# Ludwig Faddeev (1934–2017) – His Work and Legacy

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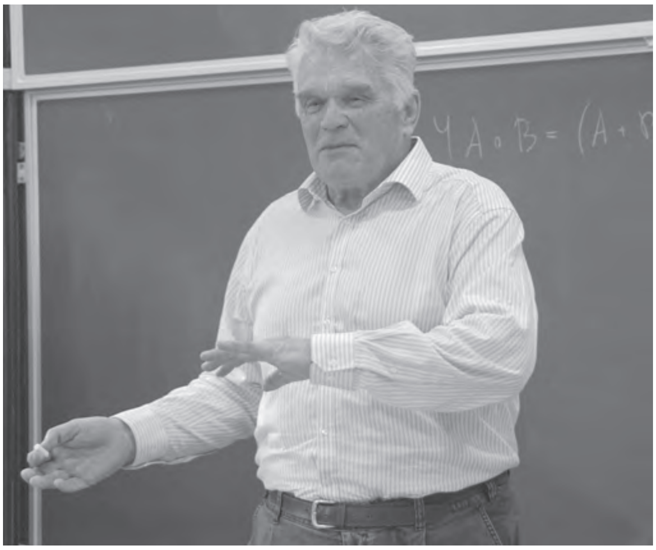



**Ludwig Faddeev. Aarhus University, August 2010.**

The mathematical community has suffered a heavy loss with the death of one of the world's greatest mathematicians and theoretical physicists Ludwig Faddeev, who passed away on 26 February after a heavy illness. Despite his ailing health, Faddeev remained active until the last months of his life. In August 2016, he attended a special meeting, the 23th European Conference on Few-Body Problems in Physics, held in Aarhus (Denmark), where a new award to recognise distinguished achievements in few-body physics, the Faddeev Medal, was inaugurated. This proved to be the last honour, among the many others, that he received in his lifetime.

Professor Ludwig Faddeev is widely known for his contributions to mathematics and theoretical physics, which have largely reshaped modern mathematical physics. His work on quantum field theory prepared the ground for the gauge fields theory revolution of the 1970s. His contributions to the many-body problem in quantum mechanics and to the inverse scattering problem belong to the deepest achievements in these areas. His pioneering work on the quantum inverse scattering method started a wide new field of research, ranging from solvable models in quantum field theory to quantum groups.

For more than 60 years, Professor Faddeev was associated with the Steklov Mathematical Institute. In 1976–2000, he was serving as Director of the Leningrad (later, St. Petersburg) branch of the Institute and Head of the Laboratory of Mathematical Problems in Physics, where he brought together a score of his pupils and colleagues. Although they are now dispersed over several countries and continents, the Faddeev school is still highly united and plays a prominent role in modern mathematical physics.

## Early years

Professor Faddeev was born in 1934 in Leningrad (now St. Petersburg) into a family of prominent Soviet mathematicians. His mother Professor V.N. Faddeeva was among the pioneers of computational methods and, for many years, headed the Laboratory of Computational Methods of the Steklov Institute in Leningrad. His father Professor D.K. Faddeev was one of the best Soviet algebraists; he is particularly remembered for his contributions to homological algebra, Galois theory and representation theory. His university teaching has shaped several generations of Soviet algebraists. He was also a distinguished musician and a brilliant pianist. The choice of the rather rare name Ludwig for his elder son reflected the parents' hope that he would one day become a professional musician. This hope would not materialise because of the hardships of wartime during his childhood but Ludwig still had a profound knowledge of classical music. Together with his father, he played Bruckner's and Mahler's symphonies (in a four-hands transcription) and his favourite composers were Berlioz and Richard Strauss.

The younger years of Faddeev were a time when the country was recovering from the ravages of war and also a time of great hope after the death of Stalin. Among his generation, there is a remarkable number of first-rate mathematicians: Arnold, Berezin, Maslov, Novikov and Sinai, to mention just a few. This spectacular explosion of talent was, to a great extent, due to a very simple circumstance: in this time of still tough ideological controls, mathematics was a domain of freedom that naturally attracted bright young people. Another crucial point was the solid scientific tradition that survived the turmoil of revolution and the repressions of the 1930s.

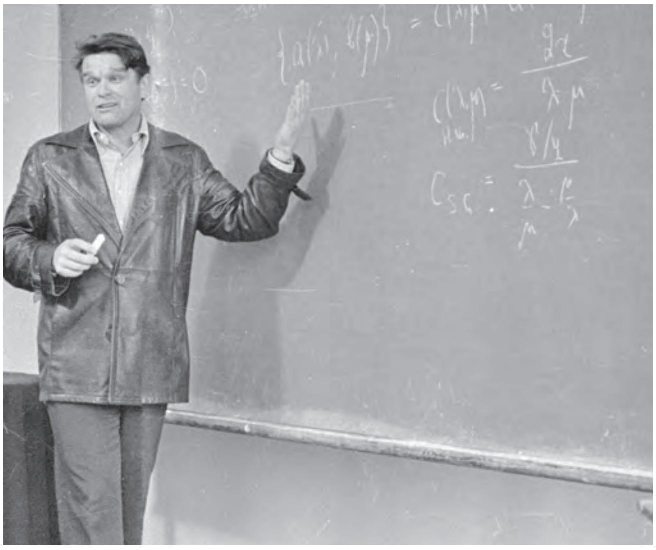
One important decision Faddeev took as a 17-year-old youth was the choice of the Physics Department of Leningrad University. At the time, his father was Dean of the Mathematics Department and Ludwig wanted to make his own way. The Physics Department had a brilliant tradition in theoretical physics and especially in general relativity and in quantum mechanics (then only 25 years old!), marked by such names as A.A. Friedmann and V.A. Fock. The higher mathematics course, which was supervised by Academician V.I. Smirnov, was directly oriented towards the needs of quantum theory, with an emphasis on operator theory, spectral theory of differential operators, etc. Starting from 1954, the mathematical education of the young theorists was mainly entrusted to Professor O.A. Ladyzhenskaya, then the youngest and most brilliant professor of the Chair of Mathematics. Before that, the Chair of Mathematics at the Physics Department, created by Academician Smirnov in the 1930s, was mainly considered an auxiliary one. Now, for the first time, it was given an independent status and was allowed to have its own students, give special courses and supervise diploma work. Faddeev was thus in the very first group of students who defended their university theses in mathematical physics. Already in 1954, Professor Ladyzhenskaya had organised a special learning seminar in quantum field theory, where Faddeev was one the main speakers. This seminar shaped his early interest in mathematical problems of quantum field theory. Professor Ladyzhenskaya also became Faddeev's thesis adviser during his PhD studentship at the Leningrad Branch of the Steklov Mathematical Institute. Her portrait remained on Faddeev's writing desk in his study at the Steklov Institute up to the last day of his life.

### First papers: quantum scattering and the inverse problem

The first published papers of Faddeev dealt with potential scattering and spectral decomposition for Schrödinger operators with continuous spectrum. His concise proof of the dispersion relations for the scattering amplitude was included in the famous Landau and Lifshitz textbook on quantum mechanics. In his PhD thesis, he gave a complete solution of the inverse scattering problem for the Schrödinger operator on the line. This work was written in the aftermath of the fundamental results on inverse scattering due to I.M. Gelfand, B.M. Levitan and V.A. Marchenko, who had solved the inverse scattering problem for the radial Schrödinger equation (which arises from the three-dimensional Schrödinger equation after a separation of variables). The case of the Schrödinger equation on the line is slightly more difficult because of the multiple continuous spectrum. Over a decade later, this paper proved to be of crucial importance as it contained all the background of the future inverse scattering method in the theory of integrable systems. In the course of this study, Faddeev also prepared a comprehensive review of quantum inverse scattering. At the invitation of Academician N.N. Bogolyubov, it was submitted at the inauguration meeting of the Laboratory of Theoretical Physics in Dubna, in the presence of Gelfand, Levitan, Krein, Marchenko and other big names; its written version, published in 1959 in Uspekhi, became a standard reference in the field.

### Quantum three-body problem

The next big subject Faddeev chose was the quantum three-body problem. At the time, he was already heavily attracted to the intricate and complicated problems of quantum field theory but believed that, before launching into the insecure waters of QFT, it was important to resolve a really difficult technical problem. While the difficulties of the quantum three-body problem are of an entirely different nature than those of its famous classical counterpart, it represents a real challenge because of the complicated structure of the continuous spectrum. Before Faddeev's work, only some partial results had been obtained by physicists (under very restrictive and not quite self-consistent assumptions on the interaction potentials). Faddeev's original approach to this problem was based on experience he had gained in his work with the so-called Friedrichs model in perturbation theory and also in the study of an instructive example from QFT, the Thirring model. The key idea consists of a clever rearrangement of the integral equations associated with the multi-body scattering problem (which basically result from the Hilbert identity for the resolvent of the Schrödinger operator) into a much more manageable and symmetric system for the so-called $T$-operators (generalising the pairwise scattering amplitudes for different particles). This new system of integral equations, called Faddeev equations, is already Fredholm, in contrast to the initial equation for the resolvent. It has become the basis of efficient numerical computations in various applications (ranging from quantum chemistry to nuclear physics). Faddeev's work on the quantum three-body problem triggered tremendous activity in the area (pursued up to the present day); his own decision, however, was definitely to move to other subjects. After the publication of his now famous monograph on three-body scattering (1963, English translation 1965), while some of his pupils continued working in this direction for another decade or more, he decided that it was now time to attack QFT.



First Conference on Quantum Solitons. Leningrad, October 1978.

### Quantum gauge theory

The bid was, in fact, a very difficult one since QFT was positively out of grace in the Soviet Union at the time. The great success of quantum electrodynamics in the late 1940s and early 1950s was followed by a decade of fruitless attempts to apply QFT to strong interactions. Still more importantly, QFT was plagued by the so-called "zero charge paradox", discovered by Landau and Pomeranchuk and believed to point out the logical inconsistency of quantum electrodynamics and QFT in general. In his short note dedicated to the memory of Wolfgang Pauli, written shortly before the tragic car accident that put an end to his scientific career, Landau insisted, with a reference to this paradox, that the Hamiltonian method in field theory was now totally dead and needed to be buried ("with all the honours it deserves"). Due to the brevity of life, he concluded, we cannot allow ourselves the luxury of spending our time on problems that do not lead to new results. Landau's words were considered by his pupils in the 1960s as the Teacher's Testament and when, in 1966, Faddeev, together with his pupil V.N. Popov, obtained a breakthrough in quantum Yang–Mills theory, their paper could not be published in any of the Soviet scientific journals nor sent abroad (for which a positive opinion of the Nuclear Physics Department of the Academy of Sciences was necessary). A short note by Faddeev and Popov was finally published in Physics Letters (with a year delay), while the full text was made available only as a preprint of the Kiev Institute of Theoretical Physics (with hand-written formulae); its English version only appeared in 1973 at the time of the big boost triggered by the gauge fields revolution of the early 1970s.

The choice of Yang–Mills theory reflected Faddeev's characteristic non-conformism but also his fundamental belief that a good physical theory should have mathematical beauty. His original idea was to understand the quantisation of general relativity, a theory of incontestable great beauty but also of notorious difficulty. Yang–Mills theory seemed, at the time, just a kind of useful model example. We know now that this example proved to be an exceptionally successful one: it allowed the generalisation of quantum electrodynamics by unifying electromagnetic and weak interactions, and the building of a consistent theory of strong interactions. Geometrically, Yang–Mills theory is, in fact, very close to general relativity (while the latter deals with the tangent bundle of the spacetime, Yang–Mills theory brings into play abstract vector bundles). All these exciting developments had already taken place in the 1970s; the key discoveries, due to G. 't Hooft, D. Gross, F. Wilczek and D. Politzer, were that Yang–Mills theory is renormalisable and free of the zero charge paradox. The culmination of this "gauge fields revolution" was the creation of the "standard model" in high energy physics. The earlier results of Faddeev and Popov provided both the technical base and the conceptual base for these developments, marked by several Nobel prizes.

Turning back to the Faddeev–Popov paper, it is worth stressing the conciseness and clarity of their approach, which was to become the basic language of the new theory. The new QFT formalism they proposed was, for the first time, entirely based on the ample use of functional integrals. Functional integrals had already been introduced into quantum mechanics by R. Feynmann in the 1940s but, for some strange reason, he never used them in quantum field theory, even though, as we understand now, they provide the easiest and most straightforward way to deduce his famous diagram expansion. In the early 1960s, Feynmann also examined the quantisation of Yang–Mills theory (he, too, regarded it as a model example before addressing quantum gravity). Feynmann discovered the inconsistency of the naive perturbative expansion for Yang–Mills theory but did not manage to resolve this problem. The use of functional integrals makes all computations completely transparent. The main point is to determine the correct symplectic measure on the quotient phase space of the theory (passing to the symplectic quotient accounts for the gauge symmetry of the theory, which is, in fact, its key property). This brings into play a specific regularised determinant of a differential operator, itself represented as an auxiliary Berezin functional integral over anti-commuting variables. The associated extra "non-physical" particles are the famous Faddeev–Popov ghosts that soon became sort of a mascot of the new method.

As the ideas of quantum field theory were spreading over new areas of mathematics (in particular, representation theory and topology), the force and flexibility of the Faddeev–Popov approach were fully confirmed once again. The refined "method of ghosts" developed into a convenient cohomology technique directly connected to supersymmetry concepts (the BRST method).

## Automorphic functions and three-dimensional inverse scattering

The work on quantum Yang–Mills theory is probably the best known of Faddeev's results of the late 1960s, although it is by no means the only one. In the aftermath of his fundamental works on perturbation theory for operators with continuous spectrum, Faddeev addressed the spectral theory of the automorphic Laplace operator on the Poincaré upper half-plane (the standard model of the non-Euclidean plane). The key problem that attracted much attention at the time was the famous trace formula found by A. Selberg, which is particularly non-trivial for discrete subgroups with a non-compact fundamental domain. At I.M. Gelfand's initiative, Faddeev applied to this problem the methods he had developed in his study of scattering theory and perturbation theory for operators with continuous spectrum. This resulted in a non-arithmetic proof of the spectral theorem for the automorphic Laplacian, followed by a proof of the Selberg trace formula (in joint work with his PhD students A. Venkov and V. Kalinin). In another development, Faddeev (together with B. Pavlov) explored the non-stationary scattering problem for the automorphic wave equation, which allows the interpretation of the zeros of Riemann's zeta function as quantum mechanical resonances.

Simultaneously, Faddeev obtained a crucial advancement in the three-dimensional inverse problem for the Schrödinger operator. The key difficulty here was to find an adequate substitute for the so-called Volterra transformation operators, which play a prominent role in the treatment of the one-dimensional inverse problem. This was done in a 1965 paper but a complete exposition had to wait for about a decade because of intensive work on other subjects. Although these results are less widely known, Faddeev considered them as his best analytic results.

During the conference "Mathematical Physics. Its Past, Present and Future". Euler Mathematical Institute, St. Petersburg, March 2014.

## Classical integrability

Another exciting area was the theory of integrable systems, which was started by the famous paper of Gardner, Greene, Kruskal and Miura (1967) on the Korteweg–de Vries equation. Faddeev learned about these developments a few years later in 1971, during a conference on the inverse scattering problem. As it turned out, the technique he had developed in his PhD thesis had now become directly relevant for the new method. His first major contribution to the new theory was his joint paper with V. Zakharov, which established that the KdV equation is completely integrable in a technical sense.

The ideological importance of this paper was immense: it provided a first ever non-trivial example of an infinite-dimensional, completely integrable system and triggered a complete change of the paradigm in the study of nonlinear evolution equations. From the very beginning, Faddeev's interest in the study of these equations was fuelled not by their role as useful models in mechanics or hydrodynamics but rather by their possible application to quantum field theory. The original KdV equation is not quite appropriate in this respect because of its non-relativistic kinematics but very quickly Faddeev came upon a truly exciting example, the now famous Sine–Gordon equation, which he studied together with his young student L. Takhtajan. Soliton solutions for this model may be interpreted as genuine relativistic particles and hence the particle content of the corresponding QFT model is much richer than suggested by naive perturbation theory. The Sine-Gordon equation was the first indication of the important role of classical quasi-particle solutions in quantum field theory and, more generally, of how rich correctly chosen non-linear QFT models could be.

The way to a full justification of these bold predictions proved to be quite long and difficult. By that time, Faddeev had created, at the Leningrad Branch of the Steklov Institute, an independent Laboratory of Mathematical Problems in Physics, which gathered his young students around him. With some pride, Faddeev defined his own role in this small team as that of a playing coach. Many results and key developments of the next two decades were largely due to their collective work. Faddeev's weekly seminar at the Steklov Institute became a focus of research activity in various aspects of integrability, in QFT and in infinite dimensional Lie groups and Lie algebras. This versatile activity and lecture courses that Faddeev delivered in the 1970s, notably at the Summer School in Les Houches, have contributed substantially to the fundamental reshaping of mathematical physics in general, with its new emphasis on interdisciplinary research and the increased role of geometric and algebraic ideas.

## The quantum inverse scattering method

Besides the study of various examples of integrable systems, the mid-1970s were also marked by the first attempts to understand quantisation of integrable models in QFT, at first at the semiclassical level. This demanded a good deal of heavy technical work, which was needed to confirm the stability of solitons, contrary to the initial scepticism of theoretical physicists. This work prepared the way for a major breakthrough at the end of the decade, when a new systematic method for solving quantum counterparts of classical integrable systems was created. This was a truly fundamental discovery that united ideas from the classical inverse scattering method, the recent developments in quantum statistical physics (due mainly to R. Baxter) and the old technical insights of quantum mechanics (the Bethe ansatz). The keystone of the new method was the beautiful algebra based on the notion of the "quantum R-matrix". One of the important examples of a quantum R-matrix was extracted from an old paper of C.N. Yang and hence the main algebraic identity satisfied by quantum R-matrices was given the name of quantum Yang–Baxter identity (a name by which it became universally known). Faddeev's programme talk with a sketch of the quantum inverse scattering method was delivered in May 1978; within a year, all his major conjectures were confirmed, with key contributions from Faddeev's pupils and collaborators: E. Sklyanin, L. Takhtajan, P. Kulish and others. One of the highlights of the new method was the solution of the quantum Sine–Gordon model.

The new algebra, focused on the quantum Yang–Baxter identity, soon led to the discovery of new algebraic objects that have subsequently been baptised quantum groups. A first example of a quantised universal enveloping algebra is due to P. Kulish and N. Reshetikhin; further examples and appropriate axiomatics are due to V. Drinfeld. Quantum groups started a new chapter in non-commutative algebra, with numerous applications ranging from knot theory and low-dimensional topology to combinatorics and representation theory. A few years later, Faddeev, together with Reshetikhin and Takhtajan, developed an original approach to the quantisation of Lie groups and Lie algebras based entirely on the use of quantum R-matrices. While the notion of quantum groups has won tremendous popularity, it should be noted that it only formalised the 'easy part' of the quantum inverse scattering method, its true core certainly being

**An excursion to the French Alps. Aiguille du Midi, March 2015.**

the algebraic and analytic technique used to solve the spectral problem for quantum Hamiltonians. In its simplest version, this was the algebraic Bethe ansatz invented by Faddeev, together with Sklyanin and Takhtajan; it was followed by the more elaborate techniques of the functional Bethe ansatz and quantum separation of variables, introduced by Sklyanin a few years later. Active work in this direction is still going on up to the present day.

### Quantum anomalies and the search of knotted solitons

The rapid development of the quantum inverse scattering method had, to a certain extent, pushed aside the four-dimensional physics in the work of Faddeev's laboratory. Still, there are quite a few important results that were obtained in this direction as well. In the 1980s, there was the joint work of Faddeev and S. Shatashvili on quantum anomalies (in particular, the Gauss law anomaly in Yang–Mills theory), which resulted in the discovery of a new, interesting cohomology class and an associated abelian extension of the three-dimensional current group. Faddeev was particularly fond of these results, since they brought to bear, rather unexpectedly, the discoveries in homological algebra of his father Professor D.K. Faddeev from the 1940s.

One more research direction was the search of non-trivial, soliton-like solutions of non-linear equations in three and four dimensions, based on the use of the Hopf invariant. In the 1990s, Faddeev's collaborator A. Niemi confirmed numerically the existence of stable "knotted" solutions of the modified Skyrme model proposed by Faddeev. These solutions play a key role in the description of the hypothetical "glueball" solutions of the Yang–Mills equations related to one of the possible scenarios of quark confinement.

### Later years

The decay of the Soviet Union and the deep crisis of the country brought about profound changes in the composition of the Faddeev group. Many of his former students and collaborators were dispersed over various laboratories and universities all over the world. There were also several early losses to deplore, provoked by the stresses of the situation in the 1990s. Those who stayed at the Steklov Institute were spending a good share of their time abroad as well. There were still quite a few gifted students but they too could only find decent jobs abroad. In the early 1990s, the support provided by the Soros Foundation was of great help but gradually it became clear that fundamental research and science in general are by no means a priority of the new Russian authorities.

During these years, Faddeev travelled a lot but his fundamental desire was to stay at home. He declined, in particular, an invitation to head the Institute of Theoretical Physics at Stony Brook after the retirement of C.N. Yang. His constant preoccupation was to save mathematics in Russia, keeping afloat both the Steklov Institute and the Mathematics Division of the Russian Academy of Sciences. Over the years, this task was getting more and more painful, causing much distress and disillusionment. He largely returned to a more solitary style of work characteristic of his younger years, in contrast to the team style of the 1980s. Still, some new fruitful collaborations emerged during this period, along with quite a few old ones. Among his important discoveries of this period, one should mention the concept of modular duality for quantum groups. This concept, which emerged from the study of integrable quantum models in discrete space-time, opened a totally new and very promising chapter in representation theory of quantum groups. While early work focused mainly on the highest weight representations of quantum groups, Faddeev's work started the study of principal series representations, which proved to be extremely rich in various interdisciplinary connections, with links to non-commutative geometry, finite difference operators, new classes of special functions, etc. Research in this area is now actively pursued by Faddeev's pupils.

Faddeev's work in mathematics and physics won him wide international recognition. He has been awarded many prestigious awards, among them the Dirac Medal (1995), the Max Planck Medal (1996), the Euler Medal (2002), the Henri Poincaré Prize (2006), the Shaw Prize (jointly with V. Arnold, 2008) and the Lomonosov Medal (2014). He was elected to leading academies including the Royal Academy of Sweden (1989), the National Academy of the USA (1990), the French Academy of Sciences (2000) and the Royal Society (2010). Since 1976, he has been a full member of the Soviet (now, Russian) Academy of Sciences. In 1986–1990, he served as President of the International Mathematical Union.

Faddeev's legacy retains all its importance for current research as well as for the future of mathematical physics. One striking example of this is given by the recent discovery of unexpected links between Yang–Mills theory and the quantum inverse scattering method. In the early years of gauge fields theory, there existed a somewhat romantic hope that Yang–Mills theory itself was

**Steklov Institute, St. Petersburg, September 2010. From left to right: I. Aref'eva, S. Shatashvili, P. Kulish, L. Faddeev, L. Takhtajan, A. Volkov.**


**At a workshop in Chamonix, March 2015. From left to right: L. Faddeev, A. Volkov, L. Takhtajan, A. Alexeev, M. Semenov-Tian-Shansky.**

integrable. This proved to be false but one of its versions, supersymmetric Yang–Mills theory, is indeed close to integrability or exact solvability. As discovered recently by Shatashvili and Nekrasov, the description of the vacuum sector in supersymmetric Yang–Mills theory (in dimension 4) directly leads to quantum integrable systems (both of standard and of new types). All the main ingredients of the quantum inverse scattering method are naturally incorporated into this new approach. This fascinating link between the seemingly very remote aspects of Faddeev's legacy is a spectacular confirmation of its depth and vitality. Our feelings now may be expressed by the line of an old Roman poet: *letum non omnia finit*. Faddeev's works and ideas remain a source of inspiration for all of us and are destined for a long and fruitful life in posterity.

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