model, and (3.4) is an analogue of (1.11)! With ingenious use of the analytic methods developed in [15], Faddeev proved that $H(s)$ is a Fredholm operator acting in the Banach space $\mathfrak{B}_{-1}$ of continuous functions $f(z)$ on F with the norm

$$\|f\| = \sup_{z\in F_0} |f(z)| + \sup_{z\in F_1} y|f(z)|$$

and depends analytically on s in the strip $0 < \operatorname{Re} s < 2$. This extends the kernel $r(z, z'; s)$ of the resolvent of A analytically to $0 < \operatorname{Re} s \leqslant 1$ with poles of finite order, and poles with $\operatorname{Re} s \geqslant 1/2$ can only lie on the line $\operatorname{Re} s = 1/2$! Analytic continuation of eigenfunctions of the continuous spectrum and their completeness also follow from the technique of perturbation theory for the continuous spectrum (developed in [15]). Since these eigenfunctions coincide for $\operatorname{Re} s > 1$ with the Eisenstein–Maass series

$$E(z, s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} y^s(\gamma z),$$

this also yields analytic continuations and functional equations for them. The reader will enjoy the detailed proofs in Faddeev's paper [19] and Lang's book [110] mentioned above.

3.2. Scattering theory for automorphic functions. Besides the general and stationary approaches to scattering theory (mentioned above), there is a more special approach proposed by Lax and Phillips in their monograph *Scattering theory* [112]. Instead of the time-dependent Schrödinger equation (1.3) they use the wave equation

$$\frac{\partial^2 u}{\partial t^2} + Hu = 0 \tag{3.5}$$

associated with H and the evolution operators $U(t)$ of Cauchy data. The latter form a group of unitary operators on the Hilbert space $\mathscr{H}_E$, the completion of the space of smooth compactly supported Cauchy data orthogonal to the negative spectrum of H with respect to the energy norm of the equation (3.5). Hence, by Stone's theorem

$$U(t) = e^{it\mathcal{L}},$$

where $\mathcal{L}$ is a self-adjoint operator acting in $\mathscr{H}_E$. Application of the Lax–Phillips approach is based on the existence of the so-called incoming and outgoing subspaces $\mathscr{D}_\pm$ in $\mathscr{H}_E$ such that

i) $U(t)\mathscr{D}_- \subset \mathscr{D}_-$ for $t < 0$ and $U(t)\mathscr{D}_+ \subset \mathscr{D}_+$ for $t > 0$;

ii) $\displaystyle\bigcap_{t<0} U(t)\mathscr{D}_- = \bigcap_{t>0} U(t)\mathscr{D}_+ = \{0\}$;

iii) $\displaystyle\bigcup_{t>0} U(t)\mathscr{D}_- = \bigcup_{t<0} U(t)\mathscr{D}_+$;

iv) $\mathscr{D}_+ \perp \mathscr{D}_-$.

The scattering operator S is now related to the contraction semigroup

$$Z(t) = PU(t)P,$$

where $t \geqslant 0$ and P is the orthogonal projection of the subspace $\mathscr{H}_E^a$ of absolutely continuous spectrum[23] of $\mathcal{L}$ onto the orthogonal complement to $\mathscr{D}_+ \oplus \mathscr{D}_-$.

In his joint paper [26] with B. S. Pavlov, Faddeev applied the Lax–Phillips scheme to the case when

$$H = A - \frac{1}{4}I,$$

where A is the Laplace operator in $\mathscr{H} = L^2(\Gamma\backslash\mathbb{H})$, which had previously been considered in [19]. Considering for simplicity the case of one cusp, they proved that the scattering operator S in the Lax–Phillips method coincides with the operator of multiplication by the reflection coefficient $c(s)$, which is determined by the asymptotic behaviour as $y \to \infty$ of the Eisenstein–Maass series

$$E(z,s) = y^s + c(s)y^{1-s} + o(1), \qquad \operatorname{Re} s = \frac{1}{2}\,.$$

In the case when $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, the classical Fourier series expansion of $E(z,s)$ (the so-called Selberg–Chowla formula) shows that

$$c(s) = \sqrt{\pi}\,\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}\,,$$

where $\Gamma(s)$ is the Euler gamma function and $\zeta(s)$ is the Riemann zeta function. Remarkably, it is proved in [26] that the Riemann hypothesis about non-trivial zeros of the zeta function is equivalent to the operator estimate

$$\limsup_{t\to\infty} \frac{1}{t} \log \|Z(t)(B+iI)^{-1}\| = -\frac{1}{4}\,, \tag{3.6}$$

where B is the generator of the semigroup $Z(t)$. This gives a purely operator-theoretic formulation of the Riemann hypothesis!

This brilliant and unexpected result inspired Lax and Phillips to give a systematic exposition of the spectral theory of the Laplace operator using

[23] $\mathscr{H}_E^a = \mathscr{H}_E$ for the Schrödinger operator (1.1) with potential (1.2).

their approach developed in [112]. In the preface to the monograph *Scattering theory for automorphic functions* [113] they write: "Our interest in harmonic analysis of $\mathrm{SL}(2,\mathbb{R})$ stems from the fascinating 1972 Faddeev–Pavlov paper [6],[24] in which they showed that the Lax–Phillips theory of scattering could be applied to the automorphic wave equation. After studying [6] we decided to redo this development entirely within the framework of our theory…". In Appendix 2 to § 7, ironically called "How not to prove the Riemann hypothesis", the authors suggest their version of the Pavlov–Faddeev criterion.

3.3. Selberg trace formula. The methods developed in [19] could naturally be used for a systematic derivation of the famous Selberg trace formula. Faddeev posed this problem to A. B. Venkov and V. L. Kalinin, his students in the Faculty of Mathematics and Mechanics at Leningrad State University. Its solution was presented in the joint paper [27], which used the diploma theses of the first two authors and Faddeev's lectures in Vilnius[25] in March 1973. This approach is based on Krein's method of the spectral shift function. Thus, suppose that H_0 and V are self-adjoint operators acting in a Hilbert space $\mathscr{H}$ and V belongs to $\mathscr{S}_1$, the Schatten–von Neumann ideal of trace-class operators. As usual, we put

$$H = H_0 + V.$$

Krein's theorem (strengthened by Birman and Solomyak) asserts that for every absolutely continuous function φ whose derivative φ' satisfies the Lipschitz condition and $\varphi' \in L^p(\mathbb{R})$, where $1 \leqslant p < \infty$, we have

$$\varphi(H) - \varphi(H_0) \in \mathscr{S}_1.$$

Moreover, there is a unique function $\xi \in L^1(\mathbb{R})$ (called the *spectral shift function* for the pair of operators H_0 and H) such that

$$\operatorname{Tr}(\varphi(H) - \varphi(H_0)) = \int_{-\infty}^{\infty} \varphi'(\lambda)\xi(\lambda)\, d\lambda$$

and, furthermore,

$$\int_{-\infty}^{\infty} \xi(\lambda)\, d\lambda = \operatorname{Tr} V.$$

The novelty of [27] is an explicit calculation of the spectral shift function for the pair $P_c A P_c$ and $\widetilde{P} B \widetilde{P}$, where P_c is the orthogonal projection operator of $\mathscr{H} = L^2(\Gamma\backslash\mathbb{H})$ onto the absolutely continuous spectral subspace of the Laplace operator A, and $\widetilde{P} = PP_1$ is the projection onto $L^2([a,\infty), y^{-2}\,dy)$ (see our synopsis of [19] above). The spectral shift function is expressed in terms of the scattering matrix of A, and the spectral trace $\operatorname{Tr}(h(A) - \widetilde{P}h(B)\widetilde{P})$ is calculated explicitly for all functions h in some explicitly given function class that is used in the derivation of the Selberg trace formula. The matrix trace of the integral operator $h(A) - \widetilde{P}h(B)\widetilde{P}$

[24]This is item [26] in the bibliography of the present survey. – *Note by L. A. T.*

[25]There, a friendship emerged between Faddeev and Askol'd Ivanovich Vinogradov, an expert in analytic number theory, who became interested in applications of the new methods of automorphic function theory to number theory.

(the integral of its kernel with coinciding arguments over the fundamental domain) is calculated following Selberg's approach described in Kubota's book [105]. Then the equality

$$\text{spectral trace} = \text{matrix trace},$$

which follows from the general theory of trace-class operators (see [99]), gives the Selberg trace formula! It is also explained in [27] that the function $Z(s)$ (introduced by Selberg and now known as the Selberg zeta function) is the regularized characteristic determinant of the Laplace operator A. Remarkably (see (2.8)), $Z(s)$ is an analogue of the transmission coefficient $a(\sqrt{\lambda}\,)$ for the one-dimensional Schrödinger equation (2.6)!

It should be noted that, although Faddeev did not publish anything else on this, he was always interested in it. For example, he believed that a new idea was required for a successful realization of his and Pavlov's approach [26] to determining the poles of the automorphic scattering matrix. Faddeev's methods in the spectral theory of automorphic functions were further developed by Venkov. The idea of regularization (either of the Hilbert identity as an equation for the resolvent of a self-adjoint operator, or of the definition of its trace and characteristic determinant) is a recurrent theme in all of Faddeev's work, from the theory of the Schrödinger operator and automorphic Laplace operator to the quantum theory of gauge fields. In 1981 Faddeev gave a plenary lecture at the conference dedicated to the 90th birthday of I. M. Vinogradov. He talked about the universal role of determinants in mathematics and theoretical physics, from the Selberg zeta function $Z(s)$ for the group $\mathrm{SL}(2,\mathbb{Z})$, with poles at $s=\rho/2$, where ρ are the non-trivial zeros of the Riemann zeta function, to the 'Faddeev–Popov ghost determinants' in the theory of Yang–Mills fields. As reported by the academician Yu. V. Prokhorov, Kolmogorov was deeply impressed by this talk by Faddeev.

4. Classical integrable equations

4.1. KdV equation. Faddeev gave a talk about his results on the inverse problem for the three-dimensional Schrödinger equation at a symposium in Novosibirsk at the beginning of 1971. V. E. Zakharov, who was then working in Novosibirsk, told him about the remarkable paper [101] by the American applied mathematicians Gardner, Greene, Kruskal, and Miura on the integration of the Korteweg–de Vries equation (KdV), a well-known equation in the theory of non-linear waves, and about Lax's interpretation of this paper in [111].

The discovery of Gardner, Greene, Kruskal, and Miura was as follows. Consider a Cauchy problem for the KdV equation:

$$u_t - 6uu_x + u_{xxx} = 0, \quad u(x,t)\big|_{t=0} = u(x), \qquad -\infty < x < \infty, \tag{4.1}$$

whose initial datum $u(x)$ decays rapidly as $|x|\to\infty$, and associate with it a one-dimensional Schrödinger operator $L(t)$ (see (2.6)) with the potential $u(x,t)$, which depends on the parameter t. Remarkably, it turns out that the non-linear evolution of $u(x,t)$ according to the KDV equation is given by surprisingly simple formulae in terms of the scattering data of the Schrödinger operator:

$$r(k,t) = e^{8ik^3t}r(k), \quad \varkappa_l(t) = \varkappa_l, \quad m_l(t) = e^{8\varkappa_l^3t}m_l, \qquad l=1,\dots,N, \tag{4.2}$$

where $(r(k), \varkappa_l, m_l)$ are the scattering data of the initial potential $u(x)$. Thus, the solution $u(x,t)$ of the Cauchy problem (4.1) is given by solving the inverse problem for the one-dimensional Schrödinger operator with scattering data (4.2)! Lax explained the formulae (4.2) by showing that the KDV equation is equivalent to the operator equation

$$\frac{dL(t)}{dt} = [L(t), A(t)], \tag{4.3}$$

now called a Lax equation,[26] where $A(t)$ is a third-order differential operator[27] depending explicitly on $u(x,t)$. Later it turned out that there is a wide class of non-linear evolution equations integrable by a similar method, which was called the 'inverse scattering method'.

Discussion of these results with Zakharov led to their joint paper [24], where the KDV equation was shown to be an infinite-dimensional completely integrable Hamiltonian system! The fundamental role of this paper cannot be overestimated. The notion of complete integrability goes back to the classical works of Euler, Lagrange, Jacobi, and Kovalevskaya on rigid-body dynamics. However, in the middle of the 20th century this theme slid into irrelevance (integrability seemed to be a very rare phenomenon) and there were no non-trivial integrable examples with infinitely many degrees of freedom. The paper [24] by Zakharov and Faddeev is extremely important. For the first time it showed the existence of an interesting and non-trivial infinite-dimensional integrable system and started the Hamiltonian theory of equations integrable by the inverse scattering method.

Namely, the phase space for the KDV equation is the set of Cauchy data, the Schwartz space $\mathscr{M} = \mathscr{S}(\mathbb{R})$ of real-valued functions. The infinite-dimensional Fréchet manifold $\mathscr{M}$ is a Poisson manifold with the Poisson bracket

$$\{F, G\}(u) = \int_{-\infty}^{\infty} \frac{d}{dx}\left(\frac{\delta F}{\delta u(x)}\right)\frac{\delta G}{\delta u(x)}\,dx \tag{4.4}$$

for smooth functionals F and G on $\mathscr{M}$, where $\delta F/\delta u(x)$ means the Fréchet derivative (variational derivative) of F. The symplectic leaves $\mathscr{M}_c$ of the Poisson bracket (4.4), referred to as the *Gardner–Zakharov–Faddeev bracket*, are the affine spaces given by the equation

$$\int_{-\infty}^{\infty} u(x)\,dx = c.$$

The corresponding symplectic form on $\mathscr{M}_c$ is

$$\Omega = \int_{-\infty}^{\infty} \mathrm{d}u(x) \wedge \left(\int_{-\infty}^{x} \mathrm{d}u(y)\,dy\right) dx,$$

where d is the exterior differentiation operator on $\mathscr{M}$. It is easy to check that the KDV equation can be written in the Hamiltonian form

$$u_t = \{H, u\} = \frac{d}{dx}\,\frac{\delta H}{\delta u(x)}\,, \quad \text{where} \quad H(u) = \int_{-\infty}^{\infty}\left(\frac{1}{2}u_x^2 + u^3\right) dx.$$

[26]The notation L and B used in Lax's original paper, was later changed to L and A, the first two letters of the name Lax! The notation L and M is also often used.

[27]Namely, $A = 4\dfrac{d^3}{dx^3} - 6u\dfrac{d}{dx} - \dfrac{\partial u}{\partial x}$.

Another parametrization of the phase space $\mathscr{M}$ is given by the scattering data $s = (r(k), \varkappa_l, m_l)$ of the potential $u(x)$, and the inverse map $\imath\colon s \to u(x)$ comes from the solution of the inverse scattering problem (discussed in § 2.3) for a one-dimensional Schrödinger operator. By an elegant use of the general Gelfand–Levitan equation for the difference of two potentials and Wronskian identities for the Jost solutions, Zakharov and Faddeev [24] calculated the pullback of the symplectic form Ω on $\mathscr{M}_c$ under the map $\imath$ and explicitly constructed canonical variables (Darboux variables). Remarkably, one has

$$\imath^*\Omega = \int_0^\infty \mathrm{d}P(k) \wedge \mathrm{d}Q(k)\, dk + \sum_{l=1}^N dp_l \wedge dq_l, \tag{4.5}$$

where

$$\begin{aligned} P(k) &= \frac{4k}{\pi} \log|a(k)|, \quad Q(k) = \arg b(k), \\ p_l &= 2\varkappa_l^2, \quad q_l = \log c_l, \qquad l = 1, \dots, N. \end{aligned} \tag{4.6}$$

Here the function $a(k)$ is recovered from the reflection coefficient $r(k)$ and the zeros $i\varkappa_l$ by using the dispersion relation (2.9), where $a(k) = 1/s_{11}(k)$, and

$$b(k) = a(k)r(k), \quad c_l = im_l\dot{a}(i\varkappa_l).$$

The variables $P(k)$ and $Q(k)$ are infinite-dimensional analogues of the classical action-angle variables in the Liouville–Arnold theorem.

Moreover, using the trace identities for the one-dimensional Schrödinger operator, which are derived similarly to the proof for the radial Schrödinger equation in § 2.2, one can express the Hamiltonian H of the KDV equation explicitly in terms of the canonical variables $P(k)$ and p_l of 'action' type! Namely, (2.9) implies that the following asymptotic expansion holds as $|k| \to \infty$ and $\operatorname{Im} k > 0$:

$$\log a(k) = \sum_{n=1}^\infty \frac{c_n}{k^n},$$

where $c_{2j} = 0$ due to the condition $r(-k) = \overline{r(k)}$. On the other hand, reducing the Schrödinger equation for the Jost solution $f_1(x,k)$ with $\operatorname{Im} k > 0$ to the Riccati equation

$$\sigma_x + \sigma^2 - u + 2ik\sigma = 0 \tag{4.7}$$

for the function

$$\sigma(x,k) = \frac{d}{dx}\log f(x,k) - ik,$$

we obtain

$$\log a(k) = \int_{-\infty}^\infty \sigma(x,k)\, dx.$$

It follows from (4.7) that the solution $\sigma(x,k)$ also has an asymptotic expansion

$$\sigma(x,k) = \sum_{n=1}^\infty \frac{\sigma_n(x)}{(2ik)^n}$$

whose coefficients $\sigma_n(x)$ are certain recursively defined polynomials in the potential $u(x)$ and its derivatives, and moreover, the $\sigma_{2j}(x)$ are total derivatives. Miraculously,

$$H=\frac{1}{2}\int_{-\infty}^{\infty}\sigma_5(x)\,dx=8\int_0^{\infty}k^3P(k)\,dk-\frac{32}{5}\sum_{l=1}^{N}p_l^{5/2}, \tag{4.8}$$

and this completes the proof of complete integrability of the KDV equation in [24]!

The contribution of the discrete spectrum to (4.8) corresponds to solitons, special localized particle-like solutions of the KDV equation. More precisely, a soliton arises in the case $N=1$ and is described by the solution

$$u(x,t)=-\frac{2\varkappa^2}{\cosh^2(\varkappa(x-vt-x_0))}\,,$$

which propagates at the speed $v=4\varkappa^2$, and the case $N>1$ corresponds to the N-soliton solution that describes the interaction of N solitons moving at velocities $v_l=4\varkappa_l^2$. The different signs before the integral and the sum on the right-hand side of (4.8) show that the continuous spectrum modes (respectively, solitons) propagate at negative (respectively, positive) velocities.

The functionals

$$I_n=\frac{1}{2}\int_{-\infty}^{\infty}\sigma_{2n+1}(x)\,dx, \qquad n=1,2,\dots,$$

are in involution with respect to the Gardner–Zakharov–Faddeev bracket and are first integrals for the KDV equation. The functional

$$I_1=-\frac{1}{2}\int_{-\infty}^{\infty}u^2\,dx$$

plays the role of the momentum, $I_2=H$, and the I_n, $n>2$, are higher integrals of motion. The method of [24] also proves complete integrability of the 'higher KDV equations'

$$u_t=\sum_n a_n\frac{d}{dx}\,\frac{\delta I_n}{\delta u(x)}\,,$$

which play a major role in Novikov's approach to the periodic problem for the KDV equation.

Faddeev repeatedly stressed that [24] combined in a miraculous way subjects that he had earlier worked on independently of each other: the inverse scattering problem for the one-dimensional Schrödinger equation, the trace identities, and Hamiltonian mechanics!

4.2. Sine-Gordon equation. At the beginning of 1972 Faddeev visited the USA, where he gave a number of talks and, in particular, told about his new work with Zakharov. The American physicist Klauder, who attended the lecture, mentioned the sine-Gordon[28] equation (SG), which originally appeared in the study of

[28] As Faddeev himself later noted, the rhyme 'sine–Klein' is rather tasteless, but contagious. However, what we call the Klein–Gordon equation, should be called the Klein–Fock equation.

surfaces of constant negative curvature and then in non-linear optics and the theory of superconductivity (Josephson effect). This equation

$$\varphi_{tt} - \varphi_{xx} + \frac{m^2}{\beta}\sin\beta\varphi = 0, \qquad -\infty < x, t < \infty, \tag{4.9}$$

is relativistically invariant and may be regarded as an essentially non-linear model of classical field theory in two-dimensional space-time.

Like the KDV equation, the SG equation is an infinite-dimensional Hamiltonian system. The phase space $\mathscr{M} = \{(\pi(x), \varphi(x))\}$ is the set of Cauchy data for (4.9):

$$\varphi(x,t)\big|_{t=0} = \varphi(x) \quad \text{and} \quad \varphi_t(x,t)\big|_{t=0} = \pi(x), \tag{4.10}$$

where $\pi(x) \in \mathscr{S}(\mathbb{R})$ and $e^{i\beta\varphi(x)} - 1 \in \mathscr{S}(\mathbb{R})$ is such that

$$\lim_{x\to-\infty}\varphi(x) = 0, \qquad \lim_{x\to\infty}\varphi(x) = \frac{2\pi}{\beta}Q.$$

The quantity $Q \in \mathbb{Z}$ plays the role of a topological charge. The symplectic form Ω on $\mathscr{M}$ is written canonically as

$$\Omega = \int_{-\infty}^{\infty} \mathrm{d}\pi(x) \wedge \mathrm{d}\varphi(x)\, dx,$$

and the SG equation takes the Hamiltonian form

$$\varphi_t = \{H, \varphi\}, \quad \pi_t = \{H, \pi\}$$

with the Hamiltonian

$$H = \int_{-\infty}^{\infty} \left(\frac{1}{2}\pi^2 + \frac{1}{2}\varphi_x^2 + \frac{m^2}{\beta^2}(1 - \cos\beta\varphi)\right) dx.$$

The functionals

$$P = -\int_{-\infty}^{\infty} \pi\varphi_x\, dx$$

(the momentum of the field $\varphi(x)$) and

$$K = \int_{-\infty}^{\infty} x\left(\frac{1}{2}\pi^2 + \frac{1}{2}\varphi_x^2 + \frac{m^2}{\beta^2}(1 - \cos\beta\varphi)\right) dx$$

(the Lorentz boost) realize a Hamiltonian action of the Lie algebra of the Poincaré group of two-dimensional space-time. This reflects the relativistic nature of the SG equation.

The SG equation attracted Faddeev's attention, and in 1973 he and his new student L. A. Takhtajan began to seek a Lax representation for (4.9).[29] Such a representation was obtained jointly with Zakharov, in the paper [29] by all three authors.

[29] A Lax representation of the form (4.3) in the light-cone coordinates $\xi = (t+x)/2$, $\eta = (t-x)/2$ was already known (see [124] and [82]). But the Hamiltonian setup of the SG equation in these coordinates differs from that given above, so we need a Lax representation in the original coordinates x, t.

The operator L in this Lax representation is a first-order matrix differential operator acting on vector-valued functions in $\mathbb{C}^4$, and the constant 4×4 matrix in front of $\dfrac{d}{dx}$ in L has rank 2. Matrix operators of this type were not previously considered in the literature. The eigenvalue equation

$$L\Psi=\lambda\Psi$$

reduces to a 2×2 matrix differential equation containing the spectral parameter λ as well as $1/\lambda$. The formalism for the direct and inverse scattering problems for L was given in [29]. As in the case of the Schrödinger equation, the scattering data $s=(r(\lambda),\zeta_j,m_j)$ consist of:

• the function $r(\lambda)=b(\lambda)/a(\lambda)$, where $a(\lambda)$ and $b(\lambda)$ are analogues of the transition coefficients satisfying the realness conditions

$$a(\lambda)=\overline{a(-\lambda)}\quad\text{and}\quad b(\lambda)=-\overline{b(-\lambda)}$$

and the condition

$$|a(\lambda)|^2+|b(\lambda)|^2=1;$$

• the zeros ζ_j, $j=1,\dots,N$, of the function $a(\lambda)$, which are symmetric with respect to the imaginary axis in the upper half-plane;

• the corresponding normalization factors m_j.

In the paper [30] by Faddeev and Takhtajan the formalism of the inverse problem, that is, inversion of the map to the scattering data

$$\imath\colon (\pi(x),\varphi(x))\to s=(r(\lambda),\zeta_j,m_j),$$

was constructed. It is based on the existence of triangular transformation operators and on integral equations of Gelfand–Levitan–Marchenko type. The main result of [30] is the proof of the complete integrability of the SG equation as an infinite-dimensional Hamiltonian system!

Namely, it was shown in [30] that[30]

$$\imath^*\Omega=\int_0^\infty \mathrm{d}\rho(\lambda)\wedge \mathrm{d}\vartheta(\lambda)\,d\lambda+\sum_{l=1}^{n_1}dp_l\wedge dq_l+\sum_{k=1}^{n_2}(d\xi_k\wedge d\eta_k+d\theta_k\wedge d\phi_k),$$

where

$$\rho(\lambda)=-\frac{8}{\pi\beta^2\lambda}\log|a(\lambda)|,\quad \vartheta(\lambda)=-\arg b(\lambda),\qquad \lambda>0,$$
$$p_l=\frac{1}{\beta^2}\log\varkappa_l,\quad q_l=8\log|c_l|,\qquad l=1,\dots,n_1,$$

$$\left.\begin{aligned}\xi_k&=\frac{4}{\beta^2}\log|\lambda_k|, & \eta_k&=4\log|d_k|,\\ \theta_k&=\arg\lambda_k, & \phi_k&=-\frac{16}{\beta^2}\arg d_k,\end{aligned}\right\}\quad k=1,\dots,n_2.$$

[30] Here and in what follows we use the notation of [30].

Here $i\varkappa_l$ (with $\varkappa_l > 0$, $l = 1, \dots, n_1$) are the zeros of $a(\lambda)$ on the imaginary axis, while $(\lambda_k, -\overline{\lambda}_k)$ (with $\operatorname{Im}\lambda_k, \operatorname{Re}\lambda_k > 0$, $k = 1, \dots, n_2$) are the pairs of zeros of $a(\lambda)$ symmetric with respect to the imaginary axis, and $N = n_1 + 2n_2$. Moreover,

$$c_l = m_l \dot{a}(i\varkappa_l) \quad \text{and} \quad d_k = m_k \dot{a}(\lambda_k).$$

Note that the corresponding Poisson brackets of the transition coefficients $a(\lambda)$ and $b(\lambda)$ take the following elegant form:

$$\{a(\lambda), a(\mu)\} = 0, \quad \{b(\lambda), b(\mu)\} = 0 \tag{4.11}$$

and

$$\{a(\lambda), b(\mu)\} = \frac{\beta^2 \lambda\mu}{4(\lambda^2 - \mu^2)} a(\lambda) b(\mu). \tag{4.12}$$

These formulae are fundamental for the quantization of the SG model (see § 5.5).

As in the case of the KDV equation (see § 4.1), the trace identities, that is, the asymptotic expansions

$$\frac{1}{i} \log a(\lambda) = \sum_{n=1}^{\infty} \frac{I_n}{\lambda^n} \qquad \text{as } \lambda \to \infty$$

and

$$\frac{1}{i} \log a(\lambda) = \sum_{n=0}^{\infty} I_{-n} \lambda^n \quad \text{as } \lambda \to 0,$$

give an infinite set $\{I_n\}$ of integrals of motion for the SG model. Here $I_0 \equiv \pi Q$ (mod 2π), and

$$P = \frac{2m}{\beta^2}(I_{-1} + I_1) \quad \text{and} \quad H = \frac{2m}{\beta^2}(I_{-1} - I_1). \tag{4.13}$$

The Hamiltonian H and the momentum P of the SG model are expressed solely in terms of 'action'-type variables by the following beautiful and transparent formulae:

$$P = \int_0^\infty p(\lambda)\rho(\lambda)\, d\lambda + \sum_{l=1}^{n_1} P_{sl} + \sum_{k=1}^{n_2} P_{bk}, \tag{4.14}$$

$$H = \int_0^\infty \sqrt{p(\lambda)^2 + m^2}\, \rho(\lambda)\, d\lambda + \sum_{l=1}^{n_1} \sqrt{P_{sl}^2 + M_s^2} + \sum_{k=1}^{n_2} \sqrt{P_{bk}^2 + M_{bk}^2}\,, \tag{4.15}$$

where

$$p(\lambda) = m\biggl(\frac{1}{8\lambda} - 2\lambda\biggr), \quad P_{sl} = \frac{m}{\beta^2}\biggl(\frac{1}{\varkappa_l} - 16\varkappa_l\biggr),$$
$$P_{bk} = \frac{m(\lambda_k - \overline{\lambda}_k)}{\beta^2}\biggl(\frac{1}{|\lambda_k|^2} - 16\biggr),$$

and

$$M_s = \frac{8m}{\beta^2}\,, \quad M_{bk} = \frac{16m}{\beta^2}\sin\theta_k. \tag{4.16}$$

These formulae admit the following interpretation in terms of the classical particles generated by the SG equation. The first summands in (4.14), (4.15) are the contributions of relativistic particles with mass m, momentum $p(\lambda)$, and density $\rho(\lambda)$, which are generated by the continuous spectrum of L. The topological charge of these particles is zero. The second summands are written as sums over particles with masses M_s and momenta P_{sl}. These new particles are generated by the discrete spectrum eigenvalues of L lying on the imaginary axis. They correspond to solitons (localized particle-like solutions of the SG equation) with topological charge

$$Q = \operatorname{sign} c_l.$$

The third summands in (4.14) and (4.15) are generated by the pairs of discrete spectrum eigenvalues of L that are symmetric with respect to the imaginary axis. These particles with mass M_{bk} and momentum P_{bk} possess an internal degree of freedom and, depending on their internal state, their mass varies from zero to twice the mass of a soliton. The corresponding solutions of the SG equation are the so-called double solitons or 'breathers'. They have topological charge zero and correspond to the bound states of solitons with antisolitons. At the level of the classical excitation spectrum, this realizes Einstein's dream that 'one non-linear self-interacting field generates several kinds of particles'. As a result, besides a scalar particle of mass m, the semiclassical spectrum contains[31] solitons and antisolitons of mass $8m/\beta^2$ as well as their bound states of masses

$$M_n = \frac{16m}{\beta^2}\sin\biggl(\biggl(n+\frac{1}{2}\biggr)\frac{\beta^2}{16}\biggr),$$

where it is assumed that the quantum theory is defined only for $\beta^2 = 8\pi/N$, N being an integer.[32]

As a continuation of [30], in the joint paper [37] with Takhtajan, Faddeev proved[33] that the Cauchy problem (4.9)–(4.10) for the SG equation in the class of rapidly decaying initial data is equivalent to the corresponding problem in the light-cone coordinates. The Hamiltonian formalism was also developed for the latter problem, and its complete integrability was established.

4.3. Hamiltonian approach in soliton theory. A refined version of the Lax representation (4.3) is the so-called zero-curvature representation

$$\frac{\partial F}{\partial x} = U(x,t,\lambda)F, \tag{4.17}$$

$$\frac{\partial F}{\partial t} = V(x,t,\lambda)F, \tag{4.18}$$

[31] These results were presented in a paper by Faddeev and Takhtajan, "The relativistic one-dimensional model, generating several particles", which was twice submitted to *Physics Letters* (in December 1973 and June 1974), but remained unpublished for reasons that are still unclear (see [33]).

[32] We shall see in § 5 that the assumption of integrality of $8\pi/\beta^2$ is not needed and the correct formula in the one-loop approximation is (5.3) without adding the 'Maslov index' 1/2.

[33] Thus correcting an inaccurate remark in [29], which did not influence the rest of the paper.

where U and V are $n \times n$ matrices depending on some functions $u_\alpha(x,t)$ and a parameter λ, the so-called spectral parameter. The compatibility condition of the system (4.17)–(4.18),

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U, V] = 0, \tag{4.19}$$

must hold for all values of λ, and this determines a system of non-linear evolution equations for the functions $u_\alpha(x,t)$.

A large class of integrable non-linear equations can be represented in this form, including the KDV equation, the SG equation, the non-linear Schrödinger equation, the Heisenberg magnet equation, and many others. For example, the SG equation has a zero-curvature representation[34] of the form

$$U(x,t,\lambda) = \frac{1}{4}\begin{pmatrix} -i\beta\pi(x) & m\lambda e^{-i\beta\varphi(x)/2} - \dfrac{m}{\lambda}e^{i\beta\varphi(x)/2} \\ \dfrac{m}{\lambda}e^{-i\beta\varphi(x)/2} - m\lambda e^{i\beta\varphi(x)/2} & i\beta\pi(x) \end{pmatrix} \tag{4.20}$$

and

$$V(x,t,\lambda) = \frac{1}{4}\begin{pmatrix} -i\beta\varphi_x(x) & m\lambda e^{-i\beta\varphi(x)/2} + \dfrac{m}{\lambda}e^{i\beta\varphi(x)/2} \\ -\dfrac{m}{\lambda}e^{-i\beta\varphi(x)/2} - m\lambda e^{i\beta\varphi(x)/2} & i\beta\varphi_x(x) \end{pmatrix}. \tag{4.21}$$

The formalism of the inverse scattering problem for the Schrödinger equation, which was used to solve the KDV equation, extends naturally to the case of (4.17) as the method of the Riemann–Hilbert matrix factorization problem. By the general Gohberg–Krein theory, this problem is reduced to a Fredholm system of integral equations of Wiener–Hopf type. In simple cases, it can also be reduced to Marchenko-type integral equations with compact integral operators.

The Hamiltonian approach to integrable non-linear equations was developed by Faddeev's students in the Laboratory of Mathematical Problems of Physics at the Leningrad Branch of the Steklov Mathematical Institute: P. P. Kulish, A. G. Reiman, N. Yu. Reshetikhin, Semenov-Tian-Shansky, Sklyanin, and Takhtajan. The Hamiltonian structure of the corresponding equations appeared to be closely related to the Poisson structure on the dual space of a Lie algebra. As shown by Semenov-Tian-Shansky, it can be elegantly written in terms of the classical r-matrix introduced by Sklyanin. For example, a connection between the Poisson structure of integrable models of classical field theory and (generalizations of) the loop algebra was established in Faddeev's joint paper [46] with Reshetikhin. All these results were reflected in the monograph [52],[35] by Faddeev and Takhtajan.

The main example in [52] is the non-linear Schrödinger equation (NS)

$$i\psi_t = -\psi_{xx} + 2\varkappa|\psi|^2\psi, \qquad -\infty < x, t < \infty, \tag{4.22}$$

which has many physical applications. In contrast to the KDV equation, the NS equation has a natural quantum analogue. It describes a many-particle system

[34] Here, compared with [52], we change λ to $1/\lambda$.
[35] Reprinted by Springer Verlag in 2007 in its series *Classics in Mathematics*.

interacting with a δ-function potential (see § 5.2). Like the KDV equation and the SG equation, the NS equation is an infinite-dimensional Hamiltonian system. Its phase space is the vector space $\mathscr{S}(\mathbb{R};\mathbb{C})$ of complex-valued functions with complex coordinates $\psi(x)$ and $\overline{\psi}(x)$ and the symplectic form

$$\Omega = \frac{1}{i}\int_{-\infty}^{\infty} \mathrm{d}\psi(x)\wedge \mathrm{d}\overline{\psi}(x)\,dx$$

with the following Poisson brackets:

$$\{\psi(x),\psi(y)\} = \{\bar\psi(x),\bar\psi(y)\} = 0, \quad \{\psi(x),\bar\psi(y)\} = \delta(x-y). \tag{4.23}$$

The NS equation can be written in the Hamiltonian form

$$\psi_t = \{H,\psi\}, \quad \bar\psi_t = \{H,\bar\psi\},$$

with the Hamiltonian

$$H = \int_{-\infty}^{\infty} (|\psi_x|^2 + \varkappa|\psi|^4)\,dx. \tag{4.24}$$

The corresponding Poisson brackets of the transition coefficients $a(\lambda)$ and $b(\lambda)$ are given by[36]

$$\{a(\lambda),a(\mu)\} = 0, \quad \{b(\lambda),b(\mu)\} = 0 \tag{4.25}$$

and

$$\{a(\lambda),b(\mu)\} = -\frac{\varkappa}{\lambda-\mu}a(\lambda)b(\mu). \tag{4.26}$$

The trace identities

$$\frac{1}{i}\log a(\lambda) = \varkappa\sum_{n=1}^{\infty}\frac{I_n}{\lambda^n} \quad \text{as } \lambda\to\infty \tag{4.27}$$

give an infinite set of integrals of motion for the NS equation, and we have

$$H = I_3. \tag{4.28}$$

These formulae are fundamental for the quantization of the NS equation (see § 5.2).

5. Quantum integrable systems

5.1. Semiclassical quantization. From the very beginning of his work with integrable equations, Faddeev realized their rich potential for quantization.[37] The most interesting equation from this point of view was the SG equation, whose quantum version was expected by Faddeev to give an example of a relativistic quantum field theory with a rich spectrum of particles (main particles, solitons, and their bound states corresponding to breathers) generated by a single field φ in the original equation (4.9). It would be natural to expect the quantum version of this model

[36] Here $b(\lambda)$ corresponds to $\overline{b}(\lambda)$ in [52], for compatibility with § 5.2.

[37] He wrote in his scientific autobiography [54]: “If in a single word I had to focus the sphere of my scientific interests, it would be quantization”.

to be exactly soluble, as is the case with the original classical model. After that, the task of constructing the theory of quantum integrable models (generalizing the inverse scattering method) became the most important direction of Faddeev's work. Developing the Hamiltonian approach to soliton theory (see § 4.3) was a natural part (and a necessary preliminary stage) of the quantization programme.

The first step in realizing this programme was quantization of the SG model in the semiclassical approximation. In 1974 Kulish, one of Faddeev's first students, deduced from the existence of infinitely many local conservation laws for the SG equation that the momenta p_{in} of incoming (as $t \to -\infty$) particles and p_{out} of outgoing (as $t \to \infty$) particles (solitons, double solitons, and main particles) satisfy the equations

$$\sum_a (p_a^{2n+1})_{\mathrm{in}} = \sum_b (p_b^{2n+1})_{\mathrm{out}}, \tag{5.1}$$

$$\sum_a (p_a^0 p_a^{2n})_{\mathrm{in}} = \sum_b (p_b^0 p_b^{2n})_{\mathrm{out}}, \tag{5.2}$$

where $p_a^0 = \sqrt{p_a^2 + m_a^2}$ and $n = 0, 1, 2, \dots$. It follows directly from these equations that the number of particles of each kind and their momenta are preserved in interactions. It is remarkable that this property remains valid after quantization: there is no multiple creation of particles in the SG model and the number of particles of each kind and their momenta are preserved under interactions! This result, which was unexpected[38] for theoretical physicists, was proved in the framework of perturbation theory by I. Ya. Aref'eva, one of Faddeev's first female students, and his new student V. E. Korepin. More delicate arguments using the locality of the integrals of motion give rise to a remarkable conclusion: the scattering is factorizable, that is, all many-particle processes are reduced to two-particle ones.

Faddeev performed a systematic quantization of this model by means of functional integration in the joint paper [31] with Korepin and Kulish, continued with Korepin in [32]. Namely, in the one-loop approximation they deduced a formula for the mass of the bound state of a soliton and an antisoliton:

$$M_n = \frac{16m}{\beta^2} \sin \frac{n\beta^2}{16} \tag{5.3}$$

(see the exact formula (5.34)). Apart from the spectrum of masses, the authors of [31] and [32] calculated the scattering matrices of solitons in the semiclassical approximation and predicted their exact expressions for the integer values of $8\pi/\beta^2$, which correspond to the so-called reflectionless case. The calculation is based on the fact that the classical limit of the two-particle scattering matrix is determined by the generating function of the canonical transformation from the initial coordinates to the final ones. An extended overview of the results for the SG equation is contained in the joint survey paper [38] with Korepin. In the joint paper [36] with Kulish and S. V. Manakov, analogous results were obtained for the NS equation, which may be regarded as a non-relativistic limit of the SG equation.

[38] In physics seminars, Faddeev was fiercely attacked by V. N. Gribov and his school, who asserted that quantum corrections destroy completely the classical integrability of the SG model.

We note that the classical limit of the scattering matrix of solitons is expressed in terms of dilogarithms. The expression proposed in [31] for the exact quantum scattering matrix in the case of integer $8\pi/\beta^2$ involves a function (now called the quantum dilogarithm), whose remarkable properties Faddeev analysed much later (see § 6.2). Exact formulae for the scattering matrix of the quantum model for all values of the coupling constant were derived by A. B. Zamolodchikov from the main kinematic postulates of quantum field theory ([129], and see also [130]).

In computing the so-called one-loop quantum corrections in [31], [32], and [38], Faddeev had another occasion to use his favorite tool: the calculation of the determinants of differential operators.[39] Namely, write the propagator G as a functional (Feynman) path integral:

$$G = \int \exp\{iS(u)\}\, du$$

with certain boundary conditions. Expanding the classical action function $S(u)$ near a stationary point, whose role is played by some trajectory $u_0(x,t)$ (for example, a multisoliton solution: $u = u_0 + \varphi$), we obtain

$$S(u) = S_0 + \varphi K \varphi + O(\phi^3), \qquad K = K_0 + \varepsilon v(x,t), \quad K_0 = \partial_t^2 - \partial_x^2 + m^2.$$

Then the one-loop approximation is given by the Gaussian integral

$$G \sim \int \exp\biggl\{ i \int \varphi K \varphi \biggr\} \prod D\varphi \sim \det{}^{1/2}(K/K_0),$$

which is formally defined as the regularized determinant of the differential operator K. As usual, this determinant is expressed via the trace of the resolvent:

$$\frac{d}{d\varepsilon} \log \det K = \operatorname{Tr} v K^{-1}.$$

Faddeev liked this trick very much and often used it (see § 2.2). Since there is a complete description of the solutions of the SG equation in terms of the scattering data (see § 4.2), we also have a complete description of all solutions of the linearized equation near a multisoliton solution, and therefore we have an explicit formula for its resolvent. Then finding the desired quantities reduces to a direct calculation. Strictly speaking, the actual calculation scheme is somewhat more complicated because K has zero eigenvalues (the so-called zero-mode problem). This problem was solved in the joint paper [35] with Korepin.

5.2. Quantum inverse problem method: first steps. Successful quantization of the SG equation in the semiclassical approximation raised hopes that this model can be quantized exactly. At the same time, the possibilities of semiclassical quantization were obviously exhausted, and in one of Faddeev's seminars[40] at the beginning of 1978 he posed the problem of extending the inverse scattering method to the quantum case.

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[39]We again cite [54]: "If in a single term I had to characterize my technical means, it would be determinants".

[40]The importance of Faddeev's seminars was described in detail by Semenov-Tian-Shansky in his introduction to the collection of papers *Faddeev's seminar on mathematical physics* [120].

The following three groups of examples of quantum systems known at the end of the 1970s could be referred to as 'integrable' in the sense that their exact solutions were known.

First, there is the quantum NS equation. Quantization replaces the fields $\psi(x)$ and $\bar{\psi}(x)$ with Poisson brackets (4.23) by operators $\Psi(x)$ and $\Psi^\dagger(x)$ with the canonical commutation relations[41]

$$[\Psi(x), \Psi(y)] = [\Psi(x)^\dagger, \Psi^\dagger(y)] = 0, \qquad [\Psi(x), \Psi^\dagger(y)] = \delta(x-y)I.$$

The rapidly decreasing case

$$\psi(x), \bar{\psi}(x) \to 0, \qquad |x| \to \infty,$$

corresponds to quantization in the Fock space (see [87]), which is an orthogonal sum of the vacuum subspace spanned by a vector Ω with $\Psi(x)\Omega = 0$ and N-particle subspaces spanned by vectors of the form

$$\biggl(\int_{\mathbb{R}^N} f(x_1, \dots, x_N)\Psi^\dagger(x_1)\cdots\Psi^\dagger(x_N)\,d^N x\biggr)\Omega.$$

The quantum Hamiltonian is obtained from (4.24) by replacing ψ and $\bar{\psi}$ by Ψ and $\Psi^\dagger$ and using the normal ordering:

$$H = \int_{-\infty}^{\infty} \bigl(-\Psi_x^\dagger(x)\Psi_x(x) + \varkappa\Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x)\bigr)\,dx. \tag{5.4}$$

On an N-particle subspace, this Hamiltonian becomes the singular differential operator

$$H_N = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + \varkappa \sum_{1\leqslant j<k\leqslant N} \delta(x_j - x_k). \tag{5.5}$$

Explicit expressions for its eigenfunctions were constructed by Berezin, McGuire, Yang, Brézin, and Zinn-Justin.

The second source of examples of quantum integrable systems was Kostant's work on quantization of the open Toda lattice using the methods of representation theory.

Third, exactly soluble lattice models of phase transitions (and closely related spin chains) were extensively studied in solid state physics and statistical mechanics following the classical papers of Ising and Bethe. In particular, an important role in the creation of the quantum inverse problem method was played by the transfer-matrix method brought to perfection by Baxter in the mid-1970s.

At first sight, all these approaches had nothing in common with the inverse scattering method in the theory of solitons. The first step towards a synthesis of classical and quantum methods was taken in Faddeev's joint paper [39] with his new student Sklyanin. Taking the example of the NS equation and the SG equation, they suggested that quantization should be based on the transition coefficients $a(\lambda)$ and $b(\lambda)$, which determine the scattering data (see § 4.2 and § 4.3).

[41] Here we set the Planck constant $\hbar$ to be 1.

The fundamental Poisson brackets for $a(\lambda)$ and $b(\lambda)$ for both equations are of the form

$$\{a(\lambda), a(\mu)\} = 0, \qquad \{a(\lambda), a(\mu)\} = 0, \tag{5.6}$$

$$\{a(\lambda), b(\mu)\} = a(\lambda)b(\mu)\rho(\lambda, \mu), \tag{5.7}$$

where for the NS equation and the SG equation we have (see (4.26) and (4.12))

$$\rho_{\mathrm{NS}}(\lambda, \mu) = \frac{c}{\lambda - \mu}\,, \qquad \rho_{\mathrm{SG}}(\lambda, \mu) = \frac{2\gamma\lambda\mu}{\lambda^2 - \mu^2}\,, \tag{5.8}$$

respectively,[42] and[43] $\gamma = \beta^2/8$. Here we also put $c = -\varkappa < 0$ and thus restrict ourselves to the case of an attracting potential in the Hamiltonian (5.5), since this is the choice of the sign of the coupling constant that makes the NS equation a non-relativistic limit of the SG equation.

It was conjectured in [39] that the correct quantum analogue of the quadratic Poisson bracket (5.7) is the quadratic commutation relation

$$A(\lambda)B(\mu) = B(\mu)A(\lambda)\check{\rho}(\lambda, \mu) \tag{5.9}$$

for some quantum operators $A(\lambda)$ and $B(\lambda)$. In the semiclassical approximation,[44]

$$\check{\rho}(\lambda, \mu) \simeq 1 - i\rho(\lambda, \mu).$$

As exact expressions for $\alpha(\lambda, \mu)$, it was proposed to use

$$\check{\rho}_{\mathrm{NS}}(\lambda, \mu) = \frac{\lambda - \mu - ic/2}{\lambda - \mu + ic/2} \tag{5.10}$$

and

$$\check{\rho}_{\mathrm{SG}}(\lambda, \mu) = \frac{\sinh(\alpha - \beta) - i\sin(\gamma/2)}{\sinh(\alpha - \beta) + i\sin(\gamma/2)}\,, \tag{5.11}$$

where[45] $\alpha = \log\lambda$ and $\beta = \log\mu$ are the so-called physical rapidities.

As in the classical case, the logarithm of $A(\lambda)$ must be the generating function for commuting local integrals of motion. It was also suggested in [39] to interpret the operators $B(\lambda)$ as creation operators for the eigenfunctions of $A(\lambda)$. Indeed, define the vacuum Ω as a common eigenvector of all $A(\lambda)$ with the eigenvalue 1: $A(\lambda)\Omega = \Omega$. Then by (5.9) the vector

$$\Psi(\lambda_1, \dots, \lambda_N) = B(\lambda_1)\cdots B(\lambda_N)\Omega$$

is also an eigenvector of all operators $A(\lambda)$ with the eigenvalue $\prod_{n=1}^{N}\check{\rho}(\lambda, \lambda_n)$.

---

[42] See the footnotes 34 and 36.

[43] This is compatible with the notation of [40] and [52]; in [30] one has $\gamma = \beta^2$.

[44] Since we have put $\hbar = 1$, it corresponds to $\varkappa \to 0$ or $\gamma \to 0$.

[45] Since we mainly use $\gamma = \beta^2/8$ as the coupling constant in the SG model, using β for the rapidity should lead to no ambiguity.

The concrete choice of the coefficients $\check{\rho}(\lambda,\mu)$ in [39] relied on a study of analytic properties of the eigenvalues of $A(\lambda)$ as functions of λ and on the requirement that the results obtained for bound states must be compatible with (5.3). Another argument in the case of the NS model came from the calculations of Sklyanin in [121], who obtained the following explicit formulae in this case:

$$A_{\mathrm{NS}}(\lambda) = {:}a_{\mathrm{NS}}\biggl(\lambda+\frac{ic}{2}\biggr){:} \quad \text{and} \quad B_{\mathrm{NS}}(\lambda) = {:}b_{\mathrm{NS}}(\lambda){:}\,,$$

where $:\Phi:$ is the operator in the Fock space whose normal symbol (see [87]) is the classical functional Φ of the canonical fields $\bar{\psi}$ and ψ that occur in the description of the NS equation.

The expressions

$$H = I_3 + \frac{c^2}{12} I$$

for the quantum Hamiltonian of the NS model, where I is the identity operator, and

$$\log A(\lambda) = -ic\sum_{n=1}^{\infty}\frac{I_n}{\lambda^n}$$

are the analogues of (4.27)–(4.28) in the classical case.

Expanding $\log\prod_{k=1}^{N}\check{\rho}(\lambda,\lambda_k)$ in powers of λ^{-1}, we obtain

$$H\Psi(\lambda_1,\dots,\lambda_N) = E(\lambda_1,\dots,\lambda_N)\Psi(\lambda_1,\dots,\lambda_N),$$

where

$$E(\lambda_1,\dots,\lambda_N) = \sum_{k=1}^{N}\lambda_k^2.$$

There was some element of luck in the case of the NS model: despite the presence of interactions, one was able to realize the quantum model in the same Hilbert space where the canonical commutation relations for free fields have been realized, that is, in the Fock space. Therefore, the problem splits into N-particle quantum-mechanical problems, which can be solved explicitly, and the operators $A(\lambda)$ and $B(\lambda)$ can be described explicitly in terms of the normal symbols. In the case of a genuinely relativistic quantum field theory (such as the SG model) with manifestly non-free representation of the canonical commutation relations as well as divergences and renormalization, there was no chance of such a simple answer, and an essentially new approach was required.

5.3. Quantum inverse problem method: the *R*-matrix. Surprisingly, among the three approaches mentioned above, Faddeev was more interested in Baxter's papers (see [86]), which were the most difficult and most remote from his direct interests. This testified to the intuitive genius of Faddeev, who had a feeling for the deep mathematical potential hidden in those papers!

Using the joint paper [41] by Faddeev and Takhtajan, one can describe Baxter's approach as follows. Let V be a finite-dimensional vector space over $\mathbb{C}$ and $R(\lambda)$

an operator[46] in $V \otimes V$ depending on a complex parameter λ. Write R_{12}, R_{13}, and R_{23} for the operators in $V \otimes V \otimes V$ corresponding to three embeddings

$$\operatorname{End}(V \otimes V) \hookrightarrow \operatorname{End}(V \otimes V \otimes V).$$

More precisely,

$$R_{12} = R \otimes I, \quad R_{23} = I \otimes R,$$

where I is the identity operator on V, and

$$R_{13}(u \otimes v \otimes w) = \sum_k u_k \otimes v \otimes w_k, \quad \text{where} \quad R(u \otimes w) = \sum_k u_k \otimes w_k.$$

With every matrix $R(\lambda)$ that satisfies

$$R_{12}(\lambda-\mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda-\mu) \tag{5.12}$$

for all λ and μ, we associate in a natural way a commutative family $t_N(\lambda)$ of operators acting in the vector space

$$V^{\otimes N} = \underbrace{V \otimes \cdots \otimes V}_{N}$$

for an arbitrary N. This family is described is follows. It is assumed in the general case that $\det R(\lambda)$ is a meromorphic function of λ.

Let P be the permutation operator in $V \otimes V$:

$$P(u \otimes v) = v \otimes u, \qquad u, v \in V.$$

We define an embedding

$$\imath_n \colon \operatorname{End} V \hookrightarrow \operatorname{End}(V^{\otimes N})$$

in such a way that, under this embedding, an operator $a \in \operatorname{End} V$ acts non-trivially only on the nth factor in the tensor product $V^{\otimes N}$, $n = 1, \dots, N$. Using this embedding, we define a map

$$\operatorname{End}(V \otimes V) \ni A = \sum_k a_k \otimes b_k \mapsto A_n = \sum_k a_k \otimes \imath_n(b_k) \in \operatorname{End} V \otimes \operatorname{End}(V^{\otimes N}).$$

In other words, A_n is a matrix in V whose entries are operators in $V^{\otimes N}$. Putting

$$L_n(\lambda) = R(\lambda+\eta)_n \quad \text{and} \quad \widehat{R}(\lambda) = PR(\lambda), \tag{5.13}$$

we see that (5.12) takes the form

$$\widehat{R}(\lambda-\mu)(L_n(\lambda) \otimes L_n(\mu)) = (L_n(\mu) \otimes L_n(\lambda))\widehat{R}(\lambda-\mu), \qquad n = 1, \dots, N. \tag{5.14}$$

[46] In Baxter's work, $V = \mathbb{C}^2$ and $R(\lambda)$ is the 4×4 matrix formed by the Boltzmann weights of the eight-vertex model.

Here $\otimes$ stands for the tensor product with respect to the 'auxiliary' space V and the operator product in the 'quantum' space $V^{\otimes N}$. Since the maps $\imath_n$ for different n commute, from (5.14) we obtain

$$\widehat{R}(\lambda-\mu)(T_N(\lambda)\otimes T_N(\mu)) = (T_N(\mu)\otimes T_N(\lambda))\widehat{R}(\lambda-\mu), \tag{5.15}$$

where

$$T_N(\lambda) = L_N(\lambda)\cdots L_1(\lambda) \tag{5.16}$$

is the 'quantum' monodromy matrix, that is, the ordered product of $L_n(\lambda)$ along the finite chain $1,\dots,N$. Equivalently,

$$R(\lambda-\mu)(T_N(\lambda)\otimes I)(I\otimes T_N(\mu)) = (I\otimes T_N(\mu))(T_N(\lambda)\otimes I)R(\lambda-\mu). \tag{5.17}$$

Putting

$$t_N(\lambda) = \operatorname{Tr}_V T_N(\lambda) \in \operatorname{End}(V^{\otimes N}), \tag{5.18}$$

from (5.15) we obtain

$$[t_N(\lambda), t_N(\mu)] = 0.$$

This property of commutativity of the 'transfer matrices' $t_N(\lambda)$ gives rise to an exact solution of the eight-vertex model and enables one to regard $t_N(\lambda)$ as a generating function of the integrals of motion for the corresponding quantum spin chain.

Remarkably, Faddeev saw the analogy between the quantum monodromy matrix $T_N(\lambda)$ and the monodromy matrix for the zero-curvature representation (4.17), (4.18)! Namely, the monodromy matrix $T(x;\lambda)$ for (4.17) is defined as a solution of the Cauchy problem for the differential equation

$$\frac{d}{dx}T(x;\lambda) = U(x;\lambda)T(x;\lambda), \qquad T(0;\lambda) = I,$$

where I is the identity matrix.

In a discrete version when the continuous variable x varies over the lattice $x_n = n\Delta$ with spacing Δ and $n\in\mathbb{Z}$, equation (4.17) takes the form

$$F_{n+1} = L_n(\lambda)F_n,$$

and the monodromy matrix is defined as follows:

$$T_N(\lambda) = L_N(\lambda)\cdots L_1(\lambda). \tag{5.19}$$

As $\Delta\to 0$, we have

$$L_n(\lambda) \approx I + \Delta U(x_n;\lambda).$$

It is now clear that the formulae (5.16) and (5.19) coincide.

These ideas underlie the joint paper [40] by Faddeev, Sklyanin, and Takhtajan, where the authors gave an exact solution of the quantum SG model and formulated the quantum inverse problem (or scattering) method (QISM), a method for the exact solution of the quantum models that correspond to the equations solvable by the classical inverse scattering method (see § 4). Before describing the results of [40], we shall state the main algebraic principles of QISM following that paper.

5.4. Quantum inverse problem method: general scheme. The *first key element* of QISM is a local L-operator, a generalization of the matrices (5.13) in Baxter's approach. Suppose that for every site n of a finite lattice of length N we are given a space of states (a Hilbert space $\mathfrak{h}_n$ of finite or infinite dimension) and a matrix $L_n(\lambda)$ in V (depending on a 'spectral' parameter λ) whose entries are operators in $\mathfrak{h}_n$. The matrix $L_n(\lambda)$ is called an L-operator if there is a matrix $\widehat{R}(\lambda,\mu) \in \operatorname{End}(V \otimes V)$ such that (5.14) holds for all λ and μ, that is,

$$\widehat{R}(\lambda,\mu)(L_n(\lambda) \otimes L_n(\mu)) = (L_n(\mu) \otimes L_n(\lambda))\widehat{R}(\lambda,\mu). \tag{5.20}$$

We stress that, in contrast to Baxter's approach, the auxiliary space V is now distinguished from the Hilbert space $\mathfrak{h}$ playing the role of a quantum state space at the nth lattice site. The equation (5.12) corresponds to the case

$$L(\lambda) = R(\lambda + \eta),$$

which was considered by Baxter. This separation of the roles of the R-matrix and the L-operator plays a key part in QISM and is a seed of the future theory of quantum groups! (See § 6.1.)

Considering the tensor product of Hilbert spaces

$$\mathfrak{H}_N = \mathfrak{h}_1 \otimes \cdots \otimes \mathfrak{h}_N$$

(the full Hilbert space on N sites) and embedding, as above, operators in $\mathfrak{h}_n$ into operators in $\mathfrak{H}_N$, we see that the matrix

$$T_N(\lambda) = L_N(\lambda) \cdots L_1(\lambda)$$

(the quantum monodromy matrix) satisfies the same relation (5.15) as that which holds for the local L-operators:

$$\widehat{R}(\lambda,\mu)(T_N(\lambda) \otimes T_N(\mu)) = (T_N(\mu) \otimes T_N(\lambda))\widehat{R}(\lambda,\mu). \tag{5.21}$$

As above, this yields the commutativity of the family of operators

$$t_N(\lambda) = \operatorname{Tr}_V T_N(\lambda)$$

in $\mathfrak{H}_N$.

The *second fundamental principle* of QISM is the algebraic Bethe ansatz (discovered in [40]) of diagonalization of the commutative family of operators $t_N(\lambda)$. We stress that, in contrast to Baxter's approach, the algebraic Bethe ansatz uses all entries of the quantum monodromy matrix T_N and their commutation relations (5.21), not only the commutativity of the traces $t_N(\lambda)$.

For simplicity we consider the important case when $V = \mathbb{C}^2$ and, following [40] and [41], assume that the matrix $\widehat{R}(\lambda,\mu)$ has the form

$$\widehat{R}(\lambda,\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\lambda,\mu) & c(\lambda,\mu) & 0 \\ 0 & c(\lambda,\mu) & b(\lambda,\mu) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.22}$$

where the functions $b(\lambda,\mu)$ and $c(\lambda,\mu)$ satisfy

$$\frac{b(\lambda,\mu)}{c(\lambda,\mu)} = -\frac{b(\mu,\lambda)}{c(\mu,\lambda)}\,.$$

We represent the matrix $T_N(\lambda)$ in the form

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix},$$

where the subscript N is omitted for simplicity of notation, and we write the commutation relations that follow from (5.21):

$$\begin{aligned} [B(\lambda), B(\mu)] &= 0, \\ A(\lambda)B(\mu) &= \frac{1}{c(\mu,\lambda)}B(\mu)A(\lambda) - \frac{b(\mu,\lambda)}{c(\mu,\lambda)}B(\lambda)A(\mu), \\ D(\lambda)B(\mu) &= \frac{1}{c(\lambda,\mu)}B(\mu)D(\lambda) - \frac{b(\lambda,\mu)}{c(\lambda,\mu)}B(\lambda)D(\mu). \end{aligned}$$

We also assume the existence of a vector $\Omega \in \mathfrak{H}_N$ (a generating vector) which is annihilated by the operator $C(\lambda)$ and is an eigenvector of $A(\lambda)$ and $D(\lambda)$ with eigenvalues $a(\lambda)$ and $d(\lambda)$, respectively. The existence of a generating vector (as well as the equality (5.21)) often follows from similar properties of the L-operator. Thus, write $L_n(\lambda)$ in the form

$$L_n(\lambda) = \begin{pmatrix} a_n(\lambda) & b_n(\lambda) \\ c_n(\lambda) & d_n(\lambda) \end{pmatrix}$$

and suppose that there is an $\omega_n \in \mathfrak{h}_n$ such that

$$a_n(\lambda)\omega_n = \alpha_n(\lambda)\omega_n, \quad d_n(\lambda)\omega_n = \delta_n(\lambda)\omega_n, \quad \text{and} \quad c_n(\lambda)\omega_n = 0. \tag{5.23}$$

Then the generating vector is given by

$$\Omega = \omega_1 \otimes \cdots \otimes \omega_N$$

and

$$a(\lambda) = \prod_{n=1}^{N} \alpha_n(\lambda) \quad \text{and} \quad d(\lambda) = \prod_{n=1}^{N} \delta_n(\lambda).$$

Finally, we seek eigenvectors of the commutative family of operators $t(\lambda) = A(\lambda) + D(\lambda)$ in the form

$$\Psi(\lambda_1,\ldots,\lambda_n) = \prod_{i=1}^{n} B(\lambda_i)\Omega. \tag{5.24}$$

We get from the above commutation relations that

$$
\begin{aligned}
t(\lambda)\Psi(\lambda_1,\dots,\lambda_n) = \biggl(a(\lambda)\prod_{i=1}^{n}\frac{1}{c(\lambda_i,\lambda)} + d(\lambda)\prod_{i=1}^{n}\frac{1}{c(\lambda,\lambda_i)}\biggr)\Psi(\lambda_1,\dots,\lambda_n)\\
+\sum_{j=1}^{n}\frac{b(\lambda,\lambda_j)}{c(\lambda,\lambda_j)}\biggl(a(\lambda_j)\prod_{\substack{1\leqslant k\leqslant n\\ k\neq j}}\frac{1}{c(\lambda_k,\lambda_j)}\\
- d(\lambda_j)\prod_{\substack{1\leqslant k\leqslant n\\ k\neq j}}\frac{1}{c(\lambda_j,\lambda_k)}\biggr)B(\lambda)\prod_{\substack{1\leqslant k\leqslant n\\ k\neq j}}B(\lambda_k)\Omega.
\end{aligned}
$$

Indeed, to prove this, it suffices to use the symmetry of the vector $\Psi(\lambda_1,\dots,\lambda_n)$ with respect to $\lambda_1,\dots,\lambda_n$. This symmetry follows from the commutativity of the 'creation operators' $B(\lambda)$. The summand containing $\Psi(\lambda_1,\dots,\lambda_n)$ appears in an obvious way, and it is also clear that there are only two summands with $B(\lambda_1)$ replaced by $B(\lambda)$, and the other summands follow by symmetry. Now, for the vector $\Psi(\lambda_1,\dots,\lambda_n)$ to be an eigenvector for the operators $t(\lambda)$ with eigenvalue

$$
\Lambda(\lambda;\lambda_1,\dots,\lambda_n) = a(\lambda)\prod_{i=1}^{n}\frac{1}{c(\lambda_i,\lambda)} + d(\lambda)\prod_{i=1}^{n}\frac{1}{c(\lambda,\lambda_i)} \tag{5.25}
$$

it suffices that the n-tuple $\lambda_1,\dots,\lambda_n$ is a solution of the system of equations

$$
\frac{a(\lambda_j)}{d(\lambda_j)} = \prod_{\substack{1\leqslant k\leqslant n\\ k\neq j}}\frac{c(\lambda_k,\lambda_j)}{c(\lambda_j,\lambda_k)}\,, \qquad j=1,\dots,n. \tag{5.26}
$$

These are the famous Bethe equations, obtained by Bethe in 1931 in another form while solving the isotropic quantum Heisenberg magnet model (see [88]). The novelty here is the compact formula (5.24) for eigenvectors and simple algebraic derivation of (5.25) and (5.26) using only the commutation relations (5.21) and the existence of a generating vector Ω.

The last, *third element* of QISM depends on the model considered and involves a passage to the 'infinite volume' limit ($N\to\infty$), and getting rid of the lattice ($\Delta\to 0$) for continuous models. It is crucial to define a full ('physical') Hilbert space $\mathscr{H}$ of the model so that the quantum integrals of motion (including the Hamiltonian H) act in $\mathscr{H}$. By the main principle of quantum field theory, the operator H must be positive definite and must annihilate the 'physical vacuum' (a special vector $\Omega_{\mathrm{phys}}\in\mathscr{H}$). The task of distinguishing a separable Hilbert space $\mathscr{H}$ in the 'large' non-separable Hilbert space $\mathfrak{H}_\infty$, finding the spectrum of the quantum integrals of motion by passing to the limit in the Bethe ansatz equations (5.26), and constructing the asymptotic states and the S-matrix often requires involved technical tools.[47]

[47]We note that the difficult analytic issues related to taking the limit as $N\to\infty$ and $\Delta\to 0$ and the construction of the subspace $\mathscr{H}\subset\mathfrak{H}_\infty$ were not discussed either in [40] or in the other papers on QISM. Their rigorous mathematical justification is an interesting and difficult problem in functional analysis.

5.5. Quantum inverse problem method: SG model. To get rid of ultraviolet divergences (that is, divergences at small distances) in quantum field theory, one must consider the theory on a finite lattice when the space variable x assumes discrete values

$$x_n = -L + (n-1)\Delta, \quad n = 1, \ldots, N, \quad \text{and} \quad x_{N+1} = L.$$

The lattice size Δ (respectively, its length $2L$) plays the role of the ultraviolet (respectively, infrared) cut-off parameter.

The classical L-operator for the lattice SG model is of the form

$$L_n^{\mathrm{cl}}(\lambda) = I + \int_{x_n}^{x_n+\Delta} U(x,\lambda)\,dx + O(\Delta^2),$$

where $U(x,\lambda)$ is given by (4.20). The following formula for the corresponding quantum L-operator was proposed in [40]:

$$L_n(\lambda) = \begin{pmatrix} e^{-i\Delta\beta\pi_n/4} & \dfrac{m\Delta}{4}\left(\lambda e^{-i\beta\varphi_n/2} - \dfrac{1}{\lambda}e^{i\beta\varphi_n/2}\right) \\ \dfrac{m\Delta}{4}\left(\dfrac{1}{\lambda}e^{-i\beta\varphi_n/2} - \lambda e^{i\beta\varphi_n/2}\right) & e^{i\Delta\beta\pi_n/4} \end{pmatrix}. \tag{5.27}$$

Here π_n and φ_n are canonical operators in $\mathfrak{h}_n = L^2(\mathbb{R})$, which are embedded in $\mathfrak{H}_N = \bigotimes_{n=1}^N \mathfrak{h}_n$ as described above and satisfy the Heisenberg commutation relations

$$[\varphi_m, \varphi_n] = [\pi_m, \pi_n] = 0 \quad \text{and} \quad [\varphi_m, \pi_n] = \frac{i}{\Delta}\delta_{mn} I$$

obtained by discretizing the relations

$$[\varphi(x), \varphi(y)] = [\pi(x), \pi(y)] = 0 \quad \text{and} \quad [\varphi(x), \pi(y)] = i\delta(x-y) I$$

for the field operators $\varphi(x)$ and $\pi(x)$. In other words, the unitary operators $u_n = e^{i\Delta\beta\pi_n/4}$ and $v_n = e^{i\beta\varphi_n/2}$ in the Schrödinger representation on $L^2(\mathbb{R})$ are of the form

$$(u_n f)(x) = f\left(x + \frac{\beta}{4}\right), \quad (v_n f)(x) = e^{i\beta x/2} f(x)$$

and yield a Weyl pair:

$$u_n v_n = q v_n u_n, \qquad q = e^{i\gamma}, \quad \gamma = \frac{\beta^2}{8}\,.$$

Remarkably, the L-operator (5.27) satisfies (5.20) with an R-matrix of the form (5.22), where

$$b(\lambda,\mu) = \frac{i\sin\gamma}{\sinh(\alpha - \beta + i\gamma)} \quad \text{and} \quad c(\lambda,\mu) = \frac{\sinh(\alpha-\beta)}{\sinh(\alpha - \beta + i\gamma)} \tag{5.28}$$

while $\alpha = \log\lambda$ and $\beta = \log\mu$. The relation (5.20) holds up to the terms of order Δ^2, which is quite sufficient for passing to the limit as $\Delta \to 0$.

It was shown in [40] that after the modification of the diagonal matrix entries of the L-operator following the rule[48]

$$(L_n(\lambda))_{11} \mapsto (\widetilde{L}_n(\lambda))_{11} = u_n^*\biggl(1 + \frac{S}{2}(qv_n^2 + q^{-1}v_n^{*2})\biggr),$$
$$(L_n(\lambda))_{22} \mapsto (\widetilde{L}_n(\lambda))_{22} = \biggl(1 + \frac{S}{2}(qv_n^2 + q^{-1}v_n^{*2})\biggr)u_n,$$

where $S = (m\Delta/4)^2$, the pairwise products $\widetilde{L}_{n+1}(\lambda)\widetilde{L}_n(\lambda)$ of L-operators satisfy the relations (5.23), where

$$\omega_{n,n+1} = \bigl[1 + S\cos(\beta(x_n + x_{n+1}))\bigr]\delta\biggl(x_{n+1} - x_n - \frac{\beta}{4} + \frac{2\pi}{\beta}\biggr)$$

is a vector in the rigged Hilbert space $L^2(\mathbb{R}^2)$ and the eigenvalues of the operators $(\widetilde{L}_{n+1}(\lambda)\widetilde{L}_n(\lambda))_{11}$ and $(\widetilde{L}_{n+1}(\lambda)\widetilde{L}_n(\lambda))_{22}$ at $\omega_{n,n+1}$ are independent of n and are given by

$$\alpha(\lambda) = 1 + S(\lambda^2q^{-1} + \lambda^{-2}q), \qquad \delta(\lambda) = 1 + S(\lambda^2q + \lambda^{-2}q^{-1}). \tag{5.29}$$

A diagonalization of the commutative family $t_N(\lambda) = \operatorname{Tr} T_N(\lambda)$, where $T_N(\lambda)$ is the monodromy matrix

$$T_N(\lambda) = L_N(\lambda)L_{N-1}(\lambda)\cdots L_1(\lambda),$$

was done in the paper [40] by using the algebraic Bethe ansatz developed there with generating vector

$$\Omega_N = \prod_{n=1}^{N/2}\omega_{n,n+1}. \tag{5.30}$$

Letting $N \to \infty$ and $\Delta \to 0$ in such a way that $2L = N\Delta$ remains constant (or using the Izergin–Korepin L-operator for finite Δ) and assuming that Ω_N tends to a vector Ω_0, we see from (5.26) and (5.28), (5.29) that

$$\Psi(\lambda_1,\dots,\lambda_n) = \prod_{l=1}^{n} B_L(\lambda_l)\Omega_0$$

[48]The modification of the L-operator and the definition of $\omega_{n,n+1}$ used in [40] are different from (but equivalent to) ours. We also mention the L-operator

$$L_n(\lambda) = \begin{pmatrix} u_n^*\rho_n & \dfrac{m\Delta}{4}\biggl(\lambda v_n^* - \dfrac{1}{\lambda}v_n\biggr) \\ \dfrac{m\Delta}{4}\biggl(\dfrac{1}{\lambda}v_n^* - \lambda v_n\biggr) & \rho_n u_n \end{pmatrix}, \qquad \rho_n = \bigl(1 + S(qv_n^2 + q^{-1}v_n^{*2})\bigr)^{1/2},$$

of the lattice SG model and the vector

$$\omega_{n,n+1} = \bigl[1 - 2S\cos(\beta(u_{2n} + u_{2n-1}))\bigr]^{-1/2}\delta\biggl(u_{2n} - u_{2n-1} - \frac{\beta}{4} + \frac{2\pi}{\beta}\biggr),$$

which were constructed in [103]. For these, the relations (5.20) and (5.23) for the pairwise products of the L-operators hold with the same $\alpha(\lambda)$ and $\delta(\lambda)$ as in (5.29), but now for all Δ.

is an eigenvector for the operators $t_L(\lambda) = \operatorname{Tr} T_L(\lambda)$ with an eigenvalue

$$\Lambda_L(\lambda;\lambda_1,\dots,\lambda_n) = \exp\left\{\frac{m_1\cosh(2\alpha - i\gamma)L}{2}\right\}\prod_{j=1}^{n}\frac{\sinh(\alpha_j - \alpha + i\gamma)}{\sinh(\alpha_j - \alpha)} + \exp\left\{\frac{m_1\cosh(2\alpha + i\gamma)L}{2}\right\}\prod_{j=1}^{n}\frac{\sinh(\alpha_j - \alpha - i\gamma)}{\sinh(\alpha_j - \alpha)}\,, \quad (5.31)$$

if $\alpha_1 = \log\lambda_1,\dots,\alpha_n = \log\lambda_n$ satisfy the system of equations

$$\exp\{-im_1\sin\gamma\sinh(2\alpha_j)L\} = \prod_{\substack{1\leqslant k\leqslant n\\ k\neq j}}\frac{\sinh(\alpha_j - \alpha_k + i\gamma)}{\sinh(\alpha_k - \alpha_j + i\gamma)}\,. \quad (5.32)$$

Here n is assumed to be finite and the 'bare mass' m in the L-operator is chosen independent of Δ so that $m_1 = m^2\Delta/4$ remains finite as $\Delta\to 0$ (physicists call the parameter $1/\Delta$ the 'cut-off momentum'). Thus, we are dealing with a system of n particles with momenta $m_1\sin\gamma\sinh(2\alpha_l)$ and energies $m_1\sin\gamma\cosh(2\alpha_l)$, where the rapidities $2\alpha_l$ satisfy (5.32).

However, this description is unsatisfactory. First, the state Ω_0 has topological charge $Q = -\infty$. Second, the excitation energy becomes negative for $\operatorname{Im}\alpha_l = \pi/2$. In the physics language, Ω_0 is a pseudovacuum and the excitations described are quasi-particles. To resolve this difficulty, it was proposed in [40] to construct the physical vacuum Ω_{phys} as a 'filled Dirac sea' of quasi-particles with negative energy. This means an application of a large number of operators $B_L(ie^{\alpha_l})$ to Ω_0, where the $\alpha_l + \pi i/2$ satisfy (5.32), which is formally written as

$$\Omega_{\mathrm{phys}} = \lim_{\substack{\Delta\to 0\\ L\to\infty}}\prod_l B_L(ie^{\alpha_l})\Omega_0.$$

Since the operators $B_L(\lambda)$ raise the topological charge by 1, Ω_{phys} has topological charge $Q = 0$. This construction requires analysis[49] of the limiting procedure $\Delta\to 0$ and $L\to\infty$ and a hypothesis on the uniform distribution of α_l as $L\to\infty$ on the interval $-\Lambda\leqslant\alpha\leqslant\Lambda$ with some density $\rho(\alpha)$, where

$$\Lambda\sim\log\frac{1}{m\Delta}\,.$$

In this limit, the system of equations (5.26) naturally becomes an integral equation for the distribution function, and the requirement of finiteness of the solution $\rho(\alpha)$ gives rise to a mass renormalization. Namely, it was shown in [40] that the bare mass m depends on Δ in the following way:

$$m = m_r^{(\pi-\gamma)/\pi}\Delta^{-\gamma/\pi}\,, \quad (5.33)$$

where the finite quantity m_r (the 'renormalized mass') gives a scale for masses in the theory, that is, the masses of physical excitations are proportional to m_r.[50]

[49] See footnote 47.

[50] From the point of view of conformal field theory, it is natural to include the factor $\Delta^{-\gamma/\pi}$ in the definition of the operators v_n and v_n^* since the operators $e^{\pm i\beta\varphi(x)/2}$ have an 'anomalous dimension'. Then the expressions $m_r^{(\pi-\gamma)/\pi}v_n$ and $m_r^{(\pi-\gamma)/\pi}v_n^*$ in the L-operator have the dimension of mass.

We have

$$\rho(\alpha) = Cm_r \cosh \alpha',$$

where the constant C depends only on γ, and

$$\alpha' = \frac{\pi\alpha}{\pi - \gamma} .$$

Physical excitations differ from the vacuum by finitely many quasi-particles that polarize the vacuum. For example, a one-particle state is obtained by adding one quasi-particle with real α. This 'pushes out' the vacuum's quasi-particles and gives rise to the formation of a 'hole' (omitting one of the $\alpha_k + \pi i/2$ with α_k close to α), which is formally written as

$$\Psi(\alpha) = \lim_{\substack{\Delta\to 0\\ L\to\infty}} B_L(e^{\alpha}) \prod_l B_L(ie^{\widetilde{\alpha}_l})\Omega_0.$$

One constructs a scattering state of n particles $\Psi(\alpha_1, \dots, \alpha_n)$ obtained by adding n quasi-particles with real $\alpha_1, \dots, \alpha_n$ to the vacuum in a similar way.[51] Their bound states $\Psi_n(\alpha)$ are obtained by adding a 'string' (a bound state of n quasi-particles with $n \leqslant [\pi/\gamma] - 1$) to the vacuum, where

$$\alpha_j = \alpha + i\gamma\biggl(j - \frac{n+1}{2}\biggr), \qquad j = 1, \dots, n,$$

and the soliton–antisoliton pair $\Psi(\alpha_s, \alpha_{\overline{s}})$ is constructed in a slightly more complicated way. Denoting the mass of a main particle by M we find that the mass of a bound state of n particles is

$$M_n = M\frac{\sin(n\gamma'/2)}{\sin(\gamma'/2)} , \tag{5.34}$$

and the mass of a soliton is given by

$$M_s = \frac{M}{2\sin(\gamma'/2)} , \tag{5.35}$$

where

$$\gamma' = \frac{\pi\gamma}{\pi - \gamma}$$

is the renormalized coupling constant. The exact formula (5.34) becomes the formula (5.3) in the one-loop approximation, and the formulae (5.34) and (5.35) tend to the classical formulae (4.16) in the semiclassical limit.[52]

[51] A construction of the 'physical' Hilbert space $\mathscr{H}$ spanned by these vectors as a subspace of the non-separable Hilbert space $\mathfrak{H}_\infty$ (the tensor product of the rigged Hilbert spaces $L^2(\mathbb{R})$ over all $x \in \mathbb{R}$) is a difficult problem in functional analysis. See also the definition of the asymptotic space $\mathscr{H}_{\mathrm{as}}$ in § 8.2.2.

[52] In the modern approach, solitons and antisolitons of mass m_s are referred to as main particles of the SG model. They have bound states (breathers) of masses $2m_s \sin(\gamma' k/2)$, where $k \leqslant [\pi/\gamma']$. These main particles correspond to $k = 1$ and are absent when $\gamma' > \pi/2$. Breathers may also be regarded as bound states of the main particles, which is in accordance with 'nuclear democracy'.

Finally, defining an operator $A(\lambda)$ as

$$A(\lambda)=\lim_{L\to\infty}\frac{t_L(\lambda^{\pi-\gamma/\pi}e^{i\gamma/2})}{\Lambda_L(\lambda^{\pi-\gamma/\pi}e^{i\gamma/2})},$$

where $\Lambda_L(\lambda)$ is the eigenvalue of the operator (5.5) at $\Psi(\alpha)$ for finite L, and defining $B(\lambda)$ as the creation operator of the main particle,

$$B(\lambda)\Psi(\alpha_1,\dots,\alpha_n)=\Psi(\alpha,\alpha_1,\dots,\alpha_n),$$

we get that $A(\lambda)$ and $B(\lambda)$ satisfy the relation (5.9) (predicted in [39]), where $\check{\rho}(\lambda,\mu)$ is given by (5.11) with γ replaced by γ'! As in the classical case, the quantum operators of momentum P and energy H can be found from the asymptotic behaviour of the operator $A(\lambda)$,

$$\log A(\lambda)=\sum_{j=1}^{\infty}I^{\pm}_{2j-1}e^{\mp\alpha(2j-1)},\qquad \alpha\to\pm\infty, \tag{5.36}$$

using formulae analogous to (4.13):

$$P=\frac{M_s}{4}(I_1^-+I_1^+),\qquad H=\frac{M_s}{4}(I_1^--I_1^+).$$

This completes our description of [40].[53]

5.6. Quantum inverse scattering method: spin chains. Besides [40], the algebraic Bethe ansatz was also described in detail in [41] along with Baxter's transfer-matrix method. Our exposition in § 5.3 is mainly based on [41]. Faddeev very much liked to present this construction in his pedagogical surveys and lectures ([42], [43], [47], [61], [68], [69], [72], [77], [79]). However, the main theme of [41] was a generalization of the algebraic Bethe ansatz to the so-called XYZ spin chain. This integrable model is given by the Hamiltonian

$$H_{XYZ}=-\frac{1}{2}\sum_{n=1}^{N}(J_x\sigma_n^1\sigma_{n+1}^1+J_y\sigma_n^2\sigma_{n+1}^2+J_z\sigma_n^3\sigma_{n+1}^3), \tag{5.37}$$

defined as an operator in the quantum space $(\mathbb{C}^2)^{\otimes N}$. Here

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad\text{and}\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

are the standard Pauli matrices and the σ_n^a, $a=1,2,3$, are their embeddings in $(\mathbb{C}^2)^{\otimes N}$ as operators acting on the nth factor of the tensor product. The periodicity condition $n+N\equiv n$ is also assumed in (5.37). When $J_x=J_y$, the Hamiltonian H_{XYZ} commutes with the so-called magnon number operator

$$M=\sum_{n=1}^{N}(\sigma_n^3-1), \tag{5.38}$$

[53] In the beginning of 1979, Faddeev sent a telegram to C.-N. Yang "Solved Sine-Gordon model by Bethe Ansatz" and received his answer "Congratulations!" (there was no e-mail at that time).

and the corresponding R-matrix is of the form (5.22), which enables us to use the algebraic Bethe ansatz with a local generating vector ω_n such that

$$(\sigma_n^1 + i\sigma_n^2)\omega_n = 0.$$

However, in the general case of unequal constants J, the R-matrix has additional non-zero entries $d(\lambda, \mu)$,

$$\widehat{R}(\lambda,\mu) = \begin{pmatrix} 1 & 0 & 0 & d(\lambda,\mu) \\ 0 & b(\lambda,\mu) & c(\lambda,\mu) & 0 \\ 0 & c(\lambda,\mu) & b(\lambda,\mu) & 0 \\ d(\lambda,\mu) & 0 & 0 & 1 \end{pmatrix}, \tag{5.39}$$

and is parametrized by elliptic functions. It follows that the Hamiltonian H_{XYZ} does not commute with M, the operators $B(\lambda)$ do not commute, there are no generating vectors annihilated by $C(\lambda)$, and the algebraic Bethe ansatz in its original form is not applicable. Baxter was able to bypass these obstacles and find the spectrum of H_{XYZ} using a very complicated *ad hoc* construction (the Q-operator method). In [41], Takhtajan and Faddeev showed that Baxter's construction can be completely immersed in QISM after a modification of the latter. Thus, by introducing gauge transformations of the local L-operators

$$L'_n(\lambda) = M_{n+1}(\lambda)L_n(\lambda)M_n(\lambda)$$

with carefully chosen matrices $M_n(\lambda)$, one can ensure that a λ-independent local generating vector exists for $L'_n(\lambda)$.[54] These gauge transformations were used in [41] to construct multiparameter operators $A_{kl}(\lambda)$, $B_{kl}(\lambda)$, $C_{kl}(\lambda)$, and $D_{kl}(\lambda)$ whose commutation relations enable one to apply the algebraic Bethe ansatz construction. It should be noted that the term the 'Baxter–Yang equation' (subsequently transformed into the now commonly accepted *Yang–Baxter equation*) for (5.12) first appeared in [41].

The papers [44] and [45] by Faddeev and Takhtajan are devoted to a study of the so-called spin XXX-chain, that is, the Hamiltonian (5.37) in the totally isotropic case

$$J_x = J_y = J_z \equiv J.$$

In this case the Hamiltonian has an additional symmetry: H_{XXX} commutes with the total spin operators

$$S^a = \frac{1}{2}\sum_{n=1}^{N} \sigma_n^a, \qquad a = 1, 2, 3. \tag{5.40}$$

As was shown in [44] and [45], this global SU(2)-symmetry implies that the operators $B(\lambda)$ generate by means of (5.24) only highest weight vectors with respect to the action (5.40) (that is, the vectors annihilated by the operator $S^+ \equiv S^1 + iS^2$).

[54] This construction was discussed from the point of view of algebraic geometry by Krichever [104].

The missing eigenvectors of the Hamiltonian H_{XXX} are obtained by applying the operator $S^- = S^1 - iS^2$ to the highest weight vectors.

The spectrum of H_{XXX} in the limit $N \to \infty$ under the antiferromagnetic condition $J > 0$ was also studied in [44] and [45]. When $J > 0$, the vector Ω generating the Bethe ansatz is not a physical vacuum (that is, an eigenvector of H_{XXX} with the minimum eigenvalue). To construct the physical vacuum and excitations, one needs a procedure for 'filling the pseudovacuum' similar to that described for the SG equation in § 5.5. An accurate study of the algebraic Bethe ansatz equations for elementary excitations (spin waves) over the physical vacuum in [44] and [45] revealed a surprising fact: the spin of a spin wave turned out to be not equal to 1 (as physicists had erroneously assumed) but to 1/2 instead. This discovery attracted the attention of experts in solid state physics, and the papers [44] and [45] were often cited subsequently.

Many of Faddeev's colleagues in theoretical physics were puzzled by his interest in integrable models: their applicability seemed to be confined to two-dimensional space-time. However, knowing the history of the development of physics and mathematics, he was fully aware of the universal importance of the exact solutions, and his intuition again proved to be infallible! The interest of theoretical physicists working on realistic models of QFT in integrable models began with Lipatov's well-known paper [115] on high-energy scattering of hadrons in special kinematics. It was shown there that an essential part of the hadron scattering amplitude is described by a quantum integrable chain. Remarkably, it was shown in the joint paper [71] by Faddeev and G. P. Korchemsky that this chain coincides with an integrable version of the isotropic Heisenberg magnet of spin $s = 0$ for the non-compact group $\mathrm{SL}(2, \mathbb{C})$. This made it possible to apply the full power of QISM to Lipatov's problem and related issues. And this was only the beginning of modern applications of the theory of integrable models to problems of QFT in four-dimensional space-time. For example, integrable models are now used to study the correspondence (discovered by Maldacena) between string theory in Lobachevskii's five-dimensional space and conformal theory on its boundary (more precisely, the absolute), and it was recently discovered by Nekrasov and S. L. Shatashvili that a description of the vacuum sector in supersymmetric quantum field theories (in particular, in four-dimensional space-time) directly gives rise to quantum integrable systems. These and other applications of the theory of quantum integrable systems along with their brief history were presented in Faddeev's last survey [79] in 2013.

6. Quantum groups

6.1. Quantization of Lie groups and algebras. The classical inverse scattering method served as a basis for new mathematical structures and notions. For example, the notion of classical r-matrix (introduced by Sklyanin) led Drinfeld to the creation of the theory of Poisson–Lie groups (Lie groups endowed with Poisson structures compatible with the group operation). In a similar way, quantum Lie groups and algebras were introduced as abstractions of concrete algebraic constructions arising within the quantum inverse problem method. We give two illustrative examples.

1. In [53], Faddeev and Takhtajan introduced the $\mathbb{C}$-algebra A_q with generators a, b, c, and d and relations

$$\begin{gathered} ab = qba, \quad ac = qca, \quad bc = cb, \quad bd = qdb, \quad cd = qdc, \\ ad - da = (q - q^{-1})bc, \end{gathered} \tag{6.1}$$

where $q \in \mathbb{C} \setminus \{0\}$. These relations possess the following remarkable property. Consider the matrices

$$T' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \quad \text{and} \quad T'' = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix},$$

where a', b', c', d' and a'', b'', c'', d'' are commuting sets of elements satisfying (6.1). Then the set a, b, c, d such that

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = T'T''$$

also satisfies these relations. In other words, the basic relations (6.1) are preserved by matrix multiplication, that is, A_q is a *bialgebra* with coproduct Δ given on the generators a, b, c, and d by

$$\Delta(T) = T \dot{\otimes} T,$$

where the symbol $\dot{\otimes}$ means the tensor product of algebras and the ordinary product of matrices. Since the generators a, b, c, d commute as $q \to 1$, the algebra A_q may be regarded as a deformation (quantization) of the commutative algebra $\mathbb{C}[a, b, c, d]$ of polynomial functions on the algebra $M_2(\mathbb{C})$ of 2×2 matrices.

2. Kulish and Reshetikhin [107] and Sklyanin [122] considered the $\mathbb{C}$-algebra $U_\hbar$ with generators H and $X^\pm$ and relations

$$[H, X^\pm] = \pm 2X^\pm \quad \text{and} \quad [X^+, X^-] = \frac{\sinh(\hbar H)}{\sinh \hbar}, \tag{6.2}$$

where $\hbar \in \mathbb{C}$ plays the role of the Planck constant and formal power series in $\hbar$ of type $e^{\pm\hbar H/2}$ are allowed in the algebra $U_\hbar$. Sklyanin [123] showed that $U_\hbar$ is also a bialgebra with coproduct

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \Delta(X^\pm) = X^\pm \otimes e^{-\hbar H/2} + e^{\hbar H/2} \otimes X^\pm.$$

As $\hbar \to 0$, the relations (6.2) tend to commutation relations for the generators of the Lie algebra $\mathfrak{sl}(2)$. Therefore, $U_\hbar$ may be regarded as a deformation (quantization) of the universal enveloping algebra $U\mathfrak{sl}(2)$ of the Lie algebra $\mathfrak{sl}(2)$. $U_\hbar$ is also commonly defined to be the $\mathbb{C}$-algebra with generators E, F, K, and K^{-1} and relations

$$KK^{-1} = K^{-1}K = 1, \;\; KE = q^2 EK, \;\; KF = q^{-2} FK, \;\; \text{and} \;\; [E, F] = \frac{K - K^{-1}}{q - q^{-1}},$$

where $q = e^\hbar$. Here one has

$$\Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F, \quad \Delta(K) = K \otimes K,$$

and the formulae

$$E = iX^{+}e^{\hbar H/2}, \quad F = -ie^{-\hbar H/2}X^{-}, \quad K^{\pm 1} = e^{\pm \hbar H} \tag{6.3}$$

establish an equivalence of the two definitions.

The second example was a model for the general definition (given by Jimbo and Drinfeld) of the quantum universal enveloping algebra $U_{\hbar}\mathfrak{g}$ of a simple Lie algebra $\mathfrak{g}$. Namely, Drinfeld observed that the theory of Hopf algebras gives the most appropriate description of the main relations for the monodromy matrices in the quantum inverse scattering method. Accordingly, he defined quantum groups as special Hopf algebras.

The main algebraic formulae of the quantum inverse problem method, which in particular give rise to the examples above, are of the form

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{6.4}$$

and

$$RT_1T_2 = T_2T_1R. \tag{6.5}$$

They are obtained from (5.12) and (5.17) by taking the appropriate limit

$$\lambda, \mu, \lambda - \mu \to \infty,$$

where we omit the subscript N of T_N and put $T_1 = T \otimes I$ and $T_2 = I \otimes T$ (see § 5.3). However, these relations were not used to full extent in the papers on quantum groups mentioned above. Therefore, these formulae were taken as a basis for the systematic definition of quantum Lie groups and algebras in the joint papers [56] and [59] with Reshetikhin and Takhtajan. This approach, also known as the Reshetikhin–Takhtajan–Faddeev (FRT in English) formalism, is widely used in the theory of quantum groups and its applications.

This approach is based on the following simple construction [56], [59]. For every non-singular $n^2 \times n^2$ matrix R there is a naturally related bialgebra $\mathcal{A}_R$, the *algebra of functions on the quantum matrix algebra of rank n associated with R*. Namely, let $\mathbb{C}\langle t_{ij}\rangle$ be a free associative $\mathbb{C}$-algebra with identity 1 and generators t_{ij}, $i, j = 1, \dots, n$, and let $\mathbb{I}_R$ be the two-sided ideal in $\mathbb{C}\langle t_{ij}\rangle$ generated by the relations (6.5). One defines $\mathcal{A}_R$ as the quotient algebra

$$\mathcal{A}_R = \mathbb{C}\langle t_{ij}\rangle / \mathbb{I}_R.$$

It is a bialgebra with coproduct Δ and co-unit ε which are defined on the generators t_{ij} by the formulae

$$\Delta(t_{ij}) = \sum_{k=1}^{n} t_{ik} \otimes t_{kj}, \qquad \varepsilon(t_{ij}) = \delta_{ij}, \quad i, j = 1, \dots, n,$$

that is,

$$\Delta(T) = T \dot{\otimes} T \quad \text{and} \quad \varepsilon(T) = I.$$

Furthermore, let $\mathbb{C}\langle x_1, \dots, x_n\rangle$ be the algebra of non-commutative polynomials in the variables $x_1, \dots, x_n$ and let P be the permutation matrix in $\mathbb{C}^n \otimes \mathbb{C}^n$:

$$P(u \otimes v) = v \otimes u, \qquad u, v \in \mathbb{C}^n.$$

For every polynomial $f(t)\in\mathbb{C}[t]$ we define $\mathbb{C}^n_{f,R}$, the *algebra of functions on the quantum n-dimensional vector space associated with the polynomial* $f(t)$ *and the matrix* R to be the quotient algebra

$$\mathbb{C}^n_{f,R}=\mathbb{C}\langle x_1,\dots,x_n\rangle/\mathbb{I}_{f,R},$$

where $\mathbb{I}_{f,R}$ is the two-sided ideal in $\mathbb{C}\langle x_1,\dots,x_n\rangle$ generated by the relations

$$f(\widehat{R})(x\otimes x)=0,\qquad \widehat{R}=PR,$$

and x is the column vector of the generators $x_1,\dots,x_n$. Defining an algebra homomorphism

$$\delta\colon \mathbb{C}^n_{f,R}\to\mathcal{A}_R\otimes\mathbb{C}^n_{f,R}$$

on the generators by

$$\delta(x_i)=\sum_{k=1}^n t_{ik}\otimes x_k,$$

so that $\delta(x)=T\dot{\otimes}x$, we easily see that the map δ determines a coaction of the bialgebra $\mathcal{A}_R$ on the algebra $\mathbb{C}^n_{f,R}$, endowing it with the structure of a left $\mathcal{A}_R$-comodule.

In the case when the matrix R satisfies the Yang–Baxter equation (6.4), a general method for constructing a Hopf algebra from the bialgebra $\mathcal{A}_R$ was given in [59]. Remarkably, the Yang–Baxter equation has a series of solutions[55] R_q parametrized by the simple Lie algebra $\mathfrak{g}$ and a parameter $q\in\mathbb{C}^*=\mathbb{C}\backslash\{0\}$. The non-commutative Hopf algebra corresponding to the bialgebra $\mathcal{A}_{R_q}$ is the desired *algebra of functions* $\mathrm{Fun}(G_q)$ *on the quantum group* G_q, where G is the Lie group with Lie algebra $\mathfrak{g}$. In the 'classical limit' as $q\to1$ the Hopf algebra $\mathrm{Fun}(G_q)$ becomes a commutative Hopf algebra: the algebra of polynomial functions on the simple Lie group G endowed with the additional structure of a Poisson–Lie group.

This procedure of defining the quantum groups G_q is similar to defining the classical Lie groups as algebraic groups (algebraic subvarieties in the space $M_n(\mathbb{C})$ of $n\times n$ matrices). In contrast to the case $q=1$, when all simple Lie groups are embedded in 'Her All-embracing Majesty' $M_n(\mathbb{C})$ (as expressed by Herman Weyl), the algebras $\mathcal{A}_R$ corresponding to distinct series of simple Lie algebras are non-isomorphic for $q\neq1$. This illustrates once again Faddeev's favourite principle: 'quantization removes degeneracy'. Hence, the case $q\neq1$ is more fundamental in a certain sense.

For example, the matrix $R=R_q$ corresponding to the series A_{n-1} is of the form

$$R_q=q\sum_{i=1}^n e_{ii}\otimes e_{ii}+\sum_{\substack{i,j=1,\dots,n\\ i\neq j}}e_{ii}\otimes e_{jj}+(q-q^{-1})\sum_{\substack{i,j=1,\dots,n\\ i>j}}e_{ij}\otimes e_{ji},\qquad q\in\mathbb{C}^*,$$

where the e_{ij} are the matrix units, and it satisfies the Hecke condition

$$\widehat{R}_q^2=(q-q^{-1})\widehat{R}_q+I,\qquad \widehat{R}_q=PR_q.$$

[55]The solutions R_q are obtained by a special limiting procedure as $\lambda\to\infty$ from the R-matrices $R(\lambda)$ corresponding to quantum integrable models. They play an important role in the construction of invariants of knots and links.

The bialgebra $\mathcal{A}_q = \mathcal{A}_{R_q}$ (the *algebra of functions on the matrix algebra of rank* n) coacts on the *algebra of functions on quantum* n*-dimensional Euclidean space*, the algebra $\mathbb{C}_q^n$ with generators $x_1, \dots, x_n$ and relations

$$x_i x_j = q x_j x_i, \qquad 1 \leqslant i < j \leqslant n,$$

as well as on the q*-exterior algebra of the quantum vector space* $\mathbb{C}_q^n$, the finite-dimensional algebra $\bigwedge^\bullet \mathbb{C}_q^n$ with generators $x_1, \dots, x_n$ and relations

$$x_i^2 = 0, \quad x_i x_j = -q^{-1} x_j x_i, \qquad 1 \leqslant i < j \leqslant n.$$

These algebras are obtained from $\mathbb{C}^n_{f,R_q}$ by specifying

$$f(t) = t - q \quad \text{and} \quad f(t) = t + q^{-1},$$

respectively (in the latter case, for $q^2 \neq -1$).

In general position (when q is not a root of unity), the centre of the corresponding algebra $\mathcal{A}_q$ is generated by the quantum determinant

$$\det_q T = \sum_{\sigma \in \mathrm{Sym}(n)} (-q)^{l(\sigma)} t_{1\sigma_1} \cdots t_{n\sigma_n},$$

where $l(\sigma)$ is the length (number of transpositions) of the permutation σ. We have

$$\Delta(\det_q T) = \det_q T \otimes \det_q T.$$

The quotient algebra of $\mathcal{A}_q$ by the relation $\det_q T = 1$ is called the *algebra of functions on the quantum group* $\mathrm{SL}_q(n)$ and is denoted by $\mathrm{Fun}(\mathrm{SL}_q(n))$. The bialgebra $\mathrm{Fun}(\mathrm{SL}_q(n))$ is a Hopf algebra with the antipode S,

$$S(t_{ij}) = (-q)^{i-j} \widetilde{t}_{ji}, \qquad i, j = 1, \dots, n,$$

where the $\widetilde{t}_{ij}$ are the so-called quantum cofactors,

$$\widetilde{t}_{ij} = \sum_{\sigma \in \mathrm{Sym}(n-1)} (-q)^{l(\sigma)} t_{1\sigma_1} \cdots t_{i-1\sigma_{i-1}} t_{i+1\sigma_{i+1}} \cdots t_{n\sigma_n},$$

and

$$\sigma = (\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_n) = \sigma(1, \dots, j-1, j+1, \dots, n).$$

The Hopf algebra $\mathrm{Fun}(\mathrm{GL}_q(n))$ is similarly defined as the quotient algebra of $\mathcal{A}_q\langle t\rangle$ by the relations

$$t t_{ij} = t_{ij} t, \quad t \det_q T = \det_q T \, t, \qquad i, j = 1, \dots, n,$$

and we have

$$S(t) = \det_q T, \quad S(t_{ij}) = t(-q)^{i-j} \widetilde{t}_{ji}.$$

In the simplest case when $n = 2$ the matrix R_q is of the form

$$R_q = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \tag{6.6}$$

It was used in [53]. Putting

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we obtain (6.1), where $\det_q T = ad - qbc$ and if $\det_q T = 1$, then

$$S(T) = \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix}.$$

Moreover, $\mathbb{C}_q^2$ is given by Weyl's commutation relations

$$uv = qvu$$

and the corresponding q-exterior algebra is given by the relations

$$\zeta^2 = \eta^2 = 0 \quad \text{and} \quad \zeta\eta = -q^{-1}\eta\zeta.$$

Real forms of the quantum group $\mathrm{SL}_q(n)$ arise when $|q| = 1$ and $q \in \mathbb{R}$. We obtain the quantum group $\mathrm{SL}_q(n, \mathbb{R})$ in the first case and $\mathrm{SU}_q(n)$ in the second. Quantum groups of the classical series B_n, C_n, D_n and their real forms were studied in detail in [59].

Let $\mathfrak{g}$ be the Lie algebra of a Lie group G, and $U\mathfrak{g}$ its universal enveloping algebra. Schwartz's classical relation realizes $U\mathfrak{g}$ in the form

$$U\mathfrak{g} \cong C_e^{-\infty}(G), \tag{6.7}$$

where $C_e^{-\infty}(G)$ is the algebra of distributions on G supported at the identity element e. The following general construction of quantization of simple Lie algebras, which was proposed in [56] and [59], takes (6.7) for a starting point. Consider the bialgebra $\mathcal{A}_R$, where R satisfies the Yang–Baxter equation and let $\mathcal{A}_R^* = \mathrm{Hom}(\mathcal{A}_R, \mathbb{C})$ be the dual space to $\mathcal{A}_R$. The coproduct Δ in $\mathcal{A}_R$ induces a product in $\mathcal{A}_R^*$ by the formula

$$(l_1 l_2, a) = (l_1 \otimes l_2, \Delta(a)), \qquad l_1, l_2 \in \mathcal{A}_R^*, \quad a \in \mathcal{A}_R,$$

and endows $\mathcal{A}_R^*$ with the structure of a $\mathbb{C}$-algebra. Putting

$$R^{(+)} = PRP, \quad R^{(-)} = R^{-1},$$

we let U_R denote the subalgebra of $\mathcal{A}_R^*$ generated by the elements $l_{ij}^{(\varepsilon)}$, $\varepsilon = \pm$, $i, j = 1, \dots, n$, which are defined as follows in terms of the matrix-valued functionals $L^{(\varepsilon)} = \{l_{ij}^{(\varepsilon)}\} \in M_n(\mathcal{A}_R^*)$:

$$(L^{(\varepsilon)}, T_1 \cdots T_k) = R_1^{(\varepsilon)} \cdots R_k^{(\varepsilon)}, \qquad \varepsilon = \pm, \tag{6.8}$$

where

$$T_i = I \otimes \dots \otimes \underbrace{T}_{i} \otimes \dots \otimes I \in M_n(\mathcal{A}_R)$$

and the matrices $R_i^{(\pm)}$ act non-trivially only on the 0th and ith factors in the tensor product $\mathbb{C}^n \otimes \dots \otimes \mathbb{C}^n$ ($k+1$ times) and coincide there with the matrices $R^{(\pm)}$.

By (6.4), the formula (6.8) is compatible with the relations in $\mathcal{A}_R$. The algebra U_R is a bialgebra with coproduct

$$\Delta(L^{(\varepsilon)}) = L^{(\varepsilon)}\dot{\otimes}L^{(\varepsilon)} \tag{6.9}$$

and relations

$$R^{(+)}L_1^{(\varepsilon)}L_2^{(\varepsilon)} = L_2^{(\varepsilon)}L_1^{(\varepsilon)}R^{(+)}, \tag{6.10}$$

$$R^{(+)}L_1^{(+)}L_2^{(-)} = L_2^{(-)}L_1^{(+)}R^{(+)}, \tag{6.11}$$

where

$$L_1^{(\varepsilon)} = L^{(\varepsilon)} \otimes I, \quad L_2^{(\varepsilon)} = I \otimes L^{(\varepsilon)} \in M_{n^2}(U_R), \quad \text{and} \quad \varepsilon = \pm.$$

When the R-matrix is associated[56] with a simple Lie algebra $\mathfrak{g}$, the bialgebra U_R is a Hopf algebra with the antipode

$$S(L^{(\varepsilon)}) = S_{q^{-1}}(L^{(\varepsilon)}),$$

where S_q is the antipode in the Hopf algebra $\mathrm{Fun}(G_q)$. Remarkably, the Hopf algebra U_R is the required completion of the quantum universal enveloping algebra $U_q\mathfrak{g}$. The matrices $L^{(+)}$ and $L^{(-)}$ are upper-triangular and lower-triangular respectively, and their non-zero entries are generators of the quantum Cartan–Weyl basis for $U_q\mathfrak{g}$. The complicated relations and formulae for the coproducts of the generators of the quantum Chevalley basis in the Drinfeld–Jimbo approach follow from the simple formulae (6.9)–(6.11).

When $n = 2$, we have

$$L^{(+)} = \begin{pmatrix} e^{\hbar H/2} & (q-q^{-1})X^+ \\ 0 & e^{-\hbar H/2} \end{pmatrix} \quad \text{and} \quad L^{(-)} = \begin{pmatrix} e^{-\hbar H/2} & 0 \\ -(q-q^{-1})X^- & e^{\hbar H/2} \end{pmatrix},$$

where $q = e^{\hbar}$ and H and $X^{\pm}$ are the generators of the Kulish–Reshetikhin algebra (6.2). When $|q| = 1$, we obtain a real form $U_q\mathfrak{sl}(2,\mathbb{R})$, the algebra $U_q\mathfrak{sl}(2)$ with anti-involution $*$ acting by the formulae

$$H^* = H \quad \text{and} \quad X_{\pm}^* = -X_{\pm},$$

which is equivalent by (6.3) to the formulae

$$E^* = E, \quad F^* = F, \quad \text{and} \quad K^* = K.$$

The infinite-dimensional case of loop groups and algebras was also considered in [56]. These particular examples first appeared in the framework of the inverse problem method and served as a basis for developing the notion of quantum group. In the semiclassical limit as $q \to 1$ ($\hbar \to 0$) the construction presented in [56] and [59] gives rise to Poisson structures on Lie groups and algebras, and these structures are naturally described in terms of the classical r-matrix. In particular, they give rise to Lie–Poisson groups and Lie bialgebras, which play an important

[56]Thus, $R = cR_q$, where $c = q^{-1/n}$ for the series A_{n-1} and $c = 1$ for the series B_n, C_n and D_n.

role in the Hamiltonian interpretation of the classical inverse scattering method presented in the monograph [52].

Faddeev liked the main relation (6.5) and often returned to it in his subsequent papers. He used it in [60] for the exchange algebra in the Wess–Zumino–Novikov–Witten model, to describe the quantum cotangent bundles of Lie groups in the joint paper [62] with A. Yu. Alekseev, as well as to construct lattice analogues of Kac–Moody algebras in [63] jointly with Alekseev and Semenov-Tian-Shansky. This relation was a basis for a series of joint papers [64], [65], and [67] with A. Yu. Volkov and [74] and [75] with R. M. Kashaev, which are devoted to the lattice Liouville model introduced in [53] and the general properties of quantum lattice models.

6.2. Quantum dilogarithm and modular double. Drinfeld showed that the R-matrices R_q in the previous subsection can be obtained as

$$R_q = (\rho \otimes \rho)(\mathcal{R}),$$

where $\mathcal{R} \in U_q\mathfrak{g} \otimes U_q\mathfrak{g}$ is the universal R-matrix he introduced, and ρ is a finite-dimensional representation of the algebra $U_q\mathfrak{g}$. Faddeev noted in his paper [73] (which became classical) that in the simplest case when $\mathfrak{g} = \mathfrak{sl}(2)$ Drinfeld's formula can be written as

$$\mathcal{R} = q^{(H\otimes H)/2} s_q\bigl\{-(q-q^{-1})(E\otimes F)\bigr\}, \tag{6.12}$$

where we use the generators E and F, and the well-known classical q-product

$$s_q(x) = \prod_{k=1}^{\infty}(1+q^{2k+1}x)$$

is absolutely convergent for $|q|<1$. Euler's identities of 1748,

$$\begin{aligned} s_q(x) &= 1 + \sum_{n=1}^{\infty} \frac{q^{n(n-1)/2}x^n}{(q^{-1}-q)\cdots(q^{-n}-q^n)} \\ &= \exp\biggl\{\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n(q^n-q^{-n})}\biggr\}, \quad \text{where } |x|, |q| < 1, \end{aligned}$$

show that the function $s_q(x)$ may be regarded as both the q-exponential function and Euler's q-dilogarithm!

Let U, V be a Weyl pair, that is, a pair of operators in a Hilbert space $\mathscr{H}$ satisfying

$$UV = q^2 VU.$$

The famous Schützenberger formula

$$s_q(U)s_q(V) = s_q(U+V)$$

confirms the interpretation of $s_q(U)$ as a non-commutative exponential function. The remarkable pentagonal relation

$$s_q(V)s_q(U) = s_q(U)s_q(q^{-1}UV)s_q(V),$$

which was derived by Faddeev and Volkov in [65], is a non-commutative analogue of the five-term relation for the Rogers dilogarithm (see the joint paper [66] with Kashaev).

The function $s_q(x)$ is defined only for $|q|<1$. This makes it inapplicable to the case when $|q|=1$ which corresponds to the quantum group $\mathrm{SL}_q(2,\mathbb{R})$ (see § 6.1). Faddeev [70] suggested considering the ratio

$$\frac{s_q(x)}{s_{\widetilde{q}}(\widetilde{x})}, \quad \text{where } q=e^{i\pi\tau},\ \widetilde{q}=e^{-\pi i/\tau},\ \text{and } \widetilde{x}=x^{1/\tau},$$

instead of $s_q(x)$. Thus, we put[57] $\tau=b^2$ and, following [70] and [73], consider the function

$$\Phi_b(z)=\frac{s_q(e^{2\pi bz})}{s_{\widetilde{q}}(e^{2\pi b^{-1}z})} \tag{6.13}$$

for $\operatorname{Im} b>0$ and $\operatorname{Re} b>0$. Remarkably, $\Phi_b(z)$ as a function of b extends analytically to the domain $b\in\mathbb{C}\setminus i\mathbb{R}$ and determines a meromorphic function of z. Indeed, when

$$\operatorname{Im} b>0, \quad \operatorname{Re} b>0, \quad \text{and} \quad |\operatorname{Im} z|<\frac{|\operatorname{Re}(b+b^{-1})|}{2},$$

(6.13) may be rewritten as

$$\Phi_b(z)=\exp\biggl\{-\frac{1}{4}\int_{-\infty}^{\infty}\frac{e^{-2itz}}{\sinh(bt)\sinh(b^{-1}t)}\,\frac{dt}{t}\biggr\}, \tag{6.14}$$

where the contour of integration circumvents the singularity at $t=0$ from above (see [70]). It follows from (6.14) that $\Phi_b(z)$ determines a meromorphic function of z for all b satisfying $\operatorname{Re} b\neq 0$. The function $\Phi_b(z)$ is the 'Faddeev quantum dilogarithm' or the quantum modular dilogarithm. It possesses the symmetry property

$$\Phi_b(z)=\Phi_{-b}(z)=\Phi_{b^{-1}}(z)$$

and satisfies the functional equations

$$\begin{aligned}\Phi_b(z+ib)&=(1+qe^{2\pi z})\Phi_b(z),\\ \Phi_b(z-ib)&=(1+q^{-1}e^{2\pi z})^{-1}\Phi_b(z).\end{aligned}$$

The function $\Phi_b(z)$ has an interesting history. Under the name of 'double sine' it appears in Shintani's paper on the Kronecker limit formula for real quadratic number fields and Kurokawa's paper on the theory of the Selberg zeta function. In Ruijsenaars' paper on quantum chains of Calogero–Moser type it is called the 'hyperbolic gamma function' and is expressed as a ratio of double gamma functions, which were introduced by V. P. Alekseevskii in 1889 and E. V. Barnes in 1899. Surprisingly, the Alekseevskii–Barnes function also occurs in the expressions for the S-matrix (obtained by Zamolodchikov) and the form-factors (obtained by Smirnov) of the quantum SG model.

[57] In his papers on this subject Faddeev liked to use Weierstrass' old normalization $\omega\omega'=-1/4$ for the periods 2ω, $2\omega'$ of elliptic functions. In our notation, $\omega'=ib/2$, $\omega=i/(2b)$, and $\tau=b^2$.

Faddeev's remarkable discovery is his invention of the modular double of the quantum algebra $U_q\mathfrak{sl}(2)$ and his explanation of the fundamental role of the function Φ_q.

Namely, for all $z \in \mathbb{C}$ the map

$$K \mapsto q^{-1}uv, \quad E \mapsto i\frac{v+u^{-1}z}{q-q^{-1}}, \quad F \mapsto i\frac{u+v^{-1}z^{-1}}{q-q^{-1}}$$

determines an algebra homomorphism $r_{q,z}\colon U_q\mathfrak{sl}(2) \to \mathbb{C}^2_{q^2}$, where $\mathbb{C}^2_{q^2}$ is also extended by adding the elements u^{-1} and v^{-1}. We have

$$r_{q,z}(C) = -(z+z^{-1}),$$

where

$$C = qK + q^{-1}K^{-1} + (q-q^{-1})^2 FE$$

is the quantum Casimir element, which generates the centre of $U_q\mathfrak{sl}(2)$ if q is not a root of unity. When $|q|=1$, that is, $q=e^{\pi i\tau}$ for some $\tau\in\mathbb{R}$, the algebra $\mathbb{C}^2_{q^2}$ admits a representation in the Hilbert space $\mathscr{H}=L^2(\mathbb{R})$ which sends the generators u and v to the following unitary operators U and V:

$$(Uf)(x) = e^{-2\pi ix}f(x) \quad \text{and} \quad (Vf)(x) = f(x+\tau), \qquad f\in\mathscr{H}.$$

If $\tau\notin\mathbb{Q}$, then the centralizer of the subalgebra $\rho(\mathbb{C}^2_{q^2})$ in the algebra $\mathcal{L}(\mathscr{H})$ of all bounded operators is the image of the dual subalgebra $\mathbb{C}^2_{\widetilde{q}^2}$ with $\widetilde{q}=e^{-\pi i/\tau}$ and with generators $\widetilde{u}$ and $\widetilde{v}$ under the representation $\widetilde{u}\mapsto\widetilde{U}$ and $\widetilde{v}\mapsto\widetilde{V}$, where

$$(\widetilde{U}f)(x) = e^{2\pi ix/\tau}f(x) \quad \text{and} \quad (\widetilde{V}f)(x) = f(x+1), \qquad f\in\mathscr{H}.$$

The corresponding homomorphism $r_{\widetilde{q},\widetilde{z}}$ of the dual Hopf algebra $U_{\widetilde{q}}\mathfrak{sl}(2)$ with generators $\widetilde{E}$, $\widetilde{F}$, and $\widetilde{K}$ to the dual algebra $\mathbb{C}^2_{\widetilde{q}^2}$ is given by the same formulae as above, where $\widetilde{z}=z^{1/\tau}$. It is convenient to put $z=e^{2\pi b\lambda}$, so that

$$\widetilde{z} = e^{2\pi\lambda/b}, \qquad \lambda\in\mathbb{C}.$$

An important fact (see [70]) is that the representation of the tensor product $\mathbb{C}^2_{q^2}\otimes\mathbb{C}^2_{\widetilde{q}^2}$ in $L^2(\mathbb{R})$ is already irreducible.

This basic observation led Faddeev to the notion of the modular double of the Hopf algebra $U_q\mathfrak{sl}(2)$. This is the Hopf algebra

$$\mathscr{D}_{\mathrm{mod}} = U_q\mathfrak{sl}(2)\otimes U_{\widetilde{q}}\mathfrak{sl}(2), \quad \text{where } q=e^{\pi i\tau} \text{ and } \widetilde{q}=e^{-\pi i/\tau},\ \tau\in\mathbb{C}. \tag{6.15}$$

Similarly to the q-dilogarithm, Faddeev's quantum dilogarithm satisfies a pentagonal relation, and as was shown in [73], the Hopf algebra $\mathscr{D}_{\mathrm{mod}}$ also admits a universal R-matrix, which is given by the analogue of (6.12) with s_q now replaced by the modular-invariant function Φ_q! The universal R-matrix is now defined for all $q\in\mathbb{C}^*$, including the case $|q|=1$, which is interesting for applications.

In the case $|q|=1$ the composition of $r_{q,z}\otimes r_{\widetilde{q},\widetilde{z}}$ with the unitary representations considered above of the algebras $\mathbb{C}^2_{q^2}$ and $\mathbb{C}^2_{\widetilde{q}^2}$ realizes a representation of the modular double $\mathscr{D}_{\mathrm{mod}}$ in $L^2(\mathbb{R})$. However, this representation does not satisfy the condition that $U_q\mathfrak{sl}(2,\mathbb{R})$ must be a real form, that is, the generators E, F, K and $\widetilde{E}$, $\widetilde{F}$, $\widetilde{K}$ of the modular double must be represented by self-adjoint operators acting in $L^2(\mathbb{R})$. To avoid this difficulty and construct the modular double representation for $U_q\mathfrak{sl}(2,\mathbb{R})$, Faddeev [73] proposed the use of representations of the algebras $\mathbb{C}^2_{q^2}$ and $\mathbb{C}^2_{\widetilde{q}^2}$ by unbounded operators! Thus, let the generators u, v and $\widetilde{u}$, $\widetilde{v}$ be represented by the unbounded positive-definite self-adjoint operators U, V and $\widetilde{U}$, $\widetilde{V}$ given by the following formulae[58] on appropriate domains in $L^2(\mathbb{R})$:

$$(Uf)(x)=f(x+ib), \qquad (Vf)(x)=e^{2\pi bx}f(x), \tag{6.16}$$

$$(\widetilde{U}f)(x)=f\left(x-\frac{i}{b}\right), \quad (\widetilde{V}f)(x)=e^{2\pi x/b}f(x). \tag{6.17}$$

On the common invariant domain consisting of linear combinations of the functions $P(x)e^{-x^2+cx}$, where $P(x)$ is a polynomial and $c\in\mathbb{C}$ is arbitrary, the operators U and V commute[59] with $\widetilde{U}$ and $\widetilde{V}$ and, for $\lambda\in\mathbb{R}$, determine a unitary modular-double representation of $U_q\mathfrak{sl}(2,\mathbb{R})$. Such representations correspond to principal series representations of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ and are parametrized by the real number λ. In [78], Faddeev constructed analogues of discrete series representations of the modular double $\mathscr{D}_{\mathrm{mod}}$ in another interesting case $|\tau|=1$.

Faddeev's papers on these topics became widely known and useful both in pure mathematics and applications to conformal field theory, including his favourite quantum lattice Liouville model. In this model one naturally encounters the functional-difference operator $U+U^{-1}+V$ acting in $L^2(\mathbb{R})$, where U and V are given by (6.16). In their recent joint paper [81], Faddeev and Takhtajan considered the spectral theory of this operator and proved an eigenfunction expansion theorem, which is a q-analogue of the well-known Kontorovich–Lebedev transform in the theory of special functions. An analogue of the modified Bessel functions in this case is Kashaev's wave function, the Fourier transform of the product of two of Faddeev's quantum dilogarithms.

7. Quantum field theory. Gauge fields

7.1. Yang–Mills theory. Faddeev's papers on the theory of Yang–Mills fields are perhaps the best known and most important of all his papers of the 1960s. They are also related to a very dramatic story, which Faddeev himself described vividly in his address to the General Meeting of the Russian Academy of Sciences of March 27, 2014, on the occasion of receiving the M. V. Lomonosov Medal, the highest award of the Academy of Sciences. He entitled this address "My life amid quantum fields" [80].

Faddeev's interest in quantum field theory was already aroused in his student years when Ladyzhenskaya organized a special seminar for students on the mathematical aspects of quantum field theory, where Faddeev was the main speaker. However, quantum field theory had difficulties in mid 1950s and even went out

[58]Here we use the same notation as in [81].

[59]We stress that these operators commute only on a common invariant domain. Their resolutions of the identity do not commute!

of fashion for a long time. The striking successes of quantum electrodynamics (QED) were followed by a decade of failed attempts to use field theory to calculate intranuclear forces. The final coup was the discovery of the so-called 'zero-charge paradox' by Landau and Pomeranchuk. This paradox, also known as the Landau pole problem (which is closer to the point), concerns QED, which worked perfectly (in contrast to the meson theory of nuclear forces) and enabled one to calculate subtle effects with unparalleled precision. Nevertheless, the result of an intricate (and correct!) calculation by Landau and Pomeranchuk seemed to reveal a direct logical contradiction in its foundations.[60]

In § 1.3 we discussed the joint paper [11] by Faddeev and Berezin. This was their contribution to the explanation of the zero-charge paradox. They constructed an example which, on the one hand, showed that this problem is not an artifact of perturbation theory and, on the other hand, left the hope of constructing non-trivial models of field theory that are free from the difficulties related to the Landau pole,[61] that is, models with $\beta(\overline{g}) < 0$. However, the consensus of theorists on the eve of the 1960s was completely to the opposite. In his last short paper "Fundamental problems" [108] in 1960, written just before the tragic car accident that cut short his scientific career, Landau wrote that "the Hamiltonian method for strong interaction is dead and must be buried, although of course with all deserved honours". "The brevity of life", concluded Landau, "does not allow us the luxury of spending time on problems which will lead to no new results". In the early 1960s Landau's followers treated these words of his as the Master's testament, and when Faddeev and Popov made a decisive step forward (on the basis of functional integration in the Lagrangian approach) in quantum Yang–Mills theory in 1966, no leading physics journal in the USSR would publish their paper — nor could it be published abroad (this required permission from the Department of Nuclear Physics of the USSR Academy of Sciences). As a result, Faddeev and Popov were only able to publish the preprint[62] [20] in the Kiev Theoretical Physics Institute of the Academy of Sciences of the Ukrainian SSR and to send a short two-page text [21] to the new European journal *Physics Letters*.

The geometric beauty of Yang–Mills theory was not immediately recognized. Faddeev originally wanted to study quantum gravitation, regarding Yang–Mills fields as a simpler model example. We know now that this example proved to be exceptionally successful. It made it possible to generalize QED, unify it with

[60]From the modern point of view, it says that the Gell-Mann–Low β-function is positive, whence the QED perturbation series becomes inapplicable at small distances starting with some finite value of energy/distance (such that the effective expansion constant is infinite); quantum theory is incomplete and needs redefinition. Theories with $\beta < 0$ (asymptotically free theories), where the Landau pole problem does not occur, were unknown at that time and perhaps conceived to be non-existent at all, otherwise quantum field theory would not have been 'forbidden' by Landau and his school. In the beginning of the 1970s Gross, Wilczek, and Politzer showed that the Yang–Mills theory is asymptotically free and the 'zero-charge paradox' does not occur.

[61]In his lectures in the Faculty of Physics Faddeev described this example and its relation to the Landau pole problem in detail. However, this was not included in the *Doklady* note, where the authors only presented their exact results on self-adjoint extensions of the Laplace operator. Certainly, this reflects the lack of communication between mathematical and theoretical physics, which was typical at that time: the results of theorists were supposedly beyond mathematical discourse while the results of mathematicians were often neglected by theorists.

[62]Translated into English only in 1972.

weak interactions, and construct the first systematic theory of strong interactions. Geometrically, Yang–Mills theory is nothing but a 'general theory of relativity in the charge space', and in this sense it is very close in spirit to Einstein's theory of gravitation. The idea that the general covariance principle (underlying the general theory of relativity) is related to gauge invariance in electrodynamics goes back to Hermann Weyl. The very term 'gauge transformation' has to do with Weyl's instructive mistake: in 1918 he suggested a geometric treatment of electromagnetic fields as connections with a one-dimensional Abelian structure group. But since quantum mechanics with its complex-valued wave functions had not yet been created, scaling transformations seemed to be the only possible group. Weyl conjectured that parallel translations in an electromagnetic field change lengths, and this was the origin of the artillery term 'Eichinvarianz' extending the existing artillery metaphor that started with the introduction of 'charges' (this artillery terminology was preserved in English and French, but Fock preferred to speak of 'gradient invariance'). Although Weyl's conjecture about scale changes in electromagnetic fields was completely wrong, he corrected it immediately after quantum mechanics came into being: the 'correct' structure group is the rotation group acting on the phases of wave functions. In 1928, Weyl and Fock, working independently, deduced Dirac's equation describing a charged electron in an electromagnetic field against the background of an arbitrary pseudo-Riemannian metric on space-time. Their formulae actually contained many elements of the future non-Abelian gauge theory. The next important step was taken by Heisenberg, who endowed field theory with the non-Abelian group of isotopic transformations that mix the wave functions of protons and neutrons. The idea of generalizing the global isotopic transformations and thus passing to a non-Abelian gauge theory was contained in an unpublished talk of Klein delivered some weeks before the beginning of World War II. However, non-Abelian gauge theory in the form used today was first formulated after the war, in the famous paper [128] (1954) by Yang and Mills. This theory was immediately subjected to almost devastating criticism from Pauli since, in its naïve form, it predicted the existence of a multiplet of massless charged particles which are not observed in nature. Thus, in the 1950s the Yang–Mills theory was little known and poorly understood despite its geometrical nature and beauty, and the problem of its quantization was not solved.

The first (not fully successful) attempt to construct a quantum Yang–Mills theory was made by Feynman in the early 1960s. Like Faddeev several years later, Feynman wanted to use the technique of quantum field theory in the general theory of relativity, but because of cumbersome calculations he decided, based on a suggestion by Gell-Mann, to start with the technically simpler Yang–Mills theory. Applying the ordinary methods of perturbation theory known from QED, Feynman realized that the naïve diagrammatic approach gives a non-unitary answer in the one-loop approximation. The unitarity-restoring correction could be interpreted as the contribution of an additional scalar particle. This fictitious particle behaved like a fermion (thus breaking the usual connection between spin and statistics).

Faddeev learned about Feynman's results from the script of his talk at the 1962 Warsaw conference on gravitation (published in *Acta Physica Polonica* in 1963). A problem was to explain these results outside perturbation theory and to calculate the corrections to the naïve theory beyond the one-loop approximation. To solve this problem, Faddeev and Popov [20], [21] used the technique of functional integration once proposed by Feynman himself.[63] Besides Feynman, these problems were studied by DeWitt in the mid-1960s, who was able to construct a correct quantization method for Yang–Mills fields and Einstein's gravitation theory, but did not introduce 'Faddeev–Popov ghosts' (see § 7.3). The introduction of ghosts (as well as the deep understanding, due to Faddeev's work, of the structure of quantum Yang–Mills theory) has continued to influence the subsequent development of this important topic until now. Faddeev used to say that the main contribution was due to Feynman, DeWitt, and Popov and himself.

Of course, as in QED, obtaining the diagram expansion is only the first step in the construction of a correct theory. The second, no less important step (already mentioned above) is the proof of renormalizability of the theory, and the construction of renormalized coupling constants and renormalized perturbation series. Renormalizability of the theory depends crucially on an explicit use of its gauge invariance. One can combine the renormalization problem with the mechanism of spontaneous symmetry breaking (the 'Higgs mechanism' proposed by Higgs and independently by Brout and Englert in 1964), by means of which some quanta of the Yang–Mills theory acquire mass. The gauge model of electromagnetic and weak interactions based on the Higgs mechanism was proposed by Weinberg in 1967 and changed the theorists' attitude towards gauge theories completely. As a result, the paper of Faddeev and Popov, published in the same year (after a year-long delay), immediately formed the focus of an explosive growth of gauge theory and became foundational for its further development. It was verified by 't Hooft and Veltman that the Yang–Mills theory with spontaneous gauge symmetry breaking remains renormalizable.[64] A key discovery of Gross, Wilczek, and Politzer in the early 1970s showed that this theory is free from Landau poles: the beta function is negative and interaction becomes weak at small distances. This made it possible to extend gauge theory to strong interactions. The result of this unparalleled development was the construction of the Standard Model of particle physics, marked by several Nobel Prizes. Unfortunately, Faddeev was not among the laureates. We, his students, are not alone in feeling that this is an evident injustice. An old friend of Faddeev's and the creator of the Yang–Mills theory, the great physicist Yang Chen-Ning wrote in this connection [80]:

> Many people, including myself, felt that Faddeev should have shared the Nobel Prize of 1999 with 't Hooft and Veltman. There is a strange cultural phenomenon among theoretical physicists in the 20th century: downplaying the importance of mathematics. In the 19th century, the papers and letters of Maxwell, Boltzmann, Gibbs, Kelvin, Helmholtz and Lorentz showed, if anything, the opposite value judgement about mathematics and physics. It seems that with the exuberance of the youthful Heisenberg and Pauli, there began the idea that mathematics is at best detrimental to originality in physics. Witness the sufferings and bitterness of Max Born or Wigner. Although the mature Heisenberg in his old age changed his views about mathematics, American hubris seemed to have taken over, to perpetuate

[63]In his famous paper of 1950, Feynman used path integrals to deduce diagram expansions in QED, but paradoxically he did not use this technique in later papers.

[64]Renormalizablility of the massless Yang–Mills theory was proved by A. A. Slavnov slightly earlier.

> the cultural phenomenon of downplaying the importance of mathematics. I speculate that may have been part of the reason that Faddeev was not included in the 1999 Prize.

It should be noted that mentioning Pauli in this context is not without personal bitterness: at Oppenheimer's seminar in 1954 it was Pauli who 'killed' the work of Yang, who talked about his theory. Only the youth of Yang and benevolence of Oppenheimer saved the day, and so the paper of Yang and Mills, which was to change fundamental physics, was published.

7.2. Some general facts and notation. Faddeev dreamt of doing quantum field theory from his student years.[65] After solving the quantum three-body problem, he decided to look at the difficult problem of quantizing gravitation, which greatly interested Dirac. As a model problem, Faddeev considered the theory of Yang–Mills fields, which has a simpler formulation. Apparently, he was the first to understand the geometric structure[66] of the Yang–Mills theory and use the appropriate mathematical tools.

Faddeev liked to start his lectures with the geometric formulation of Yang–Mills theory.[67] Let M_4 be the space-time with coordinates

$$x=(x^0,x^1,x^2,x^3)$$

and the Minkowski metric

$$ds^2=\eta_{\mu\nu}\,dx^\mu\,dx^\nu,\qquad \eta_{\mu\nu}=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$$

(here and in what follows we use summation over repeated indices). Let G be a compact Lie group, $\mathfrak g$ its Lie algebra, and $P\to M_4$ a principal G-bundle over M_4.[68] We write $\mathcal A$ for the infinite-dimensional affine space of connections in the adjoint bundle $\operatorname{ad}P=P\times_G\mathfrak g$, and $\mathcal G=\operatorname{Map}(M_4,G)$ for the infinite-dimensional group of gauge transformations. The gauge group $\mathcal G$ acts as follows on the space of connections $\mathcal A$:[69]

$$(g,A)\mapsto A^g=gAg^{-1}+dgg^{-1},\quad \text{where } A\in\mathcal A,\ g\in\mathcal G. \tag{7.1}$$

The curvature of a connection A is defined by

$$F=(d-A)^2=-dA+A\wedge A$$

[65]We assume that the reader is familiar with the basics of quantum field theory. See, for example, the monograph [57] by Slavnov and Faddeev and the textbook [90] by Bogolyubov and Shirkov.

[66]It should be noted that in the 1970s Simons and Yang launched a seminar on physics and mathematics devoted to the mathematical statement of gauge theories at Stony Brook University.

[67]He liked to reminisce that A. Lichnerowicz's book *Global theory of connections and holonomy groups* [114], whose Russian translation he purchased for 46 kopeks in 1964 in a second-hand bookstore on Nevsky prospect, was a helpful tool in his understanding of the geometric formulation of Yang–Mills theory.

[68]When the Yang–Mills field interacts with 'matter', one also uses the complex vector bundle $E\to M_4$ associated with a given finite-dimensional representation $\rho\colon G\to\operatorname{End}V$ of G. In physical applications to the Weinberg–Salam–Glashow Standard Model, G is equal to $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$.

[69]As accepted by the physicists and in [57], we denote the covariant derivative of a section s of E by $(d-A)s$ instead of the more commonly used notation $(d+A)s$.

and is a 2-form on M_4 with values in the bundle $\operatorname{ad} P$. In a local frame

$$A = A_\mu\,dx^\mu \quad \text{and} \quad F = \frac{1}{2}F_{\mu\nu}\,dx^\mu \wedge dx^\nu,$$

where

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu + [A_\mu, A_\nu], \qquad \partial_\mu = \frac{\partial}{\partial x^\mu}\,.$$

The Lagrangian of the Yang–Mills theory is of the form

$$\mathscr{L}(A)(x) = \frac{1}{8\mathrm{g}^2}\operatorname{Tr} F_{\mu\nu}F^{\mu\nu}, \quad \text{where} \quad F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}. \tag{7.2}$$

Here $\operatorname{Tr}\colon \operatorname{End}\mathfrak{g} \to \mathbb{R}$ is the matrix trace (the Killing form) and g plays the role of a coupling constant. Redefining $A \to \mathrm{g}A$, one can remove the coupling constant g from (7.2), but then it occurs before the non-linear term in the definition of curvature. Thus, the interaction is determined by the quadratic term of the curvature form and we can put $\mathrm{g} = 1$ without loss of generality. The Yang–Mills equations arise as the Euler–Lagrange equations for the action functional[70]

$$\mathcal{S}(A) = \int_{M_4} \mathscr{L}(A)\,d^4x \tag{7.3}$$

and have form

$$\partial_\mu F^{\mu\nu} - [A_\mu, F^{\mu\nu}] = 0.$$

The Yang–Mills theory differs significantly by its gauge invariance from the other models of quantum field theory. The history of its applications to high-energy physics is instructive. Feynman, on the advice of Weisskopf, decided to explore the quantum Einstein gravitation theory, and Gell-Mann proposed concentrating first on the more convenient Yang–Mills model. Using the standard approach of perturbation theory, Feynman found that this approach gives a non-unitary answer in the so-called one-loop approximation. Surprisingly, in his approach to Yang–Mills theory, Feynman did not use the method of functional integration, which he had himself formulated and which was used as a starting point in the Faddeev–Popov approach [20], [21] to be presented below.

7.3. Quantization of Yang–Mills fields: Lagrangian approach. According to Feynman, the matrix entries of the scattering matrix in quantum field theory are given by a path integral

$$\langle \mathrm{in}|S|\mathrm{out}\rangle = \int e^{i\mathcal{S}(A)}\,d\mu(A), \tag{7.4}$$

where the 'integration' is over all connections A satisfying certain asymptotic conditions as $t = x_0 \to \pm\infty$. These conditions characterize the 'in/out' scattering states.[71] Since the action $\mathcal{S}(A)$ and the 'integration measure'

$$d\mu(A) = \prod_{x\in M_4} dA(x) \tag{7.5}$$

[70] Under appropriate conditions of decay of $A_\mu(x)$ as $|x| \to \infty$.

[71] Feynman did not indicate how to choose these asymptotic conditions. A correct definition of the asymptotic conditions in the path integral for the S-matrix was given in Faddeev's lectures [34] at the famous physics school in Les Houches, France. Several generations of theoretical physicists grew up on these lectures. See the monograph [57] for their detailed exposition.

are invariant under gauge transformations, the path integral (7.4) has to be made well-defined not as an 'integral' over the whole space of connections $\mathcal{A}$ (which is certainly divergent because of the infinite volume of the gauge group $\mathcal{G}$), but as a path integral over the orbit space $\mathcal{A}/\mathcal{G}$. This problem was solved in [20] and [21] using a remarkable approach known as the 'Faddeev–Popov trick'.

The main idea of the Faddeev–Popov trick can be easily explained based on a finite-dimensional example. Consider an n-dimensional Riemannian manifold X with a measure $d\mu$ introduced by the volume form of the Riemannian metric. Assume that we are given a free action of a compact Lie group G by isometries on X:

$$X \times G \ni (g, x) \mapsto T_x(g) = g \cdot x \in X.$$

The Faddeev–Popov approach consists in reducing the integral

$$\int_X f(x)\, d\mu(x)$$

of a G-invariant function $f(x)$ on X to an integral over the quotient space X/G (the space of orbits $\mathcal{O}_x = G \cdot x$ of the group G).

Namely, suppose that X/G can be realized by means of a submanifold Y of X that intersects each orbit exactly once and is given by the equations $F(x) = 0$, where $F\colon X \to \mathbb{R}^k$ is a smooth map whose differential has rank k for all $x \in X$. Let dg be the Haar measure on G normalized by

$$\int_G dg = 1.$$

We define a G-invariant function $A_F(x)$ on X by

$$A_F(x) \int_G \delta(F(g \cdot x))\, dg = 1. \tag{7.6}$$

Remarkably,

$$\begin{aligned}\int_X f(x)\, d\mu(x) &= \int_X f(x) A_F(x) \delta(F(x))\, d\mu(x) &(7.7)\\ &= \int_Y f(y) \det(F_y \mathcal{T}_y)\, d\nu(y), &(7.8)\end{aligned}$$

where $d\nu$ is the measure on Y associated with the induced volume form, F_y is the differential of F at a point $y \in Y$, and $\mathcal{T}_y$ is the differential of the map $g \mapsto T_y(g)$ at the identity e of the group G.

Indeed, using the Faddeev–Popov trick ('insertion of the unity' via (7.6)), we get by Fubini's theorem that

$$\begin{aligned}\int_X f(x)\, d\mu(x) &= \int_G \biggl(\int_X f(x) A_F(x) \delta(F(g \cdot x))\, d\mu(x)\biggr)\, dg\\ &= \int_G \biggl(\int_X f(x) A_F(x) \delta(F(x))\, d\mu(x)\biggr)\, dg\\ &= \int_X f(x) A_F(x) \delta(F(x))\, d\mu(x).\end{aligned}$$

In the second equation we made a change of variables $x \mapsto g \cdot x$ and used the G-invariance of the functions $f(x)$ and $A_F(x)$ and the measure $d\mu$. This proves (7.7), and (7.8) follows since $A_F(y) = \det(F_y \mathcal{T}_y)$, which is obtained from the familiar formula for a change of variable in the δ-function.[72]

Faddeev and Popov [20], [21] applied this trick to the path integral for the S-matrix in the Yang–Mills theory.[73] Instead of the manifold X acted on by a Lie group G, we now consider the infinite-dimensional affine space $\mathcal{A}$ of connections on a principal bundle P, acted on by the infinite-dimensional gauge group $\mathcal{G}$. As in the finite-dimensional case, we assume that the quotient space $\mathcal{A}/\mathcal{G}$ can be realized as a submanifold of $\mathcal{A}$ by means of a gauge-fixing condition, for example, using the Lorentz gauge,[74]

$$\chi^L(A)(x) = \partial_\mu A^\mu(x) = 0, \qquad A^\mu = \eta^{\mu\nu} A_\nu.$$

Applying the equalities (7.7) and (7.8) formally, we have

$$\langle \mathrm{in}|S|\mathrm{out}\rangle = \int e^{i\mathcal{S}(A)} \det M^L(A) \prod_{x\in M_4} \delta(\partial_\mu A^\mu(x))\, d\mu(A), \tag{7.9}$$

where

$$M^L(A) = \partial_\mu \partial^\mu - [A^\mu, \partial_\mu], \qquad \partial^\mu = \eta^{\mu\nu}\partial_\nu,$$

is a differential operator acting on the elements u in the adjoint representation of the Lie algebra $\mathfrak{G}$ of the gauge group $\mathcal{G}$, that is, on functions $u(x)$ on M_4 with values in the adjoint representation of the Lie algebra $\mathfrak{g}$ of the group G. The regularized determinant $\det M^L(A)$ of the differential operator $M^L(A)$ (the famous Faddeev–Popov determinant) is defined, for example, using the method of the ζ-function.

Faddeev was familiar with F. A. Berezin's papers[75] about integration with respect to Grassmann (anticommuting) variables (see [87]). Remarkably, the analogue of the Gaussian integral in this approach contains the determinant of the corresponding quadratic form in the numerator instead of the denominator. Faddeev and Popov realized that the determinant $\det M^L(A)$ appearing in (7.9) can be rewritten as an integral with respect to anticommuting variables if we introduce fictitious anticommuting variables $\bar{c}(x)$ and $c(x)$ with values in the adjoint representation of the Lie algebra $\mathfrak{g}$. These are the famous Faddeev–Popov ghosts![76] Thus,

$$\det M^L(A) = \int \exp\biggl\{ i \int_{M_4} \langle \bar{c}(x), M^L(A) c(x)\rangle\, d^4x \biggr\} \prod_{x\in M_4} d\bar{c}(x)\, dc(x), \tag{7.10}$$

[72] See, for example, the monograph *Generalized functions and operations over them* [98] by Gelfand and Shilov.

[73] We refer the reader to the monograph [57] for the main definitions and concepts of quantum field theory.

[74] It must be verified that the manifold given by the gauge-fixing condition intersects each orbit of the gauge group exactly once, so that in the Lorentz gauge the operator $M^L(A)$ must always have trivial kernel. In 1978 Gribov showed that this condition does not hold on a submanifold of Yang–Mills fields of codimension zero, and then Singer proved that there is no unique choice of gauge on S^4 and 'Gribov ambiguities' always exist. However, these ambiguities do not influence the correctness of perturbation theory.

[75] Faddeev met Berezin in 1958, at Gelfand's famous seminar, and they were associated from that time by a professional friendship.

[76] Note that the Faddeev–Popov ghosts do not occur in the asymptotic states and are not subject to any boundary conditions as $t \to \pm\infty$.

where $\langle\,\cdot\,,\cdot\,\rangle$ is the Killing form in the adjoint representation of the Lie algebra $\mathfrak{g}$. Thus we obtain

$$\langle \mathrm{in}|S|\mathrm{out}\rangle = \int \exp\biggl\{i\mathcal{S}(A) + i\int_{M_4} \langle \bar{c}(x), M^L(A)c(x)\rangle\, d^4x\biggr\}\, d\mu^L(A,\bar{c},c), \tag{7.11}$$

where

$$d\mu^L(A,\bar{c},c) = \prod_{x\in M_4} \delta(\partial_\mu A^\mu(x))\, d\bar{c}(x)\, dc(x)\, d\mu(A). \tag{7.12}$$

Note that when the gauge group is Abelian (as in Maxwell's theory and QED), the operator M^L is independent of A, and therefore ghosts do not interact with the gauge field and are not needed for constructing perturbation theory. In non-Abelian theories, the Faddeev–Popov ghosts play a key role.

The formulae (7.11) and (7.12) underlie the Feynman rules for perturbation theory in the Yang–Mills theory and are now explained in all textbooks of quantum field theory. The corresponding diagram technique for perturbation theory, along with propagators of gauge fields and ghosts and all interaction vertices, was first formulated in the Kiev preprint [20] by Faddeev and Popov. The use of these Faddeev–Popov rules underlies the remarkable achievements in high-energy physics that in the 1970s led to the formulation of the Standard Model of particle physics. For example, using the Faddeev–Popov formalism, 't Hooft and Veltman proved the renormalizability of the quantum Yang–Mills theory and the Weinberg–Salam model, while Gross, Wilczek, and Politzer showed that the quantum Yang–Mills theory is asymptotically free.[77]

7.4. Feynman integral for constrained systems. After the famous paper [21] of Faddeev and Popov on quantization of gauge fields in the formalism of functional integration, theoretical physicists naturally asked whether the proposed formalism was unitary. Faddeev was well aware that using the Hamiltonian formalism instead of the Lagrangian approach to functional integration guarantees unitarity. But this was difficult because in the Hamiltonian approach the classical dynamics of Yang–Mills fields as well as Einstein's gravitation theory are generalized Hamiltonian systems with constraints. Although Dirac proposed (see [94]) a general formalism for a classical description of constrained systems and their quantization in the operator formalism, the problem of quantization of Hamiltonian systems with first-class constraints in the Feynman integral formalism in the Hamiltonian approach remained open.

In his paper [22], which became classical, Faddeev gave an elegant mathematical interpretation of Dirac's formalism for first-class constraints and used it to solve the quantization problem for such systems in the formalism of functional integration. In the context of Yang–Mills theory, this proves the unitarity of the formalism proposed in [21] and [20].

Following [22], we begin with a finite-dimensional phase space, that is, a symplectic manifold Γ with symplectic form ω. For simplicity we consider the case when $\Gamma = \mathbb{R}^{2n}$ with coordinates

$$\boldsymbol{q} = (q^1,\dots,q^n), \quad \boldsymbol{p} = (p_1,\dots,p_n),$$

[77] For these works, their authors were awarded the Nobel Prizes in physics in 1999 and 2004, respectively.

the canonical symplectic form

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq^i$$

and a real-valued function $H(\boldsymbol{p}, \boldsymbol{q})$ playing the role of the Hamiltonian of a classical system. Let $\boldsymbol{H}$ be the Hamiltonian of the corresponding quantum system. This is a self-adjoint operator in $L^2(\mathbb{R}^n)$ defined, for example, by means of Weyl quantization. According to Feynman, the matrix entries of the unitary evolution operator

$$\exp\biggl\{-\frac{i}{\hbar}(t''-t')\boldsymbol{H}\biggr\}$$

(the S-matrix is also expressible in terms of it) are given by the following path integral in the phase space:

$$\biggl\langle \boldsymbol{q}''\biggl|\exp\biggl\{-\frac{i}{\hbar}(t''-t')\boldsymbol{H}\biggr\}\biggr|\boldsymbol{q}'\biggr\rangle \\ = \int \exp\biggl\{-\frac{i}{\hbar}\int_{t'}^{t''}\Bigl(\boldsymbol{p}\dot{\boldsymbol{q}} - H(\boldsymbol{p},\boldsymbol{q})\Bigr)\,dt\biggr\} \prod_{t'\leqslant t\leqslant t''} d\boldsymbol{p}(t)\,d\boldsymbol{q}(t), \tag{7.13}$$

where

$$\boldsymbol{p}\dot{\boldsymbol{q}} = \sum_{i=1}^{n} p_i \dot{q}^i, \quad d\boldsymbol{p}(t)\,d\boldsymbol{q}(t) = \prod_{i=1}^{n} \frac{dp_i(t)\,dq^i(t)}{2\pi\hbar}$$

and the exponent on the right-hand side contains the classical action evaluated on a trajectory $(\boldsymbol{p}(t), \boldsymbol{q}(t))$, $t' \leqslant t \leqslant t''$, with conditions $\boldsymbol{q}(t') = \boldsymbol{q}'$ and $\boldsymbol{q}(t'') = \boldsymbol{q}''$, which determine special Lagrangian submanifolds of Γ (see Chap. 1 of the monograph [57] by Slavnov and Faddeev).

We now consider the case when the canonical variables $\boldsymbol{p}$ and $\boldsymbol{q}$ do not vary in the whole phase space but are subject to constraints

$$\varphi^a(\boldsymbol{p}, \boldsymbol{q}) = 0, \qquad a = 1, \dots, m, \tag{7.14}$$

which determine a smooth submanifold M of dimension $2n - m$ in the phase space. Only first-class constraints are considered in [22]. They satisfy the relations[78]

$$\{\varphi^q, \varphi^b\}(\boldsymbol{p}, \boldsymbol{q}) = \sum_{c=1}^{m} h_c^{ab}(\boldsymbol{p}, \boldsymbol{q})\varphi^c(\boldsymbol{p}, \boldsymbol{q}), \qquad a, b = 1, \dots, m, \tag{7.15}$$

for some functions $h_c^{ab}(\boldsymbol{p}, \boldsymbol{q})$, where $\{\,\cdot\,,\cdot\,\}$ is the Poisson bracket corresponding to the symplectic form ω. In other words, the Poisson brackets of first-class constraints vanish when restricted to M. We also assume that

$$\{H, \varphi^a\}(\boldsymbol{p}, \boldsymbol{q}) = \sum_{c=1}^{m} h_c^a(\boldsymbol{p}, \boldsymbol{q})\varphi^c(\boldsymbol{p}, \boldsymbol{q}) \tag{7.16}$$

[78] It follows from (7.15) that $m \leqslant n$.

for some functions $h^a_c(\boldsymbol{p},\boldsymbol{q})$, that is, the Poisson brackets of first-class constraints with the Hamiltonian H also vanish when restricted to M.

According to Dirac, the equations of motion for such a generalized Hamiltonian system are obtained from the variational principle for the generalized action

$$S(\boldsymbol{p},\boldsymbol{q},\boldsymbol{\lambda})=\int_{t'}^{t''}\biggl(\boldsymbol{p}\dot{\boldsymbol{q}}-H(\boldsymbol{p},\boldsymbol{q})-\sum_{a=1}^{m}\lambda_a(t)\varphi^a(\boldsymbol{p},\boldsymbol{q})\biggr)\,dt.$$

Besides the canonical variables $\boldsymbol{p}$ and $\boldsymbol{q}$, it also involves independent functions

$$\boldsymbol{\lambda}(t)=(\lambda_1(t),\dots,\lambda_m(t)),$$

which play the role of Lagrange multipliers. The corresponding equations of motion consist of the canonical equations

$$\dot{p}^i=-\frac{\partial H}{\partial q_i}-\sum_{a=1}^{m}\lambda_a\frac{\partial\varphi^a}{\partial q_i}\,,\quad \dot{q}_i=\frac{\partial H}{\partial p^i}+\sum_{a=1}^{m}\lambda_a\frac{\partial\varphi^a}{\partial p^i}\,,\qquad i=1,\dots,n, \tag{7.17}$$

and the conditions (7.14). It follows from (7.15) and (7.16) that a trajectory beginning on M never leaves this submanifold. Thus, (7.17) determines a transformation law for coordinates on M, and the description of this law involves not only the Hamiltonian H but also m arbitrary functions $\lambda_a(t)$. Therefore, not all the functions on M should be regarded as observables, but only those whose smooth extension f to the whole of the phase space Γ satisfies

$$\{f,\varphi^a\}(\boldsymbol{p},\boldsymbol{q})=\sum_{b=1}^{m}g^a_b(\boldsymbol{p},\boldsymbol{q})\varphi^b(\boldsymbol{p},\boldsymbol{q}) \tag{7.18}$$

for some functions $g^a_b(\boldsymbol{p},\boldsymbol{q})$. By (7.15), the conditions (7.18) are independent of the way in which we extend a given function from M to Γ. The equations of motion for such functions are of the form

$$\dot{f}=\{H,f\}+\sum_{a=1}^{m}\lambda_a\{\varphi^a,f\},$$

so that λ_a-dependent terms disappear on M.

The functions on M whose extensions to Γ satisfy (7.18) may be regarded as arbitrary functions on some *reduced phase space* Γ^* of dimension $2n-2m$. Dirac showed that one can take for Γ^* the intersection of M with the submanifold of Γ determined by the equations

$$\chi_a(\boldsymbol{p},\boldsymbol{q})=0,\qquad a=1,\dots,m,$$

which are called additional conditions. The functions χ_a must satisfy the condition

$$\det\|\{\varphi^a,\chi_b\}\|(\boldsymbol{p},\boldsymbol{q})\neq 0. \tag{7.19}$$

Faddeev [22] suggested the following elegant mathematical interpretation of Dirac's formalism.[79] Let $\widetilde{\omega}$ be a restriction of the symplectic form ω to M, and let

[79] This was aided by Faddeev's conversations with his old friend V. I. Arnold.

X_f be a Hamiltonian vector field corresponding to the function f on Γ. The condition (7.15) is equivalent to saying that all vectors $X_{\varphi^a}(m)$, $m \in M$, are tangent to M and are zero-vectors of the 2-form $\widetilde{\omega}$. The condition $d\widetilde{\omega} = 0$ means that these vectors form an involutive distribution P on M. By the Frobenius theorem, this distribution determines a foliation on M whose leaves are integral manifolds of P. The relations (7.18) mean that f is constant along these leaves. The additional conditions determine a submanifold Γ^* of M, and the condition (7.19) means that Γ^* is transversal to the integral manifolds of P and the restriction of $\widetilde{\omega}$ to Γ^* is non-degenerate. In the case when each integral manifold intersects Γ^* at a single point, this foliation is a fibration and the reduced phase space Γ^* is its base.

An important particular case of this construction is the method of Hamiltonian reduction, which is widely used in modern mathematics. Thus, consider a Hamiltonian action ρ of a compact Lie group G on a symplectic manifold Γ. In other words, the action of G preserves the symplectic form ω, and the corresponding action of the Lie algebra $\mathfrak{g}$ is given by Hamiltonian vector fields. For every $\xi \in \mathfrak{g}$ we have a vector field

$$X_{H_\xi} = \{H_\xi, \cdot\} \quad \text{on } \Gamma, \quad \text{where} \quad dH_\xi = i_{\rho(\xi)}\omega.$$

We assume the functions H_ξ to be chosen so that the resulting map $\mathfrak{g} \to C^\infty(\Gamma)$ is a homomorphism of Lie algebras:

$$H_{[\xi,\eta]} = \{H_\xi, H_\eta\} \quad \text{for all } \xi, \eta \in \mathfrak{g}. \tag{7.20}$$

In this situation the moment map $\mu\colon \Gamma \to \mathfrak{g}^*$ is defined by

$$\mu(\xi) = H_\xi, \qquad \xi \in \mathfrak{g},$$

where $\mathfrak{g}^*$ is the dual space to the Lie algebra $\mathfrak{g}$. Suppose that 0 is a regular value of the map μ. Then $\mu^{-1}(0)$ is a smooth submanifold of Γ. If the action of G on $\mu^{-1}(0)$ is free, then the quotient space $\mu^{-1}(0)/G$ is a smooth manifold endowed with a natural symplectic form, whose pull-back on $\mu^{-1}(0)$ under the projection map is equal to the restriction of ω to $\mu^{-1}(0)$. The symplectic manifold $\mu^{-1}(0)/G$ is denoted by $\Gamma//G$ and is called the *symplectic quotient* or *Marsden–Weinstein quotient* (see [117]). The procedure itself is referred to as Hamiltonian reduction. Hamiltonian reduction is a particular case of Dirac's formalism when the constraints φ^a are the Hamiltonians H_{ξ_a} of the vector fields $\rho(\xi_a)$, where the ξ_a are generators of the Lie algebra $\mathfrak{g}$. Then $\mu^{-1}(0) = M$, and the additional conditions characterize an embedding of M/G in M that determines the reduced phase space Γ^*.

In the case when

$$\{\chi_a, \chi_b\}(\boldsymbol{p}, \boldsymbol{q}) = 0, \qquad a, b = 1, \dots, m, \tag{7.21}$$

canonical variables on Γ^* can be introduced in the following simple way. By (7.21), one can choose new coordinates on Γ in which the additional conditions take the form

$$\chi_a(\boldsymbol{p}, \boldsymbol{q}) = p_a,$$

where p_a now stands for certain canonical momenta of the new system of variables. Let q^a be the coordinates conjugate to them, and let

$$\boldsymbol{p}^* = (p^*_1, \dots, p^*_{2n-2m}), \quad \boldsymbol{q}^* = (q^{*1}, \dots, q^{*2n-2m})$$

be the other canonical variables. Then (7.19) is written in the new variables as

$$\det\left\|\frac{\partial\varphi^a}{\partial q^b}\right\|(\boldsymbol{p},\boldsymbol{q})\neq 0.$$

Hence the equations (7.14) can be solved with respect to q^a. As a result, the submanifold Γ^* is given by the equations

$$p_a=0,\quad q^a=q^a(\boldsymbol{p}^*,\boldsymbol{q}^*),\qquad a=1,\dots,m,$$

and $\boldsymbol{p}^*$, $\boldsymbol{q}^*$ are canonical coordinates for the restriction of the symplectic form ω to Γ^*:

$$\omega\big|_{\Gamma^*}=\sum_{i=1}^{2n-2m} dp_i^*\wedge dq^{*i}.$$

Thus, the generalized Hamiltonian system on the phase space Γ with the canonical coordinates $\boldsymbol{p}$, $\boldsymbol{q}$, constraints (7.14), and Hamiltonian $H(\boldsymbol{p},\boldsymbol{q})$ is reduced to a Hamiltonian system on the reduced phase space Γ^* with the canonical coordinates $\boldsymbol{p}^*$, $\boldsymbol{q}^*$ and Hamiltonian

$$H^*(\boldsymbol{p}^*,\boldsymbol{q}^*)=H(\boldsymbol{p},\boldsymbol{q})\big|_{\Gamma^*}.$$

According to Feynman, its quantization is given by the path integral (7.13):

$$\int\exp\biggl\{-\frac{i}{\hbar}\int_{t'}^{t''}\bigl(\boldsymbol{p}^*\dot{\boldsymbol{q}}^*-H^*(\boldsymbol{p}^*,\boldsymbol{q}^*)\bigr)\,dt\biggr\}\prod_{t'\leqslant t\leqslant t''}d\boldsymbol{p}^*(t)\,d\boldsymbol{q}^*(t).\tag{7.22}$$

However, finding the canonical variables on the reduced phase space is a very difficult problem, which can rarely be solved exactly.

Remarkably, Faddeev proved that the expression (7.22) can be rewritten exactly in terms of the generalized constrained Hamiltonian system on Γ by the following formula:

$$\begin{aligned}\int\exp\biggl\{-\frac{i}{\hbar}\int_{t'}^{t''}\Bigl(\boldsymbol{p}\dot{\boldsymbol{q}}-H(\boldsymbol{p},\boldsymbol{q})-\sum_{a=1}^{m}\lambda_a(t)\varphi^a(\boldsymbol{p},\boldsymbol{q})\Bigr)\,dt\biggr\}\prod_{t'\leqslant t\leqslant t''}d\boldsymbol{\lambda}(t)\,d\mu(\boldsymbol{p}(t),\boldsymbol{q}(t))\\ =\int\exp\biggl\{-\frac{i}{\hbar}\int_{t'}^{t''}\bigl(\boldsymbol{p}\dot{\boldsymbol{q}}-H(\boldsymbol{p},\boldsymbol{q})\bigr)\,dt\biggr\}\prod_{t'\leqslant t\leqslant t''}d\widetilde{\mu}(\boldsymbol{p}(t),\boldsymbol{q}(t)),\end{aligned}\tag{7.23}$$

where

$$d\boldsymbol{\lambda}=\prod_{a=1}^{m}\frac{d\lambda_a}{2\pi},\qquad d\mu(\boldsymbol{p},\boldsymbol{q})=(2\pi)^m\prod_{a=1}^{m}\delta(\chi_a)\det\|\{\varphi^a,\chi_b\}\|(\boldsymbol{p},\boldsymbol{q})\,d\boldsymbol{p}\,d\boldsymbol{q},$$

$$d\widetilde{\mu}(\boldsymbol{p},\boldsymbol{q})=d\mu(\boldsymbol{p},\boldsymbol{q})\prod_{a=1}^{m}\delta(\varphi^a),$$

and $\delta(F)$ is the δ-function of the hypersurface $F=0$ in Γ (for example, see [98]). The formula (7.23) is the desired generalization of Feynman's path integral for constrained systems. An important result proved in [22] says that the expression (7.23) is independent of the choice of the additional conditions $\chi_a(\boldsymbol{p},\boldsymbol{q})=0$.

7.5. Quantization of Yang–Mills fields: Hamiltonian approach. The formula (7.23) can be generalized to the infinite-dimensional case and enables one to write the path integral for Yang–Mills fields in the Hamiltonian approach. Following [22], we suppose that G is a simple compact Lie group of dimension n and the t^a are generators of its Lie algebra $\mathfrak{g}$ in the adjoint representation such that

$$\operatorname{Tr} t^a t^b = -2\delta^{ab}, \qquad a, b = 1, \dots, n. \tag{7.24}$$

It follows from (7.24) that the structure constants f^{abc} of the Lie algebra $\mathfrak{g}$,

$$[t^a, t^b] = \sum_{c=1}^{n} f_c^{ab} t^c,$$

form a totally antisymmetric tensor. So they can be written as $f^{abc} = f_c^{ab} = f_{bc}^a$.

We put

$$A_\mu(x) = \sum_{a=1}^{n} A_\mu^a(x) t^a \quad \text{and} \quad F_{\mu\nu}(x) = \sum_{a=1}^{n} F_{\mu\nu}^a(x) t^a.$$

The action (7.2)–(7.3) of the Yang–Mills theory can be rewritten in the form

$$\mathcal{S}(A) = -\frac{1}{2} \sum_{k=1}^{3} \int_{M_4} \operatorname{Tr}\Big(E_k \partial_0 A_k - \frac{1}{2}(E_k^2 + B_k^2) + A_0 G \Big)\, d^4 x, \tag{7.25}$$

where

$$E_k = F_{k0}, \quad B_k = -\frac{1}{2} \sum_{i,j=1}^{3} \varepsilon_{ijk} F_{ij}, \quad G = \sum_{k=1}^{3} (\partial_k E_k - [A_k, E_k]), \qquad k = 1, 2, 3,$$

and ε_{ijk} is the totally antisymmetric tensor with $\varepsilon_{123} = 1$. Taking the Legendre transform of (7.25), we obtain an infinite-dimensional generalized Hamiltonian system with first-class constraints. Thus, its phase space is the Fréchet space

$$\mathscr{X} = \mathscr{S}(\mathbb{R}^3, \mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^3, \mathbb{R}^n)$$

with canonical coordinates $E_k^a(\boldsymbol{x})$, $A_k^a(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^3$, and symplectic form

$$\Omega = \sum_{k=1}^{3} \sum_{a=1}^{n} \int_{\mathbb{R}^3} \mathrm{d}E_k^a(\boldsymbol{x}) \wedge \mathrm{d}A_k^a(\boldsymbol{x})\, d^3\boldsymbol{x}.$$

The Hamiltonian H and the constraints $G^a(\boldsymbol{x}) = 0$ (referred to as the Gauss law in the physics literature) are given by

$$H = \frac{1}{2} \sum_{k=1}^{3} \int \operatorname{Tr}\big(E_k^2(\boldsymbol{x}) + B_k^2(\boldsymbol{x})\big)\, d^3\boldsymbol{x} \tag{7.26}$$

and

$$G^a(\boldsymbol{x}) = \sum_{k=1}^{3} \big(\partial_k E_k^a(\boldsymbol{x}) - f_{bc}^a A_k^b(\boldsymbol{x}) E_k^c(\boldsymbol{x})\big), \qquad a = 1, \dots, n. \tag{7.27}$$

The variables $A^a_0(\boldsymbol{x})$ play the role of Lagrange multipliers. Making careful use of the canonical Poisson brackets

$$\{E^a_k(\boldsymbol{x}), A^b_l(\boldsymbol{y})\} = \delta_{kl}\delta^{ab}\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{7.28}$$

one can show that the Poisson brackets of the Hamiltonian and the constraints take the form (7.15)–(7.16):

$$\{H, G^a(\boldsymbol{x})\} = 0 \quad \text{and} \quad \{G^a(\boldsymbol{x}), G^b(\boldsymbol{y})\} = \sum_{c=1}^{n} f^{ab}_c G^c(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{y}), \quad a, b = 1, \dots, n.$$

The phase space $\mathscr{X}$ is acted on by the gauge group $\mathcal{G} = \mathrm{Map}(\mathbb{R}^3, G)$:

$$\boldsymbol{A}^g = g\boldsymbol{A}g^{-1} + dgg^{-1}, \quad \boldsymbol{E}^g = g\boldsymbol{E}g^{-1}, \qquad g \in \mathcal{G}, \tag{7.29}$$

where

$$\boldsymbol{A}(\boldsymbol{x}) = \sum_{a=1}^{n} A^a_k(\boldsymbol{x})t^a\, dx^k, \quad \boldsymbol{E}(\boldsymbol{x}) = \sum_{a=1}^{n} E^a_k(\boldsymbol{x})t^a\, dx^k.$$

The action of $\mathcal{G}$ on $\mathscr{X}$ is Hamiltonian. The Hamiltonians corresponding to the elements $u = \sum_{a=1}^n u^a(\boldsymbol{x})t^a$ of the Lie algebra $\mathfrak{G}$ of $\mathcal{G}$ are the functionals

$$H_u = \sum_{a=1}^{n} \int_{\mathbb{R}^3} G^a(\boldsymbol{x})u^a(\boldsymbol{x})\, d^3\boldsymbol{x}, \tag{7.30}$$

which satisfy the Poisson brackets (7.20):

$$\{H_u, H_v\} = H_{[u,v]}.$$

Thus, the Dirac formalism presented above is completely applicable to the Yang–Mills theory if the additional condition is chosen to be the so-called Coulomb gauge

$$\chi^C_a(\boldsymbol{x}) = \sum_{k=1}^{3} \partial_k A^a_k(\boldsymbol{x}) = 0.$$

It follows from (7.28) that the distribution $\{G^a(\boldsymbol{x}), \chi_b(\boldsymbol{y})\}$ is the kernel $M^C(\boldsymbol{x},\boldsymbol{y})^a_b$ of the differential operator

$$M^C(A)^a_b = -\delta^a_b\Delta + \sum_{c=1}^{n}\sum_{k=1}^{3} f^a_{bc}A^c_k(\boldsymbol{x})\partial_k, \tag{7.31}$$

where

$$\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$$

is the Laplace operator in $\mathbb{R}^3$. Thus, the regularized determinant $\det M^C(A)$ of the operator $M^C(A)$ is an analogue of $\det\|\{\varphi^a, \chi_b\}\|$ in the finite-dimensional case.

The path integral for the S-matrix in the Hamiltonian approach finally takes the form

$$\int \exp\left\{-\frac{i}{2}\sum_{k=1}^{3}\int \mathrm{Tr}\left(E_k\partial_0 A_k - \frac{1}{2}(E_k^2 + B_k^2) + A_0 G\right) d^4x\right\} d\mu(A, E),$$

where

$$d\mu(A,E) = d\mu(A)\det M^C(A) \prod_{x\in M_4}\prod_{a=1}^{n}\prod_{i=1}^{3}\delta(\chi_a^C(x))\,dE_i^a(x)$$

and the 'measure' $d\mu(A)$ was introduced in (7.5). Performing the Gaussian integration with respect to the variables $E_i^a(\boldsymbol{x})$, we obtain

$$\begin{aligned}\langle \mathrm{in}|S|\mathrm{out}\rangle &= \int \exp\{i\mathcal{S}(A)\}\det M^C(A)\prod_{x\in M_4}\prod_{a=1}^{n}\delta(\chi_a^C(x))\,d\mu(A)\\ &= \int \exp\biggl\{i\mathcal{S}(A) + i\int_{M_4}\langle \bar{c}(x), M^C(A)c(x)\rangle\,d^4x\biggr\}\,d\mu^C(A,\bar{c},c),\end{aligned} \tag{7.32}$$

where we have introduced the Faddeev–Popov ghosts and put

$$d\mu^C(A,\bar{c},c) = \prod_{x\in M_4}\prod_{a=1}^{n}\delta(\chi_a^C(x))\,d\bar{c}(x)\,dc(x)\,d\mu(A). \tag{7.33}$$

The path integral (7.32)–(7.33) generates Feynman's diagrams of perturbation theory for the S-matrix, which satisfies the unitarity condition by construction. Although the Coulomb gauge is not explicitly Lorentz-invariant, Faddeev proved in [22] that

$$\begin{aligned}\langle \mathrm{in}|S|\mathrm{out}\rangle &= \int e^{i\mathcal{S}(A)}\det M^L(A)\prod_{x\in M_4}\delta(\partial_\mu A^\mu(x))\,d\mu(A)\\ &= \int e^{i\mathcal{S}(A)}\det M^C(A)\prod_{x\in M_4}\prod_{a=1}^{n}\delta(\chi_a^C(x))\,d\mu(A),\end{aligned}$$

so that the resulting perturbation theory is Lorentz-invariant.

In other words, Faddeev [22] deduced Faddeev–Popov quantization [21] using the manifestly unitary Hamiltonian approach. The paper [22], which opened the first issue of the new journal *Theoretical and Mathematical Physics*, founded by Bogolyubov in 1969, immediately became classical. It had a huge influence not only on the development of the theory of gauge fields but also on theoretical physics as a whole.

7.6. Quantum anomalies. Quantum anomalies in the four-dimensional quantum field theory were discovered at the end of the 1960s[80] by Adler, Bell, and Jackiw. The presence of an anomaly means that the gauge symmetry of classical theory is not preserved under quantization, being destroyed by quantum corrections.[81]

More precisely, a quantum (gauge) anomaly arises in the following situation. Consider a principal G-bundle $P\to M^{2n}$ over a $2n$-dimensional manifold M^{2n}

[80] Before that, in 1949 Steinberger (a future Nobel laureate in experimental physics) studied the neutral pion decay and made a calculation where he foresaw the existence of a quantum gauge anomaly.

[81] A basic principle of internal self-consistency in quantum field theory (for example, in the Standard Model) is the absence of anomalies.

(this manifold plays the role of space-time), where G is again a compact Lie group. Let $E \to M^{2n}$ be the complex vector bundle associated with a finite-dimensional representation of G. We write $\mathcal{A}$ for the infinite-dimensional affine space of connections on E. A quantum anomaly is the assertion that the partition function of the theory of Weyl fermions in an external gauge field $A \in \mathcal{A}$, which is formally given by the path integral[82]

$$Z(A) = \int \exp\biggl\{ i \int_{M^{2n}} \psi^\dagger(x) \not{D}(A) \psi(x)\, d^{2n}x \biggr\} \prod_{x \in M^{2n}} \mathscr{D}\psi(x)\, \mathscr{D}\psi^\dagger(x), \tag{7.34}$$

is not invariant under gauge transformations[83]

$$A \mapsto A^g = g^{-1}Ag + g^{-1}\,dg, \quad \text{where} \quad g \in \mathcal{G} = \operatorname{Map}(M^{2n}, G).$$

Here $\not{D}(A)$ is the projection of the Dirac operator in the external field onto the Weyl spinors. Physicists like to say that a quantum gauge anomaly is the assertion of the non-invariance of the fermionic integration measure in (7.34), since the integrand does not change under the gauge transformation

$$A \mapsto A^g, \quad \psi \mapsto g^{-1}\psi, \quad \psi^\dagger \mapsto \psi^\dagger g, \qquad g \in \mathcal{G}.$$

Mathematically, (7.34) should be understood as a 'regularized determinant' of the operator $\not{D}(A)$, where $Z(A)$ is not a function but a section of the determinant bundle over $\mathcal{A}$. Thus, $Z(A)$ is a 'square root' of the determinant of the Dirac operator in the external gauge field A. One easily sees that there is a regularization (for example, using the ζ-function) under which the determinant of the Dirac operator is gauge invariant. It follows that the partition function $Z(A)$ is defined up to a phase.

The presence of an anomaly may be expressed by the simple and beautiful formula

$$Z(A^g) = \exp\{-iW(A,g)\} Z(A), \qquad g \in \mathcal{G}, \tag{7.35}$$

where $W(A,g)$ is the so-called Wess–Zumino action functional (it is defined only modulo $2\pi\mathbb{Z}$). We note that the physicists derived an infinitesimal version of this formula first. Thus, let $\mathfrak{G}$ be the Lie algebra of the gauge group $\mathcal{G}$ acting on $\mathcal{A}$ by

$$A \mapsto du + [A,u], \qquad u \in \mathfrak{G}.$$

Putting $g = e^{tu}$ in (7.35) and differentiating with respect to t at $t=0$, we obtain

$$\int_{M^{2n}} \biggl\langle \nabla_A \frac{\delta Z(A)}{\delta A(x)} + i\mathfrak{A}(x), u(x) \biggr\rangle\, d^{2n}x = 0$$

for all $u \in \mathfrak{G}$. Here $\delta/\delta A(x)$ is the Fréchet derivative, $\langle\,\cdot\,,\cdot\,\rangle$ is the Killing form in the given representation of G, and

$$\frac{d}{dt}\bigg|_{t=0} W(A, e^{tu}) = \int_{M^{2n}} \langle \mathfrak{A}(x), u(x) \rangle\, d^{2n}x.$$

[82] In the Euclidean path integral one should replace $\psi^\dagger$ by $\chi^\dagger$, a ψ-independent Weyl fermion of opposite chirality.

[83] Here, in contrast to § 7.3 and 7.5, we follow [48] and write connections in the form $d + A$ while A^g stands for the right action of the gauge group $\mathcal{G}$.

Using the generators t^a of the Lie algebra $\mathfrak{g}$ and the coefficients of the connection $A^a_\mu(x)$, we can rewrite the resulting relation as

$$(T^a(x)+i\mathfrak{A}^a(x))Z(A)=0, \tag{7.36}$$

where the polynomial $\mathfrak{A}^a(x)$ in the connection coefficients A^a_μ and their first derivatives at $x\in M^{2n}$ is the gauge anomaly, and

$$T^a(x)=-(\nabla_A)_\mu\frac{\delta}{\delta A^a_\mu(x)}=-\partial_\mu\frac{\delta}{\delta A^a_\mu}-\sum_{c=1}^{n}f^{ab}_c A^c_\mu\frac{\delta}{\delta A^b_\mu} \tag{7.37}$$

is the generator of gauge transformation in the space of functionals on $\mathcal{A}$. The generators $T^a(x)$ are the Gauss law operators in the $(2n{+}1)$-dimensional Yang–Mills theory.

The presence of divergences in quantum field theory inevitably gives rise to various regularization schemes. Changing the regularization scheme is expressed by the change

$$Z(A)\mapsto \exp\{i\beta(A)\}Z(A),$$

where the 'counterterm' $\beta(A)$ is a local functional (given by the integral over M^{2n} of some density which is locally determined by the coefficients of the connection A). Then we have

$$W(A,g)\to W(A,g)+\delta\beta(A,g),\quad \text{where}\quad \delta\beta(A,g)=\beta(A^g)-\beta(A). \tag{7.38}$$

If $W(A,g)$ can be cancelled out by the redefinition (7.38), then the section $Z(A)$ is well defined and invariant. Such a cancellation is possible for some representations of G, but not in the general situation. Thus, there is generally no regularization scheme that would preserve the gauge symmetry in the path integral (7.34) over Weyl fermions.

The formula (7.36) attracted the attention of Faddeev and his young student Shatashvili. In the mid-1980s they began to study the mathematical aspects of anomalies in gauge theories. Their first paper [48] on this subject already contained the key observation that the Wess–Zumino action is a 1-cocycle on the gauge transformation group $\mathcal{G}$ (and the anomaly is a 1-cocycle on the Lie algebra $\mathfrak{G}$) which acts on the functionals on the space $\mathcal{A}$ of Yang–Mills fields! This discovery was very influential in the development of this important theme, which is of current interest because of the need to study gauge and gravitation anomalies in quantum field theories with space-time of dimension greater than 4 and in superstring theory. The so-called 'descent method' for calculating cocycles in the gauge group was proposed in [48] and became popular.[84] The authors of [48] also found the 2-cocycle corresponding to the Abelian extension of the infinite-dimensional Lie algebra (the so-called 'three-dimensional current algebra') of equal-time commutation relations for the quantum Gauss law in four-dimensional space-time. Faddeev was very proud of having applied to quantum field theory results of his father, Dmitry Konstantinovich Faddeev, who discovered group cohomology in the 1940s!

[84] This descent procedure, combined with a result of Álvarez-Gaumé and Witten on gravitation anomalies (1983), was used in the famous paper by Green and Schwarz to prove the existence of mathematically self-consistent models of superstring theory.

Namely, let G be an abstract group and M a right G-module. Consider the complex $C^\bullet(G,M)$ with cochains $C^n(G,M)$, the spaces of functions on $M\times G^n$, $n=0,1,2,\dots$, and with the differential

$$\delta_n\colon C^n(G,M)\to C^{n+1}(G,M)$$

defined by the formula

$$\begin{aligned}(\delta_n\alpha_n)(m,g_1,\dots,g_{n+1}) = \alpha_n(m\cdot g_1,g_2,\dots,g_{n+1})\\ +\sum_{i=1}^{n}(-1)^i\alpha_n(m,g_1,\dots,g_{i-1},g_ig_{i+1},g_{i+2}\dots,g_{n+1})\\ +(-1)^{n+1}\alpha_n(m;g_1,\dots,g_n),\end{aligned} \tag{7.39}$$

so that $\delta_{n+1}\circ\delta_n=0$. This complex was used in [48] in the case when G is the group $\mathcal{G}$ of gauge transformations acting on the space $M=\mathcal{A}$ of Yang–Mills fields (connections on a bundle $E\to M^{2n}$) and the cochains $\alpha_n(A,g_1,\dots,g_n)$ are local functionals of A and g. Comparing these formulae with (7.35), we arrive at the key observation made in [48]: the Wess–Zumino action $W(A,g)$ is a 1-cocycle with non-zero cohomology class on the gauge group $\mathcal{G}$, and its infinitesimal version $\mathfrak{A}^a(A)$ is a 1-cocycle on the corresponding Lie algebra $\mathfrak{G}$. In the context of the action of $\mathcal{G}$ on functionals of A, which is given by

$$U(g)\Psi(A)=e^{iW(A,g)}\Psi(A^g),$$

the gauge invariance is restored:

$$U(g)Z(A)=Z(A).$$

It was conjectured in [48] (and discussed in detail in [49]) that the corresponding 2-cocycle should arise in the Hamiltonian approach to the full quantum theory, where the gauge field A is not external but dynamical. The corresponding generators of gauge transformations occurring in the Gauss law can be obtained from the generators $T^a(x)$ in (7.37) by means of a shift by a generator of gauge transformations in the fermionic Fock space, that is, by the fermionic current $J^a(x)$:

$$G^a(x)=T^a(x)+J^a(x),\quad \text{where}\quad J^a(x)=\psi^\dagger(x)t^a\psi(x),$$

and they realize a projective representation of the Lie algebra $\mathfrak{G}$.

To construct the Wess–Zumino action (that is, the 1-cocycle) and the 2-cocycle related to the Gauss law, the paper [48] suggested a completely general and beautiful method using the bicomplex with operators δ and d (d is the exterior differentiation operator arising in the theory of Chern–Simons secondary characteristic classes[85]).

[85]This approach was partially motivated by works about three-dimensional Abelian gauge theories with additional Chern–Simons terms by Deser, Jackiw, and Templeton in the early 1980s. The subject of [48] is the physical non-Abelian Yang–Mills theory in odd-dimensional spaces with an additional Chern–Simons action, and it was analysed in the framework of the Hamiltonian approach. The method of [48] is used in modern investigations in solid state physics, high-energy physics, and string theory.

Namely, we begin with a manifold M^{2n+2} of dimension $2n+2$ and consider an invariant polynomial[86] $w_{2n+2}(F)$ of degree $n+1$ in the curvature form

$$F = dA + A^2$$

of a connection A. Since the form $\omega_{2n+2} = w_{2n+2}(F)$ satisfies

$$d\omega_{2n+2}(F) = 0 \quad \text{and} \quad \delta\omega_{2n+2}(F) = 0, \tag{7.40}$$

locally on M^{2n+2}, we can find a differential form ω_{2n+1} of degree $2n+1$ such that

$$d\omega_{2n+1} = \omega_{2n+2}.$$

According to Novikov, globally on M^{2n+2} it is a multivalued form

$$\omega_{2n+1} = d^{-1}\omega_{2n+2}.$$

Let $M^{2n+1} = \partial B^{2n+2}$ be a $(2n+1)$-dimensional cycle in M^{2n+2} which is the boundary of a $(2n+2)$-dimensional submanifold B^{2n+2} of M^{2n+2}. The Chern–Simons functional $I_{\mathrm{CS}}(A)$ for Yang–Mills fields on M^{2n+1} is defined by

$$I_{\mathrm{CS}}(A) = 2\pi \int_{B^{2n+2}} \omega_{2n+2}(A') = 2\pi \int_{M^{2n+1}} \omega_{2n+1}(A), \tag{7.41}$$

where one chooses any extension A' of the connection A from M^{2n+1} to B^{2n+2}. For two such extensions, the difference

$$\int_{B^{2n+2}} \omega_{2n+2}(A') - \int_{B^{2n+2}} \omega_{2n+2}(A'')$$

is the integral of ω_{2n+2} over a top-dimensional cycle in M^{2n+2}. Therefore, assuming the integrality of the cohomology class

$$[\omega_{2n+2}] \in H^{2n+2}(M^{2n+2}, \mathbb{R}),$$

we see that the functional $e^{iI_{\mathrm{CS}}(A)}$ is well defined and depends only on the values of the gauge field on M^{2n+1}. It is easily verifiable that the $(2n+1)$-form

$$\delta\omega_{2n+1} = \omega_{2n+1}(A^g) - \omega_{2n+1}(A)$$

is closed:

$$d\delta\omega_{2n+1} = d\delta d^{-1}\omega_{2n+2} = \delta\omega_{2n+2} = 0. \tag{7.42}$$

It follows that, in contrast to the Chern–Simons action (7.41), its gauge variation

$$\alpha_1(A, g) = I_{\mathrm{CS}}(A^g) - I_{\mathrm{CS}}(A)$$

is well defined (modulo $\mathbb{Z}$) also in the case when M^{2n+1} is not a cycle but a chain with boundary: $\partial M^{2n+1} = M^{2n}$. Hence

$$\alpha_1(A, g) = 2\pi \int_{M^{2n+1}} \delta\omega_{2n+1}(A) = 2\pi \int_{M^{2n}} d^{-1}\delta\omega_{2n+1}(A) \tag{7.43}$$

[86] For example, $\mathrm{Tr}(iF/(2\pi))^{n+1}$.

is the required 1-cocycle, $\delta\alpha_1 = 0 \bmod \mathbb{Z}$. This means that α_1 is a multivalued functional in the sense of Novikov.

Remarkably, $e^{i\alpha_1(A,g)}$ depends only on the values of A and g on M^{2n} and does not depend on their extension to M^{2n+1}. Indeed, it follows from an explicit calculation that the dependence on A in (7.43) is polynomial and is a local functional of A on M^{2n}. However, the A-independent term $\alpha_1(0,g)$ in (7.43) is non-local and cannot be written only in terms of an element of the gauge group on M^{2n}. This term is the famous Wess–Zumino–Novikov–Witten action, which is proportional in our notation to

$$\int_{M^{2n}} d^{-1}\,\mathrm{Tr}(dgg^{-1})^{2n+1}. \tag{7.44}$$

Similarly to the Chern–Simons action, the multivaluedness of the action (7.44) disappears under exponentiation: the difference of two extensions from M^{2n} to M^{2n+1} is proportional to

$$\int_{M^{2n+1}} \mathrm{Tr}(dgg^{-1})^{2n+1}$$

and hence is an integer (the coefficient of proportionality is determined by the initial normalization of ω_{2n+2}).

The descent procedure just described can be continued. At the $(k+1)$st step we obtain a closed $(2n+1-k)$-form

$$\delta d^{-1}\delta\cdots d^{-1}\delta d^{-1}\omega_{2n+2},$$

and its integral over a cycle of appropriate dimension is a k-cocycle. Explicit formulae for the group cocycles and Lie-algebraic cocycles were first obtained in [48]. For example, the formula for $k=2$ and $n=2$ is

$$\mathfrak{A}_2(A;u,v) = \frac{1}{12\pi^2}\int_{M^3} \mathrm{Tr}(du\wedge dv\wedge A), \qquad u,v\in\mathfrak{G}, \tag{7.45}$$

where M^3 is a three-dimensional manifold. The formulae discovered in [48] gave rise to a new direction in the representation theory of infinite-dimensional Lie groups and algebras. The corresponding 2-cocycle (7.45) and its group analogue are now known as Mickelson–Faddeev–Shatashvili cocycles.

The following conjecture was stated in [48]:

$$[G(u),G(v)] = G([u,v]) + \mathfrak{A}_2(A,u,v), \quad G(u) = \int_{M^3} u^a(x)G^a(x)\,d^3x. \tag{7.46}$$

It is important to note that since the 2-cocycle depends on the gauge field, the ordinary fermion current algebra (in the external field A) should be replaced by the Gauss law with a dynamical gauge field. Moreover, in contrast to the case $n=1$, which corresponds to a one-dimensional central extension of the loop group LG of G, for $n=2$ we are dealing with a new mathematical object: an Abelian extension of the gauge group $\mathcal{G}$ (the group of maps from M^3 to G). Although (7.46) was not deduced in [51] in the operator approach to quantum field theory,[87] the authors

[87]It is still unknown how to do this.

of [51] developed an effective method for quantizing systems with second-class constraints, applicable to the quantization of anomalous gauge theories and now known as the Faddeev–Shatashvili quantization. In the path integral formulation, the formula (7.46) was proved in [55].

7.7. Orbit method and functional integration. The orbit method was developed by Kirillov to describe unitary representations of unipotent Lie groups. The innovative idea of this method involves relating unitary representations (algebraic objects) to coadjoint orbits (geometric objects). This approach to representation theory proved to be very fruitful. It underlies geometric representation theory.

In greater detail, let G be a connected compact reductive Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^*$ the dual space. The orbits of the coadjoint action of G on $\mathfrak{g}^*$ are parametrized by the positive Weyl chamber W_+. To every point $\lambda \in W_+$ there corresponds an orbit O_λ. For example, when $G = \mathrm{U}(n)$, the positive Weyl chamber is parametrized by ordered tuples of real numbers

$$\lambda = \{\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n\}.$$

The space $\mathfrak{g}^*$ may be identified with Hermitian $n \times n$ matrices, and the orbits O_λ consist of the matrices with given eigenvalues λ.

Coadjoint orbits carry a canonical symplectic form (the Kirillov form). For simplicity of notation we assume that G is a matrix group and its Lie algebra $\mathfrak{g}$ is a matrix algebra. Then every element $x \in O_\lambda$ of an orbit can be represented in the form

$$x = g^{-1}\lambda g,$$

and the Kirillov form can be written as

$$\omega = \frac{1}{2}\langle\lambda, [dgg^{-1}, dgg^{-1}]\rangle = \frac{1}{2}\langle x, [g^{-1}dg, g^{-1}dg]\rangle.$$

When λ is a weight of G, the form ω belongs to an integer cohomology class and determines a positive line bundle L_λ over the orbit. By the Borel–Weil–Bott theorem, the orbit O_λ induces an irreducible representation V_λ, which can be realized in the cohomology of L_λ. Kirillov proposed a universal formula for characters of the irreducible representation V_λ in terms of integrals of exponential functions over the coadjoint orbit endowed with the Liouville measure of the canonical symplectic form.

Faddeev and his students Shatashvili and Alekseev posed the problem of obtaining the character formula for the representation V_λ in terms of functional integration. This is important for several reasons. On the one hand, quantum-mechanical description of characters by means of functional integration gives a fresh view on the representation theory (and answers a question posed by Kirillov). On the other hand, such a formalism enables one to consider particles with spin and holonomies of connections (Wilson loops) in non-Abelian gauge theories in the framework of classical field theory.

Here is the answer obtained in [58]:

$$Z(A) = \int \mathcal{D}g \exp\biggl\{i\int_\gamma \langle\lambda, dgg^{-1} + gAg^{-1}\rangle\biggr\}, \tag{7.47}$$

where γ is a one-dimensional manifold (an interval or a circle) and $A \in \Omega^1(\gamma, \mathfrak{g})$ is a gauge field. The formula (7.47) admits several interesting interpretations.

First, when γ is a circle, the action functional

$$S(g, A) = \int_\gamma \langle \lambda, dgg^{-1} + gAg^{-1} \rangle$$

can be written, using the Stokes theorem, as a two-dimensional integral of the form

$$S(g, A) = \int_\Sigma \langle \lambda, F \rangle,$$

where Σ is a two-dimensional surface with boundary γ and $F = da$ is the curvature of the connection a for the group G_λ preserving λ. We note that $F/(2\pi i)$ is the first Chern form. Thus, $S(g, A)$ is the simplest example of a Chern–Simons action obtained by the descent method!

Second, when γ is a circle, it is natural to conjecture that

$$Z(A) = \chi_\lambda(\mathrm{Hol}(A, \gamma)),$$

where χ_λ is the character of an irreducible representation of G with highest weight λ, and $\mathrm{Hol}(A, \gamma) \in G$ is the holonomy of A along γ. This conjecture was proved in [58] in the cases when $G = \mathrm{SU}(n)$ and $G = \mathrm{SO}(n)$ using the Gelfand–Tsetlin integrable systems[88] on the orbits O_λ in order to introduce the 'action-angle' variables and compute the path integral.

Third, it was later shown by Alekseev and Shatashvili that (7.47) can easily be extended to more complicated examples of infinite-dimensional Lie algebras and groups. In particular, the action functional for affine Lie algebras coincides with the Wess–Zumino–Novikov–Witten chiral action, which is obtained from the second Chern class by the descent method described in the previous subsection. In the case of the Virasoro algebra one obtains the Polyakov action describing one of the two-dimensional gravitation models which is currently used in probability theory.

Fourth, (7.47) gives an interesting opportunity to study the nature of functional integration. On the one hand, this is a simple quantum-mechanical system with a compact phase space. On the other hand, for the partition function the answer is given by the character of an irreducible representation of a reductive Lie algebra, and this expression contains a very rich combinatorics. Finally, the third approach enables one to interpret this system as a two-dimensional topological quantum field theory.

Like other formulae due to Faddeev, the expression (7.47) is simple looking but full of unexpected depth and never ceases to amaze.

8. Quantum field theory. Problems of scattering theory

8.1. Wave operators in quantum field theory. Throughout his scientific activities, Faddeev thought about dealing with inevitable divergences in the ordinary approach to quantum field theory. He cited Van Hove [125], [118], who argued

[88]These integrable systems were introduced by Guillemin and Sternberg.

that renormalizations in field theory arise because of the use of eigenfunction expansion of the free Hamiltonian. In [14], Faddeev proposed a method for eliminating the eigenfunctions of the free Hamiltonian from calculations of the S-matrix. The main idea of this method is as follows. Suppose that the Hamiltonian H is given in the representation of second quantization as

$$H = H_0 + V$$

and the interaction V is a Wick polynomial whose coefficient functions involve a δ-function that guarantees conservation of momentum:

$$V = \sum_{n,m} V_{n,m},$$

$$V_{n,m} = \int_{\mathbb{R}^{n+m}} v_{n,m}(k_1,\dots,k_n;p_1,\dots,p_m)\prod_{i=1}^{n} a^\dagger(k_i)\,dk_i \prod_{j=1}^{m} a(p_j)\,dp_j,$$

$$v_{n,m}(k_1,\dots,k_n;p_1,\dots,p_m) = \delta\biggl(\sum_{i=1}^{n} k_i - \sum_{j=1}^{m} p_j\biggr)\widetilde{v}_{n,m}(k_1,\dots,k_n;p_1,\dots,p_m),$$

where $a^\dagger(k)$ and $a(k)$ are the creation and annihilation operators. The presence of the δ-function enables one to consider the dynamics in a subspace with fixed momentum, and if $v_{n,1} \neq 0$ for $n > 1$, then there is an interaction between the discrete and continuous spectra. On the other hand, the presence of the δ-function in the terms $v_{n,0}$, $n > 1$, makes it difficult to define V as an operator. By [14], when V contains terms with fewer than two annihilation operators, the construction of scattering theory first requires one to find a unitary operator W (which can naturally be referred to as the dressing operator in the sense of Greenberg–Schweber [100]) such that

$$H' = W^{-1}HW = H_0' + V' + cI,$$

where H_0' is the renormalized energy operator, c is a constant, and V' contains only potential terms, that is, V' is represented by a Wick polynomial having no terms with fewer than two annihilation operators. The eigenfunctions of H_0' can be taken as asymptotic states. The paper [14] proposed considering the wave operators for the pair of operators H_0' and $H_0' + V'$. They are defined in the standard way by the formula (1.4) as the strong limits

$$U_\pm' = \lim_{t\to\pm\infty} e^{it(H_0'+V_0')}e^{-itH_0'},$$

and the S-matrix is defined as the operator

$$S = U_+'^{*}U_-'.$$

In this approach, the scattering matrix is unitary from the very beginning (hence there is no need to renormalize the wave functions) and has non-trivial matrix entries only between states containing two or more particles. This programme was realized for the Lee model [84]. The divergences related to charge renormalization are of a different mathematical nature [11].

8.2. Infrared divergences and asymptotic conditions in quantum electrodynamics. The presence of infrared divergences (the infrared catastrophe) was known in QED since the classical paper [89] by Bloch and Nordsieck (1937). Textbooks on QED propose overcoming infrared divergences by summing the probabilities of the transition from the given initial state to all final states containing an arbitrary number of 'soft' photons in addition to detectable particles. But this commonly accepted formal treatment of the infrared catastrophe is not completely satisfying, because the initial and final states are not treated symmetrically, and the scattering operator is not defined at all. In their joint paper [23], Kulish and Faddeev asked whether the absence of the scattering operator is unavoidable and follows from the physical essence of the problem, or whether there is an alternative approach to infrared divergences enabling one to define the scattering operator. It was shown in [23] that infrared divergences can be avoided by modifying simultaneously the space of asymptotic states and the definition of the scattering operator. This is because the asymptotic dynamics should be taken properly into account in order to have well-defined wave operators. The procedure proposed in [23] was suggested by the theory of non-relativistic scattering for long-range potentials and has a simple physical interpretation. Therefore, following [23], we begin with this case.

8.2.1. *Non-relativistic Coulomb scattering.* We explain the main idea of the approach in [23] using the example of scattering of a non-relativistic particle by the long-range Coulomb potential with Hamiltonian

$$H = H_0 + V = -\frac{1}{2m}\Delta + \frac{g}{|\boldsymbol{r}|}\,.$$

In the interaction representation, the potential

$$V(t) = e^{-iH_0 t} V e^{iH_0 t}$$

has the following asymptotics as $|t| \to \infty$:

$$V_{\mathrm{as}}(t) = \frac{mg}{p|t|}\,, \quad \text{where } p = \sqrt{-\Delta}\,.$$

Since $V_{\mathrm{as}}(t)$ is non-integrable with respect to t in a neighbourhood of ∞, the asymptotic dynamics is described not by the free Hamiltonian H_0, but by the explicitly time-dependent operator

$$H_{\mathrm{as}}(t) = H_0 + V_{\mathrm{as}}.$$

The solutions $\psi(t)$ of the asymptotic Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = H_{\mathrm{as}}(t)\psi$$

with initial condition $\psi(\boldsymbol{r}, t)\big|_{t=t_0} = \psi(\boldsymbol{r})$ are represented in the form

$$\psi(t) = U_{\mathrm{as}}(t)\psi, \quad \text{where} \quad U_{\mathrm{as}}(t) = e^{-iH_0(t-t_0)} \exp\biggl\{-i\frac{mg}{p}(\operatorname{sign} t)\log\biggl|\frac{t}{t_0}\biggr|\biggr\}. \quad (8.1)$$

Dollard [95] showed that the following strong limits exist in $L^2(\mathbb{R}^3)$:

$$U_\pm = \lim_{t\to\pm\infty} e^{iH\,t} U_{\mathrm{as}}(t).$$

These are analogues of the wave operators (1.4) in the rapidly decaying case. The scattering operator is accordingly defined by

$$S = U_+^* U_-.$$

It is independent of the choice of t_0 and gives familiar expressions for the differential cross sections of scattering by the Coulomb potential. It was stressed in [23] that one should derive the Hamiltonian of the asymptotic dynamics from the physical meaning of the problem[89] rather than blindly copy the patterns of scattering theory for rapidly decaying potentials (see § 1). These ideas were applied in [23] to relativistic quantum electrodynamics.

8.2.2. *Construction of $V_{\mathrm{as}}(t)$ and the S-matrix in quantum electrodynamics.* To be specific, the authors of [23] consider spinor electrodynamics [83] describing a system of interacting electrons, positrons, and photons: $\bar\psi(\boldsymbol{x})$ and $\psi(\boldsymbol{x})$ are the electron-positron field operators, $A_\mu(\boldsymbol{x})$ is the electromagnetic field operator, and

$$b_i^\dagger(\boldsymbol{p}),\ b_i(\boldsymbol{p}), \quad d_i^\dagger(\boldsymbol{p}),\ d_i(\boldsymbol{p}), \quad a_\mu^\dagger(\boldsymbol{k}),\ a_\mu(\boldsymbol{k}), \qquad i=1,2, \quad \mu=0,1,2,3,$$

are the creation and annihilation operators for electrons, positrons, and photons, respectively. The field operators $\bar\psi$, ψ, and A_μ are expressed in terms of the creation and annihilation operators by

$$\psi(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \left(\frac{m}{p_0}\right)^{1/2} \sum_{l=1}^{2} (b_l(\boldsymbol{p}) w_l(\boldsymbol{p}) e^{i(\boldsymbol{p},\boldsymbol{x})} + d_l^\dagger(\boldsymbol{p}) v_l(\boldsymbol{p}) e^{-i(\boldsymbol{p},\boldsymbol{x})})\, d^3\boldsymbol{p},$$

$$\bar\psi(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \left(\frac{m}{p_0}\right)^{1/2} \sum_{l=1}^{2} (b_l^\dagger(\boldsymbol{p}) \overline{w}_l(\boldsymbol{p}) e^{-i(\boldsymbol{p},\boldsymbol{x})} + d_l(\boldsymbol{p}) \overline{v}_l(\boldsymbol{p}) e^{i(\boldsymbol{p},\boldsymbol{x})})\, d^3\boldsymbol{p},$$

$$A_\mu(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} (a_\mu^\dagger(\boldsymbol{k}) e^{-i(\boldsymbol{k},\boldsymbol{x})} + a_\mu(\boldsymbol{k}) e^{i(\boldsymbol{k},\boldsymbol{x})})\, \frac{d^3\boldsymbol{k}}{\sqrt{2k_0}}\,,$$

where

$$p_0 = \sqrt{\boldsymbol{p}^2 + m^2}\,, \quad k_0 = |\boldsymbol{k}|,$$

and $(\,\cdot\,,\cdot\,)$ is the standard scalar product in $\mathbb{R}^3$. The corresponding interaction operator is of the form

$$V = -e \int_{\mathbb{R}^3} {:}\bar\psi(\boldsymbol{x}) \gamma^\mu \psi(\boldsymbol{x}){:}\, A_\mu(\boldsymbol{x})\, d^3\boldsymbol{x},$$

where e is the electron charge, the γ^μ are Dirac's gamma matrices and $:\cdot:$ means the normal ordering with respect to the creation and annihilation operators (see § 5.2). Remarkably, it turns out that

$$e^{-iH_0 t} V(t) e^{iH_0 t} = V_{\mathrm{as}}(t) + o(1), \qquad t\to\pm\infty,$$

[89] In analysing the Dollard formula for $U_{\mathrm{as}}(t)$, Faddeev and Buslaev proposed a general scheme for constructing wave operators for long-range potentials using the asymptotics of the classical motion as $|t|\to\infty$. This beautiful idea was implemented in [91].

where $V_{\mathrm{as}}(t)$ can be represented in the form

$$V_{\mathrm{as}}(t)=\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}J^{\mu}_{\mathrm{as}}(\boldsymbol{k},t)\big(a^{\dagger}_{\mu}(-\boldsymbol{k})+a_{\mu}(\boldsymbol{k})\big)\frac{d^3\boldsymbol{k}}{\sqrt{2k_0}}$$

with the current operator

$$J^{\mu}_{\mathrm{as}}(\boldsymbol{k},t)=-e\int_{\mathbb{R}^3}p^{\mu}\,e^{i(\boldsymbol{p},\boldsymbol{k})/p_0}\rho(\boldsymbol{p})\,\frac{d^3\boldsymbol{p}}{2p_0}$$

and the density

$$\rho(\boldsymbol{p})=\sum_{l=1}^{2}(b^{\dagger}_{l}(\boldsymbol{p})b_l(\boldsymbol{p})-d^{\dagger}_{l}(\boldsymbol{p})\,d_l(\boldsymbol{p})).$$

The states of charged particles with given momenta are eigenstates of the operator $J^{\mu}_{\mathrm{as}}(\boldsymbol{k},t)$.

As in the case considered above, the asymptotic dynamics of the system is described by the Hamiltonian

$$H_{\mathrm{as}}(t)=H_0+V_{\mathrm{as}}(t)$$

and the evolution operator $U_{\mathrm{as}}(t)$,

$$i\frac{dU_{\mathrm{as}}(t)}{dt}=H_{\mathrm{as}}(t)U_{\mathrm{as}}(t).$$

The general solution of this equation is shown in [23] to be

$$U_{\mathrm{as}}(t)=e^{-iH_0t}Z(t),$$

where

$$Z(t)=\exp\biggl\{-i\int^{t}e^{iH_0\tau}V_{\mathrm{as}}(\tau)e^{-iH_0\tau}\,d\tau-\frac{1}{2}\int^{t}\biggl(\int^{\tau}[V^{I}_{\mathrm{as}}(\tau),V^{I}_{\mathrm{as}}(s)]\,ds\biggr)\,d\tau\biggr\}$$

and the lower limit of integration is chosen in such a way[90] that

$$\int^{t}e^{is\tau}\,d\tau=\frac{1}{is}e^{ist}.$$

The operator $Z(t)$ can be rewritten as

$$Z(t)=\exp\{R(t)\}\exp\{i\Phi(t)\},$$

where

$$\begin{aligned}R(t)=\frac{e}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{p^{\mu}}{k\cdot p}\biggl(&a^{\dagger}_{\mu}(\boldsymbol{k})\exp\Bigl\{i\frac{k\cdot p}{p_0}t\Bigr\}\\&-a_{\mu}(k)\exp\Bigl\{-i\frac{k\cdot p}{p_0}t\Bigr\}\biggr)\rho(\boldsymbol{p})\,d^3\boldsymbol{p}\,\frac{d^3\boldsymbol{k}}{\sqrt{2k_0}}\,,\\ \Phi(t)=\frac{e^2}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}{:}\rho(\boldsymbol{p})\rho(\boldsymbol{q}){:}\,&\frac{p\cdot q}{((p\cdot q)^2-m^4)^{1/2}}(\operatorname{sign}t)\log\frac{|t|}{t_0}\,d^3\boldsymbol{p}\,d^3\boldsymbol{q},\end{aligned}$$

[90]In particular, this corresponds to the condition that the integrals of $V_{\mathrm{as}}(t)$ over finite t make no contribution to the expression for $U_{\mathrm{as}}(t)$.

and

$$k \cdot p = k^\mu p_\mu, \quad p \cdot q = p^\mu q_\mu, \quad \text{where} \quad q_0 = \sqrt{\boldsymbol{q}^2 + m^2}\,.$$

The operator Φ is a natural analogue of the phase in (8.1).

The operators $R(t)$ and $\Phi(t)$ commute, and the final expression for the evolution operator of asymptotic dynamics takes the form

$$U_{\mathrm{as}}(t) = \exp\{-iH_0 t\}\exp\{R(t) + i\Phi(t)\}$$

while the Kulish–Faddeev scattering operator is defined by

$$S = \lim_{\substack{t_1\to\infty \\ t_2\to-\infty}} S(t_1, t_2),$$

where

$$S(t_1,t_2) = U^\dagger_{\mathrm{as}}(t_1)\exp\{-iH(t_1-t_2)\}U_{\mathrm{as}}(t_2).$$

The expression $S(t_1, t_2)$ differs from Dyson's S-matrix for finite times

$$S_{\mathrm{D}}(t_1,t_2) = \exp\{iH_0 t_1\}\exp\{-iH(t_1-t_2)\}\,\exp\{-iH_0 t_2\}$$

by the augmenting factors $\exp\{R(t) + i\Phi(t)\}$.

To construct a meaningful theory, one must find the Hilbert space in which the operator S acts. A non-trivial fact is that it acts in the separable Hilbert space $\mathscr{H}_{\mathrm{as}}$ of asymptotic states,[91] which differs from the Fock space $\mathscr{H}_{\mathrm{F}}$ for charged particles and photons. To explain this in greater detail, we consider the operator

$$W(t) = \exp\{R(t)\}.$$

Since it preserves the number, momenta, and spins of charged particles, its action on the vector

$$b^\dagger_{s_1}(\boldsymbol{p}_1)\cdots b^\dagger_{s_n}(\boldsymbol{p}_n) d^\dagger_{i_1}(\boldsymbol{q}_1)\cdots d^\dagger_{i_m}(\boldsymbol{q}_m)|0\rangle \otimes \Psi_\gamma$$

is determined by the action on Ψ_γ (the state vector of photons) and is given by the operator

$$W_{n,m}(t) = \exp\biggl\{\frac{e}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}(f^\mu_{n,m}(\ldots|\boldsymbol{k},t)a^\dagger_\mu(\boldsymbol{k}) - \overline{f}^\mu_{n,m}(\ldots|\boldsymbol{k},t)a_\mu(\boldsymbol{k}))\,\frac{d^3\boldsymbol{k}}{\sqrt{2k_0}}\biggr\},$$

where the dots $\ldots$ stand for the dependence on $\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n,\boldsymbol{q}_1,\ldots,\boldsymbol{q}_m$ and

$$\begin{aligned} f^\mu_{n,m}(\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n,\boldsymbol{q}_1,\ldots,\boldsymbol{q}_m \mid \boldsymbol{k},t) = \sum_{i=1}^n \frac{p_i^\mu}{k\cdot p_i}\exp\biggl\{i\frac{k\cdot p_i}{p_{i0}}t\biggr\} \\ - \sum_{i=1}^m \frac{q_i^\mu}{k\cdot q_i}\exp\biggl\{i\frac{k\cdot q_i}{q_{i0}}t\biggr\}. \end{aligned} \tag{8.2}$$

[91] In 1969 Faddeev was presenting the definition of the space of asymptotic states in quantum electrodynamics to Vladimir Aleksandrovich Fock. Their conversation took place in the Rector's quarters of Leningrad State University, then the location of the Department of Theoretical Physics in the Faculty of Physics. "There are as many vectors in the space of asymptotic states as in yours", Faddeev explained to Fock!

Thus, $W(t)$ is an operator of the type 'exponential of a linear form' in the creation and annihilation operators for photons. Its discrete analogue has the form

$$\begin{aligned} W_\alpha &= \exp\biggl\{\sum_i (\overline{\alpha}_i a_i - \alpha_i a_i^\dagger)\biggr\} \\ &= \exp\biggl\{-\frac{1}{2}\sum_i |\alpha_i|^2\biggr\} \exp\biggl\{-\sum_i \alpha_i a_i^\dagger\biggr\} \exp\biggl\{\sum_i \overline{\alpha}_i a_i\biggr\} \end{aligned}$$

and is meaningless in the initial Fock space if

$$\sum_i |\alpha_i|^2 = \infty.$$

However, it can be defined as an operator acting from the Fock space to a new space. This new space may naturally be regarded as the image $\mathscr{H}_\alpha$ of the Fock space $\mathscr{H}_{\mathrm{F}}$ under the action of W_α. The properties of $\mathscr{H}_\alpha$ and its continual analogues were studied in detail in [23] as a part of the definition of the space of asymptotic states in quantum electrodynamics, and also by Aref'eva and Kulish [85] (in a more general context). It was shown in [23] that $\mathscr{H}_\alpha$ is naturally acted on by a representation of the Poincaré group [23] and contains a Lorentz-invariant and gradient-invariant subspace with non-negative metric, that is, the physical asymptotic subspace $\mathscr{H}_{\mathrm{as}}$. By [23],

$$\mathscr{H}_{\mathrm{as}} = W(t)\mathscr{H}_{\mathrm{F}} \quad \text{for all } t.$$

The scattering operator can now be precisely characterized as an operator acting in $\mathscr{H}_{\mathrm{as}}$. Namely, writing $S(t_1, t_2)$ in the form

$$S(t_1, t_2) = W^\dagger(t_1)\widetilde{S}_{\mathrm{D}}(t_1, t_2)W(t_2),$$

where

$$\widetilde{S}_{\mathrm{D}}(t_1, t_2) = \exp\{-i\Phi(t)\}\, S_{\mathrm{D}}(t_1, t_2) \exp\{i\Phi(t)\},$$

we obtain the following sequence of maps:

$$\mathscr{H}_{\mathrm{as}} \xrightarrow{W} \mathscr{H}_{\mathrm{F}} \xrightarrow{\widetilde{S}} \mathscr{H}_{\mathrm{F}} \xrightarrow{W^\dagger} \mathscr{H}_{\mathrm{as}}.$$

Since the definition of the operator S involves the operators $W^\dagger(t_1)$ and $W(t_2)$, we must use the space $\mathscr{H}_{\mathrm{as}}$ of asymptotic states in the definition of the S-matrix. It was noted in [23] that, along with charged particles, asymptotic states must contain infinitely many photons, whose low-frequency spectrum is determined by the state of the charges. Redefining the scattering operator amounts to distinguishing the 'phase' operator factors. The first circumstance is of a relativistic nature, and the second is needed even in the case of non-relativistic scattering by a Coulomb potential.

The authors of [23] also compare their results with Chung's [92], who deduced the phase factors $\exp\{\pm i\Phi(t)\}$ from an analysis of Feynman diagrams. This enabled them to verify that the matrix entries $\langle\Psi|S|\Psi'\rangle$ of the scattering operator S between arbitrary states $|\Psi\rangle$ and $|\Psi'\rangle$ contain no infrared divergences in all orders of expansion with respect to the charge e^2.

Kulish [106] considered in a similar way the scattering of soft gravitons and obtained factors analogous to those found earlier by Weinberg [126]. The methods of the paper [23] by Faddeev and Kulish are now used in various areas of theoretical physics (see, for example, [102]). The formula (8.2) entails the appearance of a pole at zero frequency k_0 of radiated photons, which in turn gives rise to the 'memory effect' in electromagnetism and gravitation (originally discovered for gravitation by Zeldovich and Polnarev in 1973).

9. Conclusion

The mathematical heritage of Ludwig Dmitrievich Faddeev will determine the development of mathematical and theoretical physics for many decades to come. For all its enormous diversity, it is united by a deep faith in the unity of mathematics and physics, in their interpenetration, and in the capability of understanding the fundamental laws of nature based on the criterion of mathematical beauty and naturalness. Faddeev recognized the importance of deep ideas coming from physics for the development of pure mathematics, and perhaps he is the first great mathematician whose work was completely based on the ideas of quantum theory. Faddeev liked to say that quantum mechanics is simpler than classical mechanics. His papers in quantum field theory prepared a revolution of gauge theories in elementary particle physics. With the advent of the models of topological quantum field theory, the same papers played a most important role in the development of topology.

His technical arsenal included both the methods of functional analysis developed in the first half of the 20th century and forming the mathematical toolbox of quantum mechanics, and the more recent methods of symplectic geometry, algebraic analysis, and quantum field theory. For example, using the method of holomorphic representation with virtuosity, Faddeev gave a definitive formulation of the scattering matrix in the formalism of functional integration. Another fundamental idea of Faddeev's was related to the role of (infinite-dimensional) group theory as a source of exact solutions in classical and quantum physics. This role of symmetry is still somewhat mysterious, but his intuition, almost always infallible, has indicated the deepest and most fruitful connections between various areas of mathematics and physics also in this respect.

Faddeev's ideas continue to play a definitive role in mathematical physics. They live in the work of his students and will live further in the work of their students and so forth, as long as mathematics and theoretical physics exist.

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