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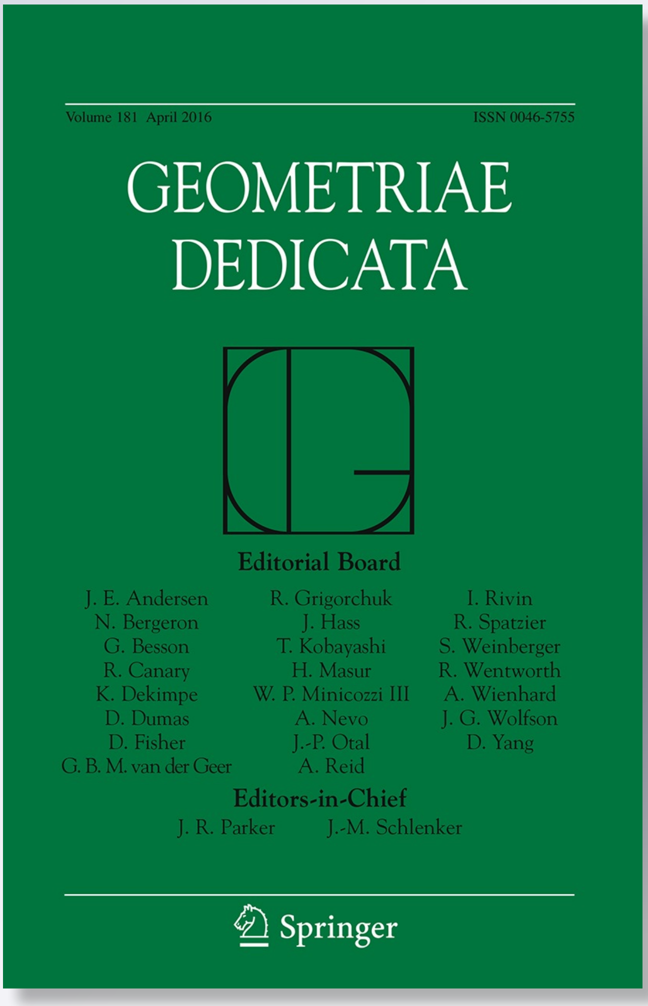

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ORIGINAL PAPER

Explicit computation of the Chern character forms

Leon A. Takhtajan[1,2]

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Abstract We propose a method for explicit computation of the Chern character form of a holomorphic Hermitian vector bundle (E, h) over a complex manifold X in a local holomorphic frame. First, we use the descent equations arising in the double complex of (p, q)-forms on X and find the explicit degree decomposition of the Chern–Simons form cs_k associated to the Chern character form ch_k of (E, h). Second, we introduce the so-called ascent equations that start from the $(2k-1, 0)$ component of cs_k, and use the Cholesky decomposition of the Hermitian metric h to represent the Chern–Simons form, modulo d-exact forms, as a ∂-exact form. This yields a formula for the Bott–Chern form bc_k of type $(k-1, k-1)$ such that $\mathrm{ch}_k = \frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial \mathrm{bc}_k$. Explicit computation is presented for the cases $k = 2$ and 3.

Keywords Chern character form · Chern-Simons form · Bott-Chern form · Ascent and descent equations · Cholesky decomposition

Mathematics Subject Classification (2000) 32L05 · 32Q

1 Introduction

Let V be a C^∞-complex vector bundle with a connection $\nabla = d + A$ over a smooth manifold X. The Chern character form $\mathrm{ch}(V, \nabla)$ for the pair (V, ∇) is defined by

$$\mathrm{ch}(V, \nabla) = \mathrm{tr}\left\{\exp\left(\frac{\sqrt{-1}}{2\pi}\nabla^2\right)\right\}.$$

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Here ∇^2 is the curvature of the connection ∇, an End V-valued 2-form on X, and tr is the trace in the endomorphism bundle End V. The Chern character form is closed, $d\,\mathrm{ch}(V,\nabla)=0$, and its cohomology class in $H^*(X,\mathbb{C})$ does not depend on the choice of ∇ (see, e.g., [1]).

Let ∇^0 and ∇^1 be two connections on V. In [2], Chern and Simons introduced secondary characteristic forms—the Chern–Simons forms $\mathrm{cs}(\nabla^1,\nabla^0)$. They are defined modulo exact forms, satisfy the equation

$$d\,\mathrm{cs}(\nabla^1,\nabla^0)=\mathrm{ch}(V,\nabla^1)-\mathrm{ch}(V,\nabla^0), \tag{1}$$

and enjoy a functoriality property under the pullbacks with smooth maps. When the bundle V is flat, putting $\nabla^1=d+A$ and $\nabla^0=d$ and using a linear homotopy $A(t)=tA$ in the Chern–Weil homotopy formula, one obtains an explicit formula for the Chern–Simons form $\mathrm{cs}(A)$ in terms of A.

Let (E,h) be a holomorphic Hermitian vector bundle—a holomorphic vector bundle of rank r over a complex manifold X, $\dim_{\mathbb{C}} X=n$, with a Hermitian metric h. The metric h induces canonical connection $d+\boldsymbol{\theta}$ in E with the curvature form $\boldsymbol{\Theta}$. In the local holomorphic frame, $\boldsymbol{\theta}=h^{-1}\partial h$ and $\boldsymbol{\Theta}=\bar\partial\boldsymbol{\theta}$ (see, e.g., [1]). Chern–Weil theory associates to every polynomial Φ on $\mathrm{GL}(r,\mathbb{C})$, invariant under conjugation, a differential form $\Phi(\boldsymbol{\Theta})$ on X. A special case of this construction is the Chern character form $\mathrm{ch}(E,h)$, defined by

$$\mathrm{ch}(E,h)=\mathrm{tr}\left\{\exp\left(\frac{\sqrt{-1}}{2\pi}\boldsymbol{\Theta}\right)\right\}=\sum_{k=0}^{n}\mathrm{ch}_k(E,h).$$

Let h_1 and h_2 be two Hermitian metrics on a holomorphic vector bundle E over a complex manifold X. In the classical paper [3], Bott and Chern showed the existence of certain secondary characteristic forms, the Bott–Chern secondary forms $\mathrm{bc}(E,h_1,h_2)$. They are defined modulo ∂ and $\bar\partial$-exact forms, satisfy the equation

$$\frac{\sqrt{-1}}{2\pi}\bar\partial\partial\,\mathrm{bc}(E,h_1,h_2)=\mathrm{ch}(E,h_1)-\mathrm{ch}(E,h_2)$$

and enjoy the functoriality property with respect to the pullbacks by holomorphic maps. Here the Chern character forms are computed for canonical connections in (E,h_1) and (E,h_2). The Bott–Chern forms have been used in geometric stability [4,5], in higher dimensional Arakelov geometry [6,7] and in physics [8] (see also [9] for their application to differential K-theory).

However, it is difficult to obtain explicit formulas for the Bott–Chern forms. It is already mentioned in the remark in [3, Sect. 3] that even for a linear homotopy h_t of Hermitian metrics, the homotopy formula in Proposition 3.15 in [3] contains the inverse metrics through $\boldsymbol{\Theta}_t=\bar\partial(h_t^{-1}\partial h_t)$ and does not allow to integrate over t in a closed form. As the result, it is difficult[1] to get explicit formulas for the Bott–Chern forms in terms of the Hermitian metrics h_1 and h_2 only. This problem manifests itself even for the case when E is a trivial bundle with metrics $h_1=h$ and $h_2=I$, the identity matrix.

Here we show how using global coordinates on the space of Hermitian positive-definite matrices associated with the Cholesky decomposition, one can obtain explicit formulas for the Bott–Chern forms on trivial bundles. Namely, in Proposition 1 we present an explicit decomposition of the Chern–Simons form cs_k associated to the Chern character form $\mathrm{ch}_k=\mathrm{ch}_k(E,h)$ into (p,q)-degrees. It is done in Sect. 2 by solving the descent equations from the double complex of (p,q)-forms on X, applied to ch_k. In Sect. 3 we introduce the so-called

[1] As was observed in [4], "One interesting feature is that we have an example of a variational problem with no simple explicit formula for the Lagrangian".

ascent equations to represent the Chern–Simons form, modulo d-exact forms, as a ∂-exact form. These equations start from the $(2k-1,0)$ component of cs_k and produce the Bott–Chern form—a form bc_k of degree $(k-1,k-1)$ such that $\mathrm{ch}_k = \frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial \mathrm{bc}_k$. It is obtained by repeatedly finding corresponding ∂-antiderivatives and seems to be very non-local. Using the Cholesky decomposition of the Hermitian metric h we explicitly solve the ascent equations in cases $k=2$ and $k=3$, and obtain explicit local formulas for the corresponding Bott–Chern forms bc_k. These formulas are presented, respectively, in Propositions 2 and 3 and constitute the main result of the paper. We believe that such explicit local formulas for the Bott–Chern forms exist for all k. In Remark 5 we prove that the form bc_2 is positive, and in Remark 7 we directly show that for bundles with upper-triangular transition functions the Euler–Lagrange functional $M_C(-,K)$ introduced in [4] is bounded below.

2 Double descent

2.1 Set-up

Let h be a Hermitian metric in a rank r trivial complex vector bundle over a complex manifold X (i.e., in general we consider a local holomorphic frame over some open neighborhood). Put (see, e.g., [1])

$$\boldsymbol{\theta} = h^{-1}\partial h \quad \text{and} \quad \boldsymbol{\Theta} = \bar{\partial}\boldsymbol{\theta}.$$

We have the following useful formulas

$$\partial\boldsymbol{\theta} = -\boldsymbol{\theta}^2, \quad \bar{\partial}\boldsymbol{\theta} = \boldsymbol{\Theta} \quad \text{and} \quad \partial\boldsymbol{\Theta} = [\boldsymbol{\Theta},\boldsymbol{\theta}], \quad \bar{\partial}\boldsymbol{\Theta} = 0, \tag{2}$$

where for the matrix-valued differential forms we write AB instead of $A\wedge B$, etc. In particular, we have

$$\partial\boldsymbol{\Theta}^k = [\boldsymbol{\Theta}^k,\boldsymbol{\theta}] \quad \text{and} \quad \partial(\boldsymbol{\theta}\boldsymbol{\Theta}^k) = -\boldsymbol{\theta}\boldsymbol{\Theta}^k\boldsymbol{\theta}. \tag{3}$$

We have, using $\imath = \frac{\sqrt{-1}}{2\pi}$,

$$\mathrm{ch}_k(h) = \frac{\imath^k}{k!}\omega_{k,k},$$

where

$$\omega_{k,k} = \mathrm{tr}\,\boldsymbol{\Theta}^k$$

is a ∂ and $\bar{\partial}$-closed real form of type (k,k); here and in what follows $\omega_{p,q}$ denotes a (p,q)-form. It follows from the Poincaré lemma that *locally* (i.e., on some polydisk coordinate chart of X) there are forms $\omega_{k+l,k-l-1}$ such that

$$\begin{aligned}
\omega_{k,k} &= \bar{\partial}\omega_{k,k-1},\\
\partial\omega_{k,k-1} &= \bar{\partial}\omega_{k+1,k-2},\\
&\ \vdots\\
\partial\omega_{2k-2,1} &= \bar{\partial}\omega_{2k-1,0},\\
\partial\omega_{2k-1,0} &= 0.
\end{aligned}$$

These descent equations[2] can be written succinctly as a single equation

$$\omega_{k,k} = (\bar{\partial} - t\partial)\left(\omega_{k,k-1} + t\omega_{k+1,k-2} + \cdots + t^{k-2}\omega_{2k-2,1} + t^{k-1}\omega_{2k-1,0}\right), \tag{4}$$

which holds for all $t \in \mathbb{R}$.

Remark 1 Putting $t = -1$ we get

$$\omega_{k,k} = d\left(\omega_{k,k-1} - \omega_{k+1,k-2} + \cdots + (-1)^{k-2}\omega_{2k-2,1} + (-1)^{k-1}\omega_{2k-1,0}\right),$$

where $d = \partial + \bar{\partial}$. This gives an explicit decomposition of the Chern–Simons secondary form cs_k into (p, q)-degrees, $p + q = 2k - 1$:

$$\mathrm{cs}_k = \frac{\imath^k}{k!}\left(\omega_{k,k-1} - \omega_{k+1,k-2} + \cdots + (-1)^{k-2}\omega_{2k-2,1} + (-1)^{k-1}\omega_{2k-1,0}\right). \tag{5}$$

It is easy to compute all these forms using (2)–(3) and equations

$$\partial\boldsymbol{\theta}^2 = 0, \quad \bar{\partial}\boldsymbol{\theta}^2 = [\boldsymbol{\Theta}, \boldsymbol{\theta}]. \tag{6}$$

First, we observe that

$$\omega_{k,k-1} = \mathrm{tr}(\boldsymbol{\theta}\boldsymbol{\Theta}^{k-1})$$

and state the following result.

Lemma 1 *We have*

$$\partial\omega_{k,k-1} = \mathrm{tr}(\boldsymbol{\theta}^2\boldsymbol{\Theta}^{k-1}) = \bar{\partial}\omega_{k+1,k-2},$$

where

$$\omega_{k+1,k-2} = \frac{1}{k+1}\,\mathrm{tr}\left\{\boldsymbol{\theta}\left(\boldsymbol{\theta}^2\boldsymbol{\Theta}^{k-2} + \boldsymbol{\Theta}\boldsymbol{\theta}^2\boldsymbol{\Theta}^{k-3} + \cdots + \boldsymbol{\Theta}^{k-3}\boldsymbol{\theta}^2\boldsymbol{\Theta} + \boldsymbol{\Theta}^{k-2}\boldsymbol{\theta}^2\right)\right\}.$$

Proof Using (2)–(3), we have

$$\partial\,\mathrm{tr}\left(\boldsymbol{\theta}\boldsymbol{\Theta}^{k-1}\right) = \mathrm{tr}\left(-\boldsymbol{\theta}^2\boldsymbol{\Theta}^{k-1} - \boldsymbol{\theta}\left(\boldsymbol{\Theta}^{k-1}\boldsymbol{\theta} - \boldsymbol{\theta}\boldsymbol{\Theta}^{k-1}\right)\right) = -\,\mathrm{tr}\left(\boldsymbol{\theta}\boldsymbol{\Theta}^{k-1}\boldsymbol{\theta}\right) = \mathrm{tr}\,\boldsymbol{\theta}^2\boldsymbol{\Theta}^{k-1}.$$

Next, using (2) and (6), we get

$$\begin{aligned}\bar{\partial}\sum_{i=0}^{k-2}\left(\boldsymbol{\theta}\boldsymbol{\Theta}^i\boldsymbol{\theta}^2\boldsymbol{\Theta}^{k-2-i}\right) &= \sum_{i=0}^{k-2}\left(\boldsymbol{\Theta}^{i+1}\boldsymbol{\theta}^2\boldsymbol{\Theta}^{k-2-i}\right) - \sum_{i=0}^{k-2}\left(\boldsymbol{\theta}\boldsymbol{\Theta}^i\left(\boldsymbol{\Theta}\boldsymbol{\theta} - \boldsymbol{\theta}\boldsymbol{\Theta}\right)\boldsymbol{\Theta}^{k-2-i}\right)\\ &= \sum_{i=0}^{k-2}\left(\boldsymbol{\Theta}^{i+1}\boldsymbol{\theta}^2\boldsymbol{\Theta}^{k-2-i}\right) + \boldsymbol{\theta}^2\boldsymbol{\Theta}^{k-1} - \boldsymbol{\theta}\boldsymbol{\Theta}^{k-1}\boldsymbol{\theta},\end{aligned}$$

since the second sum telescopes. Using the cyclic property of the trace, we obtain the formula for $\omega_{k+1,k-2}$.

Observe that $\omega_{k,k-1}$ is the constant term a_0 in the polynomial

$$F_k(t) = \mathrm{tr}\left\{\boldsymbol{\theta}\left(\boldsymbol{\Theta} + t\boldsymbol{\theta}^2\right)^{k-1}\right\} = a_0 + a_1 t + \cdots + a_{k-1}t^{k-1}, \tag{7}$$

[2] Compare with the double descent in [10] and with the holomorphic descent in [8].

while by Lemma 1,

$$\omega_{k+1,k-2} = \frac{1}{k+1} a_1.$$

This suggests to consider all coefficients a_l of $F_k(t)$—differential forms of degrees $(k+l, k-l-1)$, $l = 0, 1, \dots, k-1$.

Lemma 2 *Put $G_k(t) = \operatorname{tr}(\boldsymbol{\Theta} + t\boldsymbol{\theta}^2)^k$. We have*

$$\bar{\partial} F_k(t) - t\partial F_k(t) = G_k(t).$$

Proof It follows from Eq. (2) and (6) that

$$\partial(\boldsymbol{\Theta} + t\boldsymbol{\theta}^2) = \left[(\boldsymbol{\Theta} + t\boldsymbol{\theta}^2), \boldsymbol{\theta}\right] \quad \text{and} \quad \bar{\partial}(\boldsymbol{\Theta} + t\boldsymbol{\theta}^2) = t\left[(\boldsymbol{\Theta} + t\boldsymbol{\theta}^2), \boldsymbol{\theta}\right],$$

which implies

$$\partial(\boldsymbol{\Theta} + t\boldsymbol{\theta}^2)^k = \left[(\boldsymbol{\Theta} + t\boldsymbol{\theta}^2)^k, \boldsymbol{\theta}\right] \quad \text{and} \quad \bar{\partial}(\boldsymbol{\Theta} + t\boldsymbol{\theta}^2)^k = t\left[(\boldsymbol{\Theta} + t\boldsymbol{\theta}^2)^k, \boldsymbol{\theta}\right].$$

Therefore,

$$\begin{aligned}\partial F_k(t) &= \operatorname{tr}\left\{-\boldsymbol{\theta}^2(\boldsymbol{\Theta} + t\boldsymbol{\theta}^2)^{k-1} - \boldsymbol{\theta}\left((\boldsymbol{\Theta} + t\boldsymbol{\theta}^2)^{k-1}\boldsymbol{\theta} - \boldsymbol{\theta}(\boldsymbol{\Theta} + t\boldsymbol{\theta}^2)^{k-1}\right)\right\} \\ &= \operatorname{tr}\left\{\boldsymbol{\theta}^2(\boldsymbol{\Theta} + t\boldsymbol{\theta}^2)^{k-1}\right\}\end{aligned}$$

and

$$\begin{aligned}\bar{\partial} F_k(t) &= \operatorname{tr}\left\{\boldsymbol{\Theta}(\boldsymbol{\Theta} + t\boldsymbol{\theta}^2)^{k-1} - t\boldsymbol{\theta}\left((\boldsymbol{\Theta} + t\boldsymbol{\theta}^2)^{k-1}\boldsymbol{\theta} - \boldsymbol{\theta}(\boldsymbol{\Theta} + t\boldsymbol{\theta}^2)^{k-1}\right)\right\} \\ &= t\operatorname{tr}\left\{\boldsymbol{\theta}^2(\boldsymbol{\Theta} + t\boldsymbol{\theta}^2)^{k-1}\right\} + \operatorname{tr}(\boldsymbol{\Theta} + t\boldsymbol{\theta}^2)^k,\end{aligned}$$

so that $(\bar{\partial} - t\partial)F_k(t) = G_k(t)$.

From here it is easy to find all descent forms $\omega_{k+l,k-l-1}$.

Proposition 1 *We have*

$$\mathrm{cs}_k = \frac{\iota^k}{k!}\left(\omega_{k,k-1} - \omega_{k+1,k-2} + \cdots + (-1)^{k-2}\omega_{2k-2,1} + (-1)^{k-1}\omega_{2k-1,0}\right),$$

where

$$\omega_{k+l,k-l-1} = \frac{k!l!}{(k+l)!} a_l, \quad l = 0, 1, \dots, k-1.$$

In particular,

$$\omega_{2k-1,0} = \frac{k!(k-1)!}{(2k-1)!} \operatorname{tr} \boldsymbol{\theta}^{2k-1}.$$

Proof Using the cyclic property of the trace and the computation in the proof of Lemma 2, we obtain

$$\frac{dG_k}{dt}(t) = k\operatorname{tr}\left\{\boldsymbol{\theta}^2(\boldsymbol{\Theta} + t\boldsymbol{\theta}^2)^{k-1}\right\} = k\partial F_k(t),$$

so that

$$G_k(t) = b_0 + k\partial a_0 \frac{t}{1} + k\partial a_1 \frac{t^2}{2} + \cdots + k\partial a_{k-1}\frac{t^k}{k},$$

where $b_0 = \operatorname{tr} \boldsymbol{\Theta}^k = \omega_{k,k}$. Now it follows from Lemma 2 that

$$\bar{\partial} a_l = \binom{k+l}{l} \partial a_{l-1}, \quad l = 1, \dots, k-1,$$

and since $a_0 = \omega_{k,k-1}$, we easily obtain

$$a_l = \frac{(k+l)\cdots(k+1)}{l!} \omega_{k+l,k-l-1}.$$

Thus for $k = 1$ we have

$$\omega_{1,0} = \operatorname{tr} \boldsymbol{\theta} = \partial \log \det h \quad \text{and} \quad \omega_{0,0} = \log \det h,$$

whereas for $k = 2$

$$\omega_{2,1} = \operatorname{tr}(\boldsymbol{\theta}\boldsymbol{\Theta}) \quad \text{and} \quad \omega_{3,0} = \frac{1}{3} \operatorname{tr} \boldsymbol{\theta}^3.$$

For $k = 3$ we have

$$\omega_{3,2} = \operatorname{tr}(\boldsymbol{\theta}\boldsymbol{\Theta}^2), \quad \omega_{4,1} = \frac{1}{2} \operatorname{tr}(\boldsymbol{\theta}^3\boldsymbol{\Theta}) \quad \text{and} \quad \omega_{5,0} = \frac{1}{10} \operatorname{tr} \boldsymbol{\theta}^5,$$

and for $k = 4$ from Proposition 1 we obtain

$$\omega_{4,3} = \operatorname{tr}(\boldsymbol{\theta}\boldsymbol{\Theta}^3), \quad \omega_{5,2} = \frac{1}{5} \operatorname{tr}\left(\boldsymbol{\theta}^3\boldsymbol{\Theta}^2 + \boldsymbol{\theta}\boldsymbol{\Theta}\boldsymbol{\theta}^2\boldsymbol{\Theta} + \boldsymbol{\theta}\boldsymbol{\Theta}^2\boldsymbol{\theta}^2\right), \quad \omega_{6,1} = \frac{1}{5} \operatorname{tr}(\boldsymbol{\theta}^5\boldsymbol{\Theta})$$

and

$$\omega_{7,0} = \frac{1}{35} \operatorname{tr} \boldsymbol{\theta}^7.$$

Remark 2 The forms $\operatorname{tr} \boldsymbol{\theta}^{2k-1}$, $k \geq 1$, where $\boldsymbol{\theta} = g^{-1}dg$ is a Maurer–Cartan form, generate the cohomology ring $H^\bullet(\mathrm{GL}(\infty, \mathbb{C}), \mathbb{Q})$ for the stabilized complex general linear group $\mathrm{GL}(\infty, \mathbb{C})$.

3 Double ascent

3.1 Set-up

From the descent equations it follows that there is a form $\omega_{2k-2,0}$ such that

$$\omega_{2k-1,0} = \partial \omega_{2k-2,0}.$$

Now going up from the bottom to the top (this explains the terminology), we get

$$\partial(\omega_{2k-2,1} + \bar{\partial}\omega_{2k-2,0}) = 0,$$

so that there is a form $\omega_{2k-3,1}$ such that

$$\omega_{2k-2,1} + \bar{\partial}\omega_{2k-2,0} = \partial \omega_{2k-3,1}.$$

Therefore

$$\partial(\omega_{2k-3,2} + \bar{\partial}\omega_{2k-3,1}) = 0$$

and there is a form $\omega_{2k-4,2}$ such that

$$\omega_{2k-3,2} + \bar{\partial}\omega_{2k-3,1} = \partial\omega_{2k-4,2}.$$

Repeating this procedure, we finally get a form $\omega_{k-1,k-1}$ such that

$$\omega_{k,k-1} + \bar{\partial}\omega_{k,k-2} = \partial\omega_{k-1,k-1}.$$

The ascent equations can be written succinctly as

$$\frac{k!}{\imath^k}\mathrm{cs}_k = \partial\omega_{k-1,k-1} - d\left(\omega_{k,k-2} - \omega_{k+1,k-3} + \cdots + (-1)^k\omega_{2k-2,0}\right). \tag{8}$$

Defining CS_k as cs_k modulo exact forms (see [11]), we can rewrite (8) as

$$\mathrm{CS}_k = \frac{\imath^k}{k!}\partial\omega_{k-1,k-1}.$$

Therefore,

$$\mathrm{ch}_k = \frac{\imath^k}{k!}\bar{\partial}\partial\omega_{k-1,k-1},$$

so that $\omega_{k-1,k-1}$ is $\dfrac{k!}{\imath^{k-1}}$ times the Bott–Chern secondary form bc_k (see [3]).

Remark 3 As a corollary, we have the following version of local "$\partial\bar{\partial}$ lemma": for each form ω of type (k,k) on a complex manifold X satisfying $d\omega = 0$ on every polydisk neighborhood $U \subset X$, there is a form θ_U on U such that $\omega|_U = \bar{\partial}\partial\theta_U$.

Solving 'explicitly' the ascent equations would give explicit local expressions of the Chern character forms ch_k in terms of the corresponding Bott–Chern forms bc_k. It is known that it is not possible to get local formulas in terms of the matrix h alone. This is because each step in the ascent procedure uses Poincaré lemma which, in general, contains an integration through the homotopy formula. However, one can solve the ascent equations explicitly by using the Cholesky decomposition!

Namely, put

$$h = b^*ab = cb, \quad c = b^*a,$$

where the matrix b is upper-triangular with 1's on the diagonal, and a is diagonal with positive entries. The matrix elements a_i and b_{ij}, $i = 1, \ldots, r$, $j > i$, are global coordinates on the homogeneous space $\mathcal{H}_r = \mathrm{GL}(r,\mathbb{C})/\mathrm{U}(r)$ of hermitian positive-definite $r \times r$ matrices. We get

$$\boldsymbol{\theta} = h^{-1}\partial h = b^{-1}\partial b + b^{-1}c^{-1}\partial c\, b = b^{-1}(\theta_1 + \theta_2)b, \tag{9}$$

where

$$\theta_1 = \partial b\, b^{-1} \quad \text{and} \quad \theta_2 = c^{-1}\partial c.$$

Introducing $\theta = \theta_1 + \theta_2$, we obtain

$$\boldsymbol{\Theta} = \bar{\partial}\boldsymbol{\theta} = b^{-1}\left(\bar{\partial}\theta - \bar{\theta}_1\theta - \theta\bar{\theta}_1\right)b, \tag{10}$$

where

$$\bar{\theta}_1 = \bar{\partial}b\, b^{-1} \quad \text{and} \quad \bar{\theta}_2 = c^{-1}\bar{\partial}c.$$

These matrix-valued 1-forms satisfy

$$\partial\theta_1 = \theta_1^2, \quad \partial\theta_2 = -\theta_2^2, \quad \bar\partial\bar\theta_1 = \bar\theta_1^2, \quad \bar\partial\bar\theta_2 = -\bar\theta_2^2, \tag{11}$$

$$\bar\partial\theta_1 = -\partial\bar\theta_1 + \theta_1\bar\theta_1 + \bar\theta_1\theta_1, \quad \bar\partial\theta_2 = -\partial\bar\theta_2 - \theta_2\bar\theta_2 - \bar\theta_2\theta_2. \tag{12}$$

Moreover, since θ_1 is nilpotent (upper-triangular with zeros on the diagonal) and

$$\theta_2 = a^{-1}\theta_1^* a + a^{-1}\partial a,$$

we have important property

$$\operatorname{tr}\left(\theta_1^{l_1}\bar\theta_1^{\bar l_1}\right) = \operatorname{tr}\left(\theta_2^{l_2}\bar\theta_2^{\bar l_2}\right) = 0 \tag{13}$$

for all $l_1 + \bar l_1 > 0$ and l_2 or $\bar l_2 > 1$. We will also be using

$$\bar{\boldsymbol\theta} = h^{-1}\bar\partial h = b^{-1}\bar\theta b, \tag{14}$$

where $\bar\theta = \bar\theta_1 + \bar\theta_2$, so that

$$\bar\partial\bar{\boldsymbol\theta} = -\bar{\boldsymbol\theta}^2 \quad \text{and} \quad \boldsymbol\Theta = -\partial\bar{\boldsymbol\theta} - \boldsymbol\theta\bar{\boldsymbol\theta} - \bar{\boldsymbol\theta}\boldsymbol\theta. \tag{15}$$

I claim that it possible to compute explicitly differential forms $\omega_{2k-2-l,l}$ as traces of polynomials of the matrix-valued 1-forms $\theta_1, \bar\theta_1, \theta_2, \bar\theta_2$ and their ∂ and $\bar\partial$ differentials. In particular, one can obtain explicit formulas for the Bott–Chern forms $\omega_{k-1,k-1}$ as traces of polynomials in these variables. Though I do not have a nice general proof of this result, the explicit computation of these forms for $k = 2$ and $k = 3$ is given below.

Remark 4 The Cholesky decomposition is useful since by the holomorphic splitting principle (see, e.g., [12, Corollary 9.26]), for every holomorphic vector bundle $E \to X$ there exists a variety Y and a flat morphism $p : Y \to X$ such that the bundle $p^*(E)$ over Y admits upper-triangular transition functions.

3.2 The case $k = 2$

Start with the form $\omega_{3,0} = \frac{1}{3}\operatorname{tr}\boldsymbol\theta^3$. Using (11), (13) and the Cholesky decomposition we have

$$\omega_{3,0} = \frac{1}{3}\operatorname{tr}\left(\theta_1^3 + 3\theta_1^2\theta_2 + 3\theta_1\theta_2^2 + \theta_2^3\right) = \operatorname{tr}\left(\theta_1^2\theta_2 + \theta_1\theta_2^2\right) = \partial\operatorname{tr}(\theta_1\theta_2),$$

so that

$$\omega_{2,0} = \operatorname{tr}(\theta_1\theta_2).$$

Using (15) we get

$$\begin{aligned}\omega_{2,1} &= \operatorname{tr}(\boldsymbol\theta\boldsymbol\Theta) = -\operatorname{tr}(\boldsymbol\theta(\partial\bar{\boldsymbol\theta} + \boldsymbol\theta(\boldsymbol\theta\bar{\boldsymbol\theta} + \bar{\boldsymbol\theta}\boldsymbol\theta)) = \partial\operatorname{tr}(\boldsymbol\theta\bar{\boldsymbol\theta}) + \operatorname{tr}(\boldsymbol\theta^2\bar{\boldsymbol\theta} - \boldsymbol\theta(\boldsymbol\theta\bar{\boldsymbol\theta} + \bar{\boldsymbol\theta}\boldsymbol\theta))\\ &= \partial\operatorname{tr}(\theta\bar\theta) - \operatorname{tr}(\theta^2\bar\theta),\end{aligned}$$

and using (12) we obtain

$$\begin{aligned}\omega_{2,1} + \bar\partial\omega_{2,0} &= \partial\operatorname{tr}(\theta\bar\theta) + \operatorname{tr}(-\theta^2\bar\theta + (\bar\partial\theta_1\theta_2 - \theta_1\bar\partial\theta_2))\\ &= \partial\operatorname{tr}(\theta\bar\theta) + \operatorname{tr}(-\theta^2\bar\theta - \partial\bar\theta_1\theta_2 + \theta_1\partial\bar\theta_2 + (\theta_1\theta_2 + \theta_2\theta_1)\bar\theta)\\ &= \partial\operatorname{tr}(\theta\bar\theta - (\bar\theta_1\theta_2 - \bar\theta_2\theta_1))\\ &\quad + \operatorname{tr}(-\theta^2\bar\theta + \theta_1^2\bar\theta_2 + \bar\theta_2\bar\theta_1^2 + (\theta_1\theta_2 + \theta_2\theta_1)\bar\theta)\end{aligned}$$

$$= \partial\,\mathrm{tr}(\theta\bar{\theta} - (\bar{\theta}_1\theta_2 - \bar{\theta}_2\theta_1)).$$

Thus

$$\omega_{1,1} = \mathrm{tr}\left(\theta\bar{\theta} - (\bar{\theta}_1\theta_2 - \bar{\theta}_2\theta_1)\right) = \mathrm{tr}\left(2\theta_2\bar{\theta}_1 + \theta_2\bar{\theta}_2\right), \tag{16}$$

and we obtain the following result.

Proposition 2 *The second Bott–Chern form* bc_2 *of a trivial Hermitian vector bundle* $(\mathbb{C}^r, h)$ *over a complex manifold* X *in Cholesky coordinates* $h = b^*ab$ *is given by the formula*

$$\mathrm{bc}_2 = \frac{\sqrt{-1}}{4\pi}\,\mathrm{tr}\left(2\theta_2\bar{\theta}_1 + \theta_2\bar{\theta}_2\right).$$

Here $\bar{\theta}_1 = \bar{\partial}bb^{-1}$, $\theta_2 = c^{-1}\partial c$ *and* $\bar{\theta}_2 = c^{-1}\bar{\partial}c$.

Remark 5 Using that $c = b^*a$, we obtain from (16) that

$$\omega_{1,1} = \mathrm{tr}\left(a^{-1}\partial a \wedge a^{-1}\bar{\partial}a + 2\varphi \wedge \varphi^*\right),$$

where $\varphi = a^{-1/2}(b^*)^{-1}\partial b^* a^{1/2}$, so that $\sqrt{-1}\,\omega_{1,1} \geq 0$.

Remark 6 When

$$a = \begin{pmatrix} 1 & 0 \\ 0 & e^{\sigma} \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & \bar{f} \\ 0 & 1 \end{pmatrix},$$

we get

$$\omega_{0,0} = \sigma \quad \text{and} \quad \omega_{1,1} = \mathrm{tr}\left(\partial\sigma \wedge \bar{\partial}\sigma + 2e^{-\sigma}\partial f \wedge \bar{\partial}\bar{f}\right),$$

so that

$$\frac{1}{2}\bar{\partial}\partial\omega_{1,1} - \frac{1}{2}(\bar{\partial}\partial\omega_{0,0})^2 = \bar{\partial}\partial\left(e^{-\sigma}\partial f \wedge \bar{\partial}\bar{f}\right),$$

in agreement with Remark 3.4 in [9].

Following Remark 4, consider a rank r Hermitian vector bundle (E, h) with the transition functions taking values in the Borel subgroup $B(r)$ of upper-triangular matrices in $\mathrm{GL}(r, \mathbb{C})$. In terms of a local trivialization of E—an open cover $\{U_\alpha\}$ of X and holomorphic transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to B(r)$, a Hermitian metric h on E is given by a collection $\{h_\alpha\}$ of positive-definite Hermitian matrices on U_α, satisfying

$$h_\beta = g^*_{\alpha\beta} h_\alpha g_{\alpha\beta} \quad \text{on} \quad U_\alpha \cap U_\beta.$$

Denote by $\mathrm{bc}_{2\alpha}$ the second Bott–Chern form on U_α and write $g_{\alpha\beta} = a_{\alpha\beta}b_{\alpha\beta}$, where $a_{\alpha\beta}$ are diagonal and $b_{\alpha\beta}$ are unipotent. From Proposition 2 we obtain

$$\mathrm{bc}_{2\beta} = \mathrm{bc}_{2\alpha} + c_{\alpha\beta} \quad \text{on} \quad U_\alpha \cap U_\beta, \tag{17}$$

where

$$c_{\alpha\beta} = \frac{\sqrt{-1}}{4\pi}\,\mathrm{tr}\left\{a^{-1}_{\alpha\beta}\partial a_{\alpha\beta} \wedge \left(\overline{a^{-1}_{\alpha\beta}\partial a_{\alpha\beta}}\right) + a^{-1}_{\alpha\beta}\partial a_{\alpha\beta} \wedge a^{-1}_{\alpha}\bar{\partial}a_\alpha + a^{-1}_{\alpha}\partial a_\alpha \wedge \left(\overline{a^{-1}_{\alpha\beta}\partial a_{\alpha\beta}}\right)\right\},$$

and depends only on $a_{\alpha\beta}$. Since $a_{\alpha\beta}$ are holomorphic, we have $\bar{\partial}\partial c_{\alpha\beta} = 0$. In particular, if transition functions are unipotent, it follows from (17) that local expressions $\{\mathrm{bc}_{2\alpha}\}$ determine a well-defined $(1, 1)$-form on X.

Remark 7 Given two Hermitian metrics h_1 and h_2 on a holomorphic vector bundle E, we define a local Bott–Chern form $\mathrm{bc}_2(h_1, h_2)$ by

$$\mathrm{bc}_2(h_1, h_2) = \mathrm{bc}_2(h_1) - \mathrm{bc}_2(h_2),$$

where $\mathrm{bc}_2(h_{1,2})$ are given in Proposition 2 with $h = h_{1,2}$. It follows from (17) that for the bundle E with upper-triangular transition functions $\mathrm{bc}_2(h_1, h_2)$ is a well-defined $(1, 1)$-form on X. In particular, for such bundles Proposition 2 provides an explicit formulas for the functionals $M_\omega(-, K)$ and $M_C(-, K)$ in Donaldson's paper [4], and from Remark 5 one gets that $M_C(-, K)$ is bounded below [4, Corollary 9]. The latter functional corresponds to the case when $X = C$, and algebraic curve, an in our notations is given by

$$M_C(h_1, h_2) = \int_C 2\pi \mathrm{bc}_2(h_1, h_2) + 4\pi\sqrt{-1}\mu(\mathscr{E})\mathrm{bc}_1(h_1, h_2)\omega.$$

Here ω is a Kähler form on C, $\mathrm{vol}(C) = 1$, and h_1, h_2 are Hermitian metrics in the holomorphic vector bundle $\mathscr{E}$ over C with the slope $\mu(\mathscr{E})$.

Remark 8 Upper triangular matrices were used to study the higher Reidemeister torsion in [13]. Though the set-up in this paper and in [13] is different, it would be interesting to compare the corresponding calculations.

3.3 The case $k = 3$

Using (13) we get

$$\begin{aligned}\omega_{5,0} &= \frac{1}{10}\operatorname{tr}\boldsymbol{\theta}^5 = \frac{1}{10}\operatorname{tr}\theta^5\\ &= \frac{1}{2}\operatorname{tr}\left(\theta_1^4\theta_2 + \theta_1^3\theta_2^2 + \theta_1^2\theta_2\theta_1\theta_2 + \theta_1\theta_2\theta_1\theta_2^2 + \theta_1^2\theta_2^3 + \theta_1\theta_2^4\right)\\ &= \frac{1}{2}\partial\operatorname{tr}\left(\theta_1^3\theta_2 + \theta_1\theta_2^3 + \frac{1}{2}(\theta_1\theta_2)^2\right),\end{aligned}$$

so that

$$\omega_{4,0} = \frac{1}{2}\operatorname{tr}\left(\theta_1^3\theta_2 + \theta_1\theta_2^3 + \frac{1}{2}(\theta_1\theta_2)^2\right).$$

We will compute $\omega_{4,1} + \bar\partial\omega_{4,0}$ and find $\omega_{3,1}$ such that

$$\omega_{4,1} + \bar\partial\omega_{4,0} = \partial\omega_{3,1}.$$

First using (15) we get

$$\begin{aligned}\omega_{4,1} &= \frac{1}{2}\operatorname{tr}(\boldsymbol{\theta}^3\boldsymbol{\Theta}) = -\frac{1}{2}\operatorname{tr}(\boldsymbol{\theta}^3(\partial\bar{\boldsymbol{\theta}} + \bar{\boldsymbol{\theta}}\boldsymbol{\theta} + \boldsymbol{\theta}\bar{\boldsymbol{\theta}}))\\ &= \frac{1}{2}\partial\operatorname{tr}(\boldsymbol{\theta}^3\bar{\boldsymbol{\theta}}) + \frac{1}{2}\operatorname{tr}(\boldsymbol{\theta}^4\bar{\boldsymbol{\theta}} - \boldsymbol{\theta}^3(\bar{\boldsymbol{\theta}}\boldsymbol{\theta} + \boldsymbol{\theta}\bar{\boldsymbol{\theta}}))\\ &= \frac{1}{2}\partial\operatorname{tr}(\theta^3\bar\theta) - \frac{1}{2}\operatorname{tr}(\theta^4\bar\theta).\end{aligned}$$

Next, using (12) we obtain

$$\bar\partial\omega_{4,0} = \frac{1}{2}\operatorname{tr}\left(I_1\bar\partial\theta_1 + I_2\bar\partial\theta_2\right)$$

$$= \frac{1}{2} \operatorname{tr} \left(I_1(-\partial\bar{\theta}_1 + \theta_1\bar{\theta}_1 + \bar{\theta}_1\theta_1) + I_2(-\partial\bar{\theta}_2 - \theta_2\bar{\theta}_2 - \bar{\theta}_2\theta_2) \right)$$
$$= \frac{1}{2}\partial \operatorname{tr} \left(I_1\bar{\theta}_1 + I_2\bar{\theta}_2 \right) + \frac{1}{2} \operatorname{tr} \left((-\partial I_1 + I_1\theta_1 + \theta_1 I_1)\bar{\theta}_1 - (\partial I_2 + I_2\theta_2 + \theta_2 I_2)\bar{\theta}_2 \right),$$

where

$$I_1 = \theta_1^3 + \theta_1^2\theta_2 + \theta_2\theta_1^2 - \theta_1\theta_2\theta_1 + \theta_2\theta_1\theta_2 + \theta_2^3 = \theta^3 - \theta\theta_2\theta_1 - \theta_1\theta_2\theta$$

and

$$I_2 = -\left(\theta_1^3 + \theta_1\theta_2\theta_1 + \theta_1\theta_2^2 - \theta_2\theta_1\theta_2 + \theta_2^2\theta_1 + \theta_2^3 \right) = -\theta^3 + \theta\theta_1\theta_2 + \theta_2\theta_1\theta.$$

Using identities

$$\partial I_1 - I_1\theta_1 - \theta_1 I_1 = -\theta^4 \quad \text{and} \quad \partial I_2 + I_2\theta_2 + \theta_2 I_2 = -\theta^4,$$

we get

$$\omega_{4,1} + \bar{\partial}\omega_{4,0} = \frac{1}{2}\partial \operatorname{tr} \left(\theta^3\bar{\theta} + I_1\bar{\theta}_1 + I_2\bar{\theta}_2 \right),$$

so that

$$\omega_{3,1} = \frac{1}{2} \operatorname{tr} \left(\theta^3\bar{\theta} + I_1\bar{\theta}_1 + I_2\bar{\theta}_2 \right).$$

Equivalently,

$$\omega_{3,1} = \frac{1}{2} \operatorname{tr} \left(2\theta^3\bar{\theta}_1 - (\theta_1\theta_2^2 + 2\theta_1\theta_2\theta_1 + \theta_2^2\theta_1)\bar{\theta}_1 + (\theta_1^2\theta_2 + 2\theta_2\theta_1\theta_2 + \theta_2\theta_1^2)\bar{\theta}_2 \right).$$

Finally, we will compute $\omega_{3,2} + \bar{\partial}\omega_{3,1}$ and find $\omega_{2,2}$ such that

$$\omega_{3,2} + \bar{\partial}\omega_{3,1} = \partial\omega_{2,2}.$$

First, using (15) we obtain

$$\partial \operatorname{tr}(\boldsymbol{\theta\Theta\bar{\theta}}) = \operatorname{tr} \left(-\boldsymbol{\theta}^2\boldsymbol{\Theta\bar{\theta}} - \boldsymbol{\theta}(\boldsymbol{\Theta\theta} - \boldsymbol{\theta\Theta})\boldsymbol{\bar{\theta}} + \boldsymbol{\theta\Theta}(\boldsymbol{\Theta} + \boldsymbol{\theta\bar{\theta}} + \boldsymbol{\bar{\theta}\theta}) \right)$$
$$= \operatorname{tr} \left(\boldsymbol{\theta\Theta}^2 + \boldsymbol{\theta}^2\boldsymbol{\Theta\bar{\theta}} \right),$$

and

$$\partial \operatorname{tr}(\boldsymbol{\bar{\theta}\Theta\theta}) = \operatorname{tr} \left(-(\boldsymbol{\Theta} + \boldsymbol{\theta\bar{\theta}} + \boldsymbol{\bar{\theta}\theta})\boldsymbol{\Theta\theta} - \boldsymbol{\bar{\theta}}(\boldsymbol{\Theta\theta} - \boldsymbol{\theta\Theta})\boldsymbol{\theta} + \boldsymbol{\bar{\theta}\Theta\theta}^2 \right)$$
$$= -\operatorname{tr} \left(\boldsymbol{\theta\Theta}^2 + \boldsymbol{\bar{\theta}\Theta\theta}^2 \right),$$

so that

$$\omega_{3,2} = \operatorname{tr}(\boldsymbol{\theta\Theta}^2) = \partial \left\{ \frac{1}{2} \operatorname{tr}(\boldsymbol{\theta\Theta\bar{\theta}} - \boldsymbol{\bar{\theta}\Theta\theta}) \right\} - \frac{1}{2} \operatorname{tr}(\boldsymbol{\theta}^2\boldsymbol{\Theta\bar{\theta}} + \boldsymbol{\bar{\theta}\Theta\theta}^2).$$

Next, we write

$$\omega_{3,1} = \frac{1}{2} \operatorname{tr} \left(\boldsymbol{\theta}^3\boldsymbol{\bar{\theta}} + \boldsymbol{I}_1\boldsymbol{\bar{\theta}}_1 + \boldsymbol{I}_2\boldsymbol{\bar{\theta}}_2 \right) = \omega_{3,1}^{(1)} + \omega_{3,1}^{(2)},$$

where $\boldsymbol{\bar{\theta}}_1 = b^{-1}\bar{\theta}_1 b$ and $\boldsymbol{\bar{\theta}}_2 = b^{-1}\bar{\theta}_2 b$ and

$$\boldsymbol{I}_1 = \boldsymbol{\theta}^3 - \boldsymbol{\theta\theta}_2\boldsymbol{\theta}_1 - \boldsymbol{\theta}_1\boldsymbol{\theta}_2\boldsymbol{\theta}, \quad \boldsymbol{I}_2 = -\boldsymbol{\theta}^3 + \boldsymbol{\theta\theta}_1\boldsymbol{\theta}_2 + \boldsymbol{\theta}_2\boldsymbol{\theta}_1\boldsymbol{\theta},$$

where $\boldsymbol{\theta}_1 = b^{-1}\theta_1 b$ and $\boldsymbol{\theta}_2 = b^{-1}\theta_2 b$. We have

$$\begin{aligned}\bar\partial\omega_{3,1}^{(1)} &= \frac{1}{2}\bar\partial\,\mathrm{tr}(\boldsymbol{\theta}^3\bar{\boldsymbol{\theta}})\\ &= \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\Theta}\boldsymbol{\theta}^2\bar{\boldsymbol{\theta}} - \boldsymbol{\theta}\boldsymbol{\Theta}\boldsymbol{\theta}\bar{\boldsymbol{\theta}} + \boldsymbol{\theta}^2\boldsymbol{\Theta}\bar{\boldsymbol{\theta}} + \boldsymbol{\theta}^3\bar{\boldsymbol{\theta}}^2)\\ &= \frac{1}{2}\,\mathrm{tr}(\bar{\boldsymbol{\theta}}\boldsymbol{\Theta}\boldsymbol{\theta}^2 + \boldsymbol{\theta}^2\boldsymbol{\Theta}\bar{\boldsymbol{\theta}} - \boldsymbol{\theta}\boldsymbol{\Theta}\boldsymbol{\theta}\bar{\boldsymbol{\theta}} + \boldsymbol{\theta}^3\bar{\boldsymbol{\theta}}^2),\end{aligned}$$

so that

$$\omega_{3,2} + \bar\partial\omega_{3,1}^{(1)} = \partial\left\{\frac{1}{2}\,\mathrm{tr}(\boldsymbol{\theta}\boldsymbol{\Theta}\bar{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}\boldsymbol{\Theta}\boldsymbol{\theta})\right\} + \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\theta}^3\bar{\boldsymbol{\theta}}^2 - \boldsymbol{\theta}\bar{\boldsymbol{\theta}}\boldsymbol{\theta}\boldsymbol{\Theta}).$$

We also have

$$\begin{aligned}\partial\,\mathrm{tr}(\boldsymbol{\theta}\bar{\boldsymbol{\theta}})^2 &= 2\,\mathrm{tr}\left((-\boldsymbol{\theta}^2\bar{\boldsymbol{\theta}} - \boldsymbol{\theta}\partial\bar{\boldsymbol{\theta}})\boldsymbol{\theta}\bar{\boldsymbol{\theta}}\right)\\ &= 2\,\mathrm{tr}\left((-\boldsymbol{\theta}^2\bar{\boldsymbol{\theta}} + \boldsymbol{\theta}\boldsymbol{\Theta} + \boldsymbol{\theta}\boldsymbol{\theta}\bar{\boldsymbol{\theta}} + \boldsymbol{\theta}\bar{\boldsymbol{\theta}}\boldsymbol{\theta})\boldsymbol{\theta}\bar{\boldsymbol{\theta}}\right)\\ &= 2\,\mathrm{tr}\left(\boldsymbol{\theta}\boldsymbol{\Theta}\boldsymbol{\theta}\bar{\boldsymbol{\theta}} + \boldsymbol{\theta}\bar{\boldsymbol{\theta}}\boldsymbol{\theta}\boldsymbol{\theta}\bar{\boldsymbol{\theta}}\right),\end{aligned}$$

so that

$$\mathrm{tr}(\boldsymbol{\theta}\bar{\boldsymbol{\theta}}\boldsymbol{\theta}\boldsymbol{\Theta}) = \partial\left\{\frac{1}{2}\,\mathrm{tr}(\boldsymbol{\theta}\bar{\boldsymbol{\theta}})^2\right\} - \mathrm{tr}\left(\boldsymbol{\theta}^2\bar{\boldsymbol{\theta}}\boldsymbol{\theta}\bar{\boldsymbol{\theta}}\right).$$

Thus we obtain

$$\omega_{3,2} + \bar\partial\omega_{3,1}^{(1)} = \partial\left\{\frac{1}{2}\,\mathrm{tr}\left(\boldsymbol{\theta}\boldsymbol{\Theta}\bar{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}\boldsymbol{\Theta}\boldsymbol{\theta} - \frac{1}{2}(\boldsymbol{\theta}\bar{\boldsymbol{\theta}})^2\right)\right\} + \frac{1}{2}\,\mathrm{tr}\left(\boldsymbol{\theta}^3\bar{\boldsymbol{\theta}}^2 + \boldsymbol{\theta}^2\bar{\boldsymbol{\theta}}\boldsymbol{\theta}\bar{\boldsymbol{\theta}}\right).$$

Note that this formula is written in terms of the matrix h only. Using Cholesky decomposition, we have

$$\omega_{3,2} + \bar\partial\omega_{3,1}^{(1)} = \partial\left\{\frac{1}{2}\,\mathrm{tr}\left(\boldsymbol{\theta}\boldsymbol{\Theta}\bar{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}\boldsymbol{\Theta}\boldsymbol{\theta} - \frac{1}{2}(\boldsymbol{\theta}\bar{\boldsymbol{\theta}})^2\right)\right\} + \frac{1}{2}\,\mathrm{tr}\left(\theta^3\bar\theta^2 + \theta^2\bar\theta\theta\bar\theta\right)$$

and it remains to compute

$$\begin{aligned}\bar\partial\omega_{3,1}^{(2)} &= \frac{1}{2}\bar\partial\,\mathrm{tr}\left(\boldsymbol{I}_1\bar{\boldsymbol{\theta}}_1 + \boldsymbol{I}_2\bar{\boldsymbol{\theta}}_2\right) = \frac{1}{2}\bar\partial\,\mathrm{tr}\left(I_1\bar\theta_1 + I_2\bar\theta_2\right)\\ &= \frac{1}{2}\,\mathrm{tr}\left(\bar\partial I_1\bar\theta_1 + \bar\partial I_2\bar\theta_2 - I_1\bar\theta_1^2 + I_2\bar\theta_2^2\right).\end{aligned}$$

By a straightforward computation using

$$\bar\partial\theta = -\partial\bar\theta + \theta_1\bar\theta_1 + \bar\theta_1\theta_1 - \theta_2\bar\theta_2 - \bar\theta_2\theta_2$$

we get

$$\begin{aligned}\bar\partial I_1\bar\theta_1 &= \mathrm{tr}\left\{\Big[\bar\partial\theta\theta^2 - \theta\bar\partial\theta\theta + \theta^2\bar\partial\theta - \bar\partial\theta\theta_2\theta_1 + \theta(\bar\partial\theta_2\theta_1 - \theta_2\bar\partial\theta_1)\right.\\ &\quad \left. - (\bar\partial\theta_1\theta_2 - \theta_1\bar\partial\theta_2)\theta - \theta_1\theta_2\bar\partial\theta\Big]\bar\theta_1\right\}\\ &= \mathrm{tr}\left\{\Big[- \partial\bar\theta\theta^2 + \theta\partial\bar\theta\theta - \theta^2\partial\bar\theta + \partial\bar\theta\theta_2\theta_1 + \theta(\theta_2\partial\bar\theta_1 - \partial\bar\theta_2\theta_1)\right.\\ &\quad \left. + (\partial\bar\theta_1\theta_2 - \theta_1\partial\bar\theta_2)\theta + \theta_1\theta_2\partial\bar\theta\Big]\bar\theta_1\right\}\end{aligned}$$

$$+\operatorname{tr}\Big\{\Big[(\theta_1\bar\theta_1+\bar\theta_1\theta_1-\theta_2\bar\theta_2-\bar\theta_2\theta_2)\theta^2-\theta(\theta_1\bar\theta_1+\bar\theta_1\theta_1-\theta_2\bar\theta_2-\bar\theta_2\theta_2)\theta$$
$$+\theta^2(\theta_1\bar\theta_1+\bar\theta_1\theta_1-\theta_2\bar\theta_2-\bar\theta_2\theta_2)-(\theta_1\bar\theta_1+\bar\theta_1\theta_1-\theta_2\bar\theta_2-\bar\theta_2\theta_2)\theta_2\theta_1$$
$$-\theta((\theta_2\bar\theta_2+\bar\theta_2\theta_2)\theta_1+\theta_2(\theta_1\bar\theta_1+\bar\theta_1\theta_1))-((\theta_1\bar\theta_1+\bar\theta_1\theta_1)\theta_2+\theta_1(\theta_2\bar\theta_2+\bar\theta_2\theta_2))\theta$$
$$-\theta_1\theta_2(\theta_1\bar\theta_1+\bar\theta_1\theta_1-\theta_2\bar\theta_2-\bar\theta_2\theta_2)\Big]\bar\theta_1\Big\}$$

and

$$\begin{aligned}\bar\partial I_2\bar\theta_2&=\operatorname{tr}\Big\{\Big[-\bar\partial\theta\theta^2+\theta\bar\partial\theta\theta-\theta^2\bar\partial\theta+\bar\partial\theta\theta_1\theta_2-\theta(\bar\partial\theta_1\theta_2-\theta_1\bar\partial\theta_2)\\&\quad+(\bar\partial\theta_2\theta_1-\theta_2\bar\partial\theta_1)\theta+\theta_2\theta_1\bar\partial\theta\Big]\bar\theta_2\Big\}\\&=\operatorname{tr}\Big\{\Big[\partial\bar\theta\theta^2-\theta\partial\bar\theta\theta+\theta^2\partial\bar\theta-\partial\bar\theta\theta_1\theta_2-\theta(\theta_1\partial\bar\theta_2-\partial\bar\theta_1\theta_2)\\&\quad-(\partial\bar\theta_2\theta_1-\theta_2\partial\bar\theta_1)\theta-\theta_2\theta_1\partial\bar\theta\Big]\bar\theta_2\Big\}\\&\quad+\operatorname{tr}\Big\{\Big[-(\theta_1\bar\theta_1+\bar\theta_1\theta_1-\theta_2\bar\theta_2-\bar\theta_2\theta_2)\theta^2+\theta(\theta_1\bar\theta_1+\bar\theta_1\theta_1-\theta_2\bar\theta_2-\bar\theta_2\theta_2)\theta\\&\quad-\theta^2(\theta_1\bar\theta_1+\bar\theta_1\theta_1-\theta_2\bar\theta_2-\bar\theta_2\theta_2)+(\theta_1\bar\theta_1+\bar\theta_1\theta_1-\theta_2\bar\theta_2-\bar\theta_2\theta_2)\theta_1\theta_2\\&\quad-\theta(\theta_1(\theta_2\bar\theta_2+\bar\theta_2\theta_2)+(\theta_1\bar\theta_1+\bar\theta_1\theta_1)\theta_2)-(\theta_2(\theta_1\bar\theta_1+\bar\theta_1\theta_1)+(\theta_2\bar\theta_2+\bar\theta_2\theta_2)\theta_1)\theta\\&\quad+\theta_2\theta_1(\theta_1\bar\theta_1+\bar\theta_1\theta_1-\theta_2\bar\theta_2-\bar\theta_2\theta_2)\Big]\bar\theta_2\Big\}.\end{aligned}$$

Thus we obtain

$$\operatorname{tr}(\bar\partial I_1\bar\theta_1+\bar\partial I_2\bar\theta_2)=J_1+J_2,$$

where

$$\begin{aligned}J_1=\operatorname{tr}\Big\{&-\partial\bar\theta_2(\theta_1^2+\theta_2^2+\theta_1\theta_2)\bar\theta_1+(\theta_1^2+\theta_2^2+\theta_1\theta_2)\partial\theta_1\bar\theta_2-(\theta_1^2+\theta_2^2+\theta_2\theta_1)\partial\bar\theta_2\bar\theta_1\\&+\partial\bar\theta_1(\theta_1^2+\theta_2^2+\theta_2\theta_1)\bar\theta_2+\partial\bar\theta_1(\theta_2\theta_1-\theta_1\theta_2)\bar\theta_1-(\theta_2\theta_1-\theta_1\theta_2)\partial\bar\theta_1\bar\theta_1\\&+\partial\bar\theta_2(\theta_2\theta_1-\theta_1\theta_2)\bar\theta_2-(\theta_2\theta_1-\theta_1\theta_2)\partial\bar\theta_2\bar\theta_2+\theta_2\partial\bar\theta_1\theta_2\bar\theta_1+\theta_2\partial\bar\theta_2\theta_2\bar\theta_1\\&-\theta_1\partial\bar\theta_1\theta_1\bar\theta_2-\theta_1\partial\bar\theta_2\theta_1\bar\theta_2+\theta_1\partial\bar\theta_1\theta_2\bar\theta_1+\theta_2\partial\bar\theta_1\theta_1\bar\theta_1\\&-\theta_1\partial\bar\theta_2\theta_2\bar\theta_2-\theta_2\partial\bar\theta_2\theta_1\bar\theta_2-\theta_1\partial\bar\theta_2\theta_1\bar\theta_1+\theta_2\partial\bar\theta_1\theta_2\bar\theta_2\Big\}\end{aligned}$$

and

$$\begin{aligned}J_2=\operatorname{tr}\Big\{&\big[(\theta_1\bar\theta_1+\bar\theta_1\theta_1-\theta_2\bar\theta_2-\bar\theta_2\theta_2)\theta^2-\theta(\theta_1\bar\theta_1+\bar\theta_1\theta_1-\theta_2\bar\theta_2-\bar\theta_2\theta_2)\theta\\&+\theta^2(\theta_1\bar\theta_1+\bar\theta_1\theta_1-\theta_2\bar\theta_2-\bar\theta_2\theta_2)\big](\bar\theta_1-\bar\theta_2)-(\theta_1\bar\theta_1+\bar\theta_1\theta_1-\theta_2\bar\theta_2-\bar\theta_2\theta_2)(\theta_2\theta_1\bar\theta_1\\&-\theta_1\theta_2\bar\theta_2)-\theta_1\theta_2(\theta_1\bar\theta_1+\bar\theta_1\theta_1-\theta_2\bar\theta_2-\bar\theta_2\theta_2)\bar\theta_1+\theta_2\theta_1(\theta_1\bar\theta_1+\bar\theta_1\theta_1-\theta_2\bar\theta_2-\bar\theta_2\theta_2)\bar\theta_2\\&-\theta((\theta_2\bar\theta_2+\bar\theta_2\theta_2)\theta_1+\theta_2(\theta_1\bar\theta_1+\bar\theta_1\theta_1))\bar\theta_1-\theta(\theta_1(\theta_2\bar\theta_2+\bar\theta_2\theta_2)+(\theta_1\bar\theta_1+\bar\theta_1\theta_1)\theta_2)\bar\theta_2\\&-(\theta_1(\theta_2\bar\theta_2+\bar\theta_2\theta_2)+(\theta_1\bar\theta_1+\bar\theta_1\theta_1)\theta_2)\theta\bar\theta_1-((\theta_2\bar\theta_2+\bar\theta_2\theta_2)\theta_1+\theta_2(\theta_1\bar\theta_1+\bar\theta_1\theta_1)\theta\bar\theta_2\Big\}.\end{aligned}$$

Simplifying and using the cyclic property of the trace, we get

$$\begin{aligned}J_1=\operatorname{tr}\Big\{&(\theta_1^2+\theta_2^2+\theta_1\theta_2)\partial(\bar\theta_1\bar\theta_2)-(\theta_1^2+\theta_2^2+\theta_2\theta_1)\partial(\bar\theta_2\bar\theta_1)-(\theta_2\theta_1-\theta_1\theta_2)\partial(\bar\theta_1^2+\bar\theta_2^2)\\&+(\theta_2\partial\bar\theta_1\theta_2\bar\theta_2+\theta_2\bar\theta_1\theta_2\partial\bar\theta_2)-(\theta_1\partial\bar\theta_1\theta_1\bar\theta_2+\theta_1\bar\theta_1\theta_1\partial\bar\theta_2)+(\theta_2\partial\bar\theta_1\theta_1\bar\theta_1+\theta_2\bar\theta_1\theta_1\partial\bar\theta_1)\\&-(\theta_2\partial\bar\theta_2\theta_1\bar\theta_2+\theta_2\bar\theta_2\theta_1\partial\bar\theta_2)+\theta_2\partial\bar\theta_1\theta_2\bar\theta_1-\theta_1\partial\bar\theta_2\theta_1\bar\theta_2\Big\}\end{aligned}$$

$$= \partial\Big\{ \operatorname{tr}\Big((\theta_1^2+\theta_2^2+\theta_1\theta_2)\bar\theta_1\bar\theta_2 - (\theta_1^2+\theta_2^2+\theta_2\theta_1)\bar\theta_2\bar\theta_1 - (\theta_2\theta_1-\theta_1\theta_2)(\bar\theta_1^2+\bar\theta_2^2)$$
$$- \theta_2\bar\theta_1\theta_2\bar\theta_2 + \theta_1\bar\theta_1\theta_1\bar\theta_2 - \theta_2\bar\theta_1\theta_1\bar\theta_1 + \theta_2\bar\theta_2\theta_1\bar\theta_2 + \tfrac12((\theta_1\bar\theta_2)^2 - (\theta_2\bar\theta_1)^2)\Big)\Big\}$$
$$+ \operatorname{tr}\big\{-(\theta_1^2\theta_2+\theta_1\theta_2^2)\bar\theta_1\bar\theta_2 - (\theta_2^2\theta_1+\theta_2\theta_1^2)\bar\theta_2\bar\theta_1 - (\theta_1^2\theta_2+\theta_1\theta_2^2+\theta_2^2\theta_1$$
$$+ \theta_2\theta_1^2)(\bar\theta_1^2+\bar\theta_2^2) - \theta_2^2\bar\theta_1\theta_2\bar\theta_2 - \theta_2^2\bar\theta_2\theta_2\bar\theta_1 - \theta_1^2\bar\theta_1\theta_1\bar\theta_2 - \theta_1^2\bar\theta_2\theta_1\bar\theta_1$$
$$-\theta_2^2\bar\theta_1\theta_1\bar\theta_1 + \theta_1^2\bar\theta_1\theta_2\bar\theta_1 + \theta_2^2\bar\theta_2\theta_1\bar\theta_2 - \theta_1^2\bar\theta_2\theta_2\bar\theta_2 - \theta_1^2\bar\theta_2\theta_1\bar\theta_2 - \theta_2^2\bar\theta_1\theta_2\bar\theta_1\big\} = J_{11} + J_{12},$$

where

$$J_{11} = \partial\Big\{ \operatorname{tr}\Big((\theta_1^2+\theta_2^2+\theta_1\theta_2)\bar\theta_1\bar\theta_2 - (\theta_1^2+\theta_2^2+\theta_2\theta_1)\bar\theta_2\bar\theta_1 - (\theta_2\theta_1-\theta_1\theta_2)(\bar\theta_1^2+\bar\theta_2^2)$$
$$- \theta_2\bar\theta_1\theta_2\bar\theta_2 + \theta_1\bar\theta_1\theta_1\bar\theta_2 - \theta_2\bar\theta_1\theta_1\bar\theta_1 + \theta_2\bar\theta_2\theta_1\bar\theta_2 + \tfrac12((\theta_1\bar\theta_2)^2 - (\theta_2\bar\theta_1)^2)\Big)\Big\}$$

and

$$J_{12} = \operatorname{tr}\big\{-(\theta_1^2\theta_2+\theta_1\theta_2^2)\bar\theta_1\bar\theta_2 - (\theta_2^2\theta_1+\theta_2\theta_1^2)\bar\theta_2\bar\theta_1 - (\theta_1^2\theta_2+\theta_1\theta_2^2+\theta_2^2\theta_1+\theta_2\theta_1^2)(\bar\theta_1^2+\bar\theta_2^2)$$
$$- \theta_2^2\bar\theta_1\theta_2\bar\theta_2 - \theta_2^2\bar\theta_2\theta_2\bar\theta_1 - \theta_1^2\bar\theta_1\theta_1\bar\theta_2 - \theta_1^2\bar\theta_2\theta_1\bar\theta_1 - \theta_2^2\bar\theta_1\theta_1\bar\theta_1 + \theta_1^2\bar\theta_1\theta_2\bar\theta_1$$
$$+ \theta_2^2\bar\theta_2\theta_1\bar\theta_2 - \theta_1^2\bar\theta_2\theta_2\bar\theta_2 - \theta_1^2\bar\theta_2\theta_1\bar\theta_2 - \theta_2^2\bar\theta_1\theta_2\bar\theta_1\big\},$$

and

$$J_2 = \operatorname{tr}\big\{(\theta^2(\bar\theta_1-\bar\theta_2)(\theta_1\bar\theta_1-\theta_2\bar\theta_2) + \theta_1\theta^2(\bar\theta_1-\bar\theta_2)\bar\theta_1 - \theta_2\theta^2(\bar\theta_1-\bar\theta_2)\bar\theta_2$$
$$- \theta(\theta_1\bar\theta_1-\theta_2\bar\theta_2)\theta(\bar\theta_1-\bar\theta_2) - \theta_1\theta(\bar\theta_1-\bar\theta_2)\theta\bar\theta_1 + \theta_2\theta(\bar\theta_1-\bar\theta_2)\theta\bar\theta_2$$
$$+ \theta^2(\theta_1\bar\theta_1-\theta_2\bar\theta_2)(\bar\theta_1-\bar\theta_2) + \theta^2(\bar\theta_1\theta_1-\bar\theta_2\theta_2)(\bar\theta_1-\bar\theta_2) - \theta_2\theta_1\bar\theta_1\theta_1\bar\theta_1$$
$$- \theta_1\theta_2\theta_1\bar\theta_1^2 + \theta_2\theta_1\bar\theta_1\theta_2\bar\theta_2 + \theta_2\theta_2\theta_1\bar\theta_1\bar\theta_2 + \theta_1\theta_2\bar\theta_2\theta_1\bar\theta_1 + \theta_1^2\theta_2\bar\theta_2\bar\theta_1$$
$$- \theta_1\theta_2\bar\theta_2\theta_2\bar\theta_2 - \theta_2\theta_1\theta_2\bar\theta_2^2 - \theta_1\theta_2(\theta_1\bar\theta_1-\theta_2\bar\theta_2)\bar\theta_1 - \theta_1\theta_2(\bar\theta_1\theta_1-\bar\theta_2\theta_2)\bar\theta_1$$
$$+ \theta_2\theta_1(\theta_1\bar\theta_1-\theta_2\bar\theta_2)\bar\theta_2 + \theta_2\theta_1(\bar\theta_1\theta_1-\bar\theta_2\theta_2)\bar\theta_2 - \theta\theta_2\bar\theta\theta_1\bar\theta_1 - 2\theta_2\theta_1\bar\theta_1\theta\bar\theta_2$$
$$- \theta\theta_2\theta_1\bar\theta_1^2 - \theta\theta_1\bar\theta\theta_2\bar\theta_2 - \theta\theta_1\theta_2\bar\theta_2^2 - 2\theta_1\theta_2\bar\theta_2\theta\bar\theta_1 - \theta_2\theta\bar\theta_1\theta_1\bar\theta_2 - \theta_2\theta\bar\theta_1\theta_1\bar\theta_1$$
$$- \theta_1\theta_2\theta\bar\theta_1^2 - \theta_1\theta\bar\theta_2\theta_2\bar\theta_2 - \theta_2\theta_1\theta\bar\theta_2^2 - \theta_1\theta\bar\theta_2\theta_2\bar\theta_1\big\}.$$

Simplifying $J_{12}+J_2$ once again and after using numerous 'miraculous cancellations', we obtain

$$J_{12} + J_2 + \operatorname{tr}\left(-I_1\bar\theta_1^2 + I_2\bar\theta_2^2\right) = -\operatorname{tr}\left(\theta^3\bar\theta^2 + \theta^2\bar\theta\theta\bar\theta\right),$$

so that finally

$$\omega_{3,2} + \bar\partial\omega_{3,1} = \partial\Big\{\frac12 \operatorname{tr}\Big(\boldsymbol{\theta\Theta\bar\theta} - \boldsymbol{\bar\theta\Theta\theta} - \frac12(\theta\bar\theta)^2 + (\theta_1^2+\theta_2^2+\theta_1\theta_2)\bar\theta_1\bar\theta_2$$
$$- (\theta_1^2+\theta_2^2+\theta_2\theta_1)\bar\theta_2\bar\theta_1 - (\theta_2\theta_1-\theta_1\theta_2)(\bar\theta_1^2+\bar\theta_2^2) - \theta_2\bar\theta_1\theta_2\bar\theta_2$$
$$+ \theta_1\bar\theta_1\theta_1\bar\theta_2 - \theta_2\bar\theta_1\theta_1\bar\theta_1 + \theta_2\bar\theta_2\theta_1\bar\theta_2 + \frac12\left((\theta_1\bar\theta_2)^2 - (\theta_2\bar\theta_1)^2\right)\Big)\Big\}.$$

Thus we obtain the following result.

Proposition 3 *The third Bott–Chern form* bc_3 *of a trivial Hermitian vector bundle* $(\mathbb{C}^r, h)$ *over a complex manifold* X *in Cholesky coordinates* $h = b^*ab$ *is given by the formula*

$$\begin{aligned}\mathrm{bc}_3 = -\frac{1}{48\pi^2}\,\mathrm{tr}\Bigg(&\boldsymbol{\theta}\boldsymbol{\Theta}\bar{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}\boldsymbol{\Theta}\boldsymbol{\theta} - \frac{1}{2}(\theta\bar{\theta})^2 + (\theta_1^2 + \theta_2^2 + \theta_1\theta_2)\bar{\theta}_1\bar{\theta}_2 \\ &- (\theta_1^2 + \theta_2^2 + \theta_2\theta_1)\bar{\theta}_2\bar{\theta}_1 - (\theta_2\theta_1 - \theta_1\theta_2)(\bar{\theta}_1^2 + \bar{\theta}_2^2) - \theta_2\bar{\theta}_1\theta_2\bar{\theta}_2 \\ &+ \theta_1\bar{\theta}_1\theta_1\bar{\theta}_2 - \theta_2\bar{\theta}_1\theta_1\bar{\theta}_1 + \theta_2\bar{\theta}_2\theta_1\bar{\theta}_2 + \frac{1}{2}\big((\theta_1\bar{\theta}_2)^2 - (\theta_2\bar{\theta}_1)^2\big)\Bigg).\end{aligned}$$

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