Quantum Mechanics for Mathematicians

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To my teacher Ludwig Dmitrievich Faddeev with admiration and gratitude.

Contents

Preface

This book is based on graduate courses taught by the author over the last fourteen years in the mathematics department of Stony Brook University. The goal of these courses was to introduce second year graduate students with no prior knowledge of physics to the basic concepts and methods of quantum mechanics. For the last 50 years quantum physics has been a driving force behind many dramatic achievements in mathematics, similar to the role played by classical physics in the seventeenth to nineteenth centuries. Classical physics, especially classical mechanics, was an integral part of mathematical education up to the early twentieth century, with lecture courses given by Hilbert and Poincaré. Surprisingly, quantum physics, especially quantum mechanics, with its intrinsic beauty and connections with many branches of mathematics, has never been a part of a graduate mathematics curriculum. This course was developed to partially fill this gap and to make quantum mechanics accessible to graduate students and research mathematicians.

L.D. Faddeev was the first to develop a course in quantum mechanics for undergraduate students specializing in mathematics. From 1968 to 1973 he regularly lectured in the mathematics department of St. Petersburg State University in St. Petersburg, Russia[1], and the author enjoyed the opportunity to take his course. The notes for this book emerged from an attempt to create a similar course for graduate students, which uses more sophisticated mathematics and covers a larger variety of topics, including the Feynman path integral approach to quantum mechanics.

[1] At that time in Leningrad, Soviet Union.

There are many excellent physics textbooks on quantum mechanics, starting with the classic texts by P.A.M. Dirac [**Dir47**], L.D. Landau and E.M. Lifshitz [**LL58**], and V.A. Fock [**Foc78**], to the encyclopedic treatise by A. Messiah [**Mes99**], the recent popular textbook by J.J. Sakurai [**Sak94**], and many others. From a mathematics perspective, there are classic monographs by J. von Neumann [**vN96**] and by H. Weyl [**Wey50**], as well as a more recent book by G.W. Mackey [**Mac04**], which deal with the basic mathematical formalism and logical foundations of the theory. There is also a monumental project [**DEF$^+$99**], created with the purpose of introducing graduate students and research mathematicians to the realm of quantum fields and strings, both from a mathematics and a physics perspective. Though it contains a very detailed exposition of classical mechanics, classical field theory, and supersymmetry, oriented at the mathematical audience, quantum mechanics is discussed only briefly (with the exception of L.D. Faddeev's elegant introduction to quantum mechanics in [**Fad99**]). Excellent lecture notes for undergraduate students by L.D. Faddeev and O.A. Yakubovskiĭ [**FY80**] seems to be the only book on quantum mechanics completely accessible to mathematicians[2]. Recent books by S.J. Gustafson and I.M. Sigal [**GS03**] and by F. Strocchi [**Str05**] are also oriented at mathematicians. The latter is a short introductory course, while the former is more an intermediate level monograph on quantum theory rather than a textbook on quantum mechanics. There are also many specialized books on different parts of quantum mechanics, like scattering theory, the Schrödinger operator, $\mathbb{C}^*$-algebras and foundations, etc.

The present book gives a comprehensive treatment of quantum mechanics from a mathematics perspective and covers such topics as mathematical foundations, quantization, the Schrödinger equation, the Feynman path integral and functional methods, and supersymmetry. It can be used as a one-year graduate course, or as two one-semester courses: the introductory course based on the material in Part 1, and a more advanced course based on Part 2. Part 1 of the book, which consists of Chapters 1-4, can be considered as an expanded version of [**FY80**]. It uses more advanced mathematics than [**FY80**], and contains rigorous proofs of all main results, including the celebrated Stone-von Neumann theorem. It should be accessible to a second-year graduate student. As in [**FY80**], we adopt the approach, which goes back to Dirac and was further developed by Faddeev, that classical mechanics and quantum mechanics are just two different realizations of the fundamental mathematical structure of a physical theory that uses the notions of observables, states, measurement, and the time evolution — dynamics. Part 2, which consists of Chapters 5-8, deals with functional methods in quantum

[2]The English translation will appear in the AMS "The Student Mathematical Library" series.

mechanics, and goes beyond the material in [**FY80**]. Exposition there is less detailed and requires certain mathematical sophistication.

Though our presentation freely uses all the necessary tools of modern mathematics, it follows the spirit and tradition of the classical texts and monographs mentioned above. In this sense it can be considered "neoclassical" (as compared with a more abstract approach in [**DF99a**]). Each chapter in the book concludes with a special *Notes and references* section, which provides references to the necessary mathematics background and physics sources. A courageous reader can actually learn the relevant mathematics by studying the main text and consulting these references, and with enough sophistication, could "translate" corresponding parts in physics textbooks into the mathematics language. For the physics students, the book presents an opportunity to become familiar with the mathematical foundations and methods of quantum mechanics on a "case by case" basis. It is worth mentioning that development of many mathematics disciplines has been stimulated by quantum mechanics.

There are several ways to study the material in this book. A casual reader can study the main text in a cursory manner, and ignore numerous remarks and problems, located at the end of the sections. This would be sufficient to obtain basic minimal knowledge of quantum mechanics. A determined reader is supposed to fill in the details of the computations in the main text (a pencil and paper are required), which is the only way to master the material, and to attempt to solve the basic problems[3]. Finally, a truly devoted reader should try to solve all the problems (probably consulting the corresponding references at the end of each section) and to follow up on the remarks, which may often be linked to other topics not covered in the main text.

The author would like to thank the students in his courses for their comments on the draft of the lecture notes. He is especially grateful to his colleagues Peter Kulish and Lee-Peng Teo for the careful reading of the manuscript. The work on the book was partially supported by the NSF grants DMS-0204628 and DMS-0705263. Any opinions, findings, and conclusions or recommendations expressed in this book are those of the author and do not necessarily reflect the views of the National Science Foundation.

[3]We leave it to the reader to decide which problems are basic and which are advanced.

Part 1

Foundations

Chapter 1

Classical Mechanics

We assume that the reader is familiar with the basic notions from the theory of smooth (that is, C^∞) manifolds and recall here the standard notation. Unless it is stated explicitly otherwise, all maps are assumed to be smooth and all functions are assumed to be smooth and real-valued. Local coordinates $\boldsymbol{q} = (q^1, \dots, q^n)$ on a smooth n-dimensional manifold M at a point $q \in M$ are Cartesian coordinates on $\varphi(U) \subset \mathbb{R}^n$, where (U, φ) is a coordinate chart on M centered at $q \in U$. For $f : U \to \mathbb{R}^n$ we denote $(f \circ \varphi^{-1})(q^1, \dots, q^n)$ by $f(\boldsymbol{q})$, and we let

$$\frac{\partial f}{\partial \boldsymbol{q}} = \left(\frac{\partial f}{\partial q^1}, \dots, \frac{\partial f}{\partial q^n} \right)$$

stand for the gradient of a function f at a point $\boldsymbol{q} \in \mathbb{R}^n$ with Cartesian coordinates $(q^1, \dots, q^n)$. We denote by

$$\mathcal{A}^\bullet(M) = \bigoplus_{k=0}^{n} \mathcal{A}^k(M)$$

the graded algebra of smooth differential forms on M with respect to the wedge product, and by d the de Rham differential — a graded derivation of $\mathcal{A}^\bullet(M)$ of degree 1 such that df is a differential of a function $f \in \mathcal{A}^0(M) = C^\infty(M)$. Let $\mathrm{Vect}(M)$ be the Lie algebra of smooth vector fields on M with the Lie bracket $[\ ,\ ]$, given by a commutator of vector fields. For $X \in \mathrm{Vect}(M)$ we denote by $\mathcal{L}_X$ and i_X, respectively, the Lie derivative along X and the inner product with X. The Lie derivative is a degree 0 derivation of $\mathcal{A}^\bullet(M)$ which commutes with d and satisfies $\mathcal{L}_X(f) = X(f)$ for $f \in \mathcal{A}^0(M)$, and the inner product is a degree -1 derivation of $\mathcal{A}^\bullet(M)$ satisfying $i_X(f) = 0$

and $i_X(df) = X(f)$ for $f \in \mathcal{A}^0(M)$. They satisfy Cartan formulas

$$\mathcal{L}_X = i_X \circ d + d \circ i_X = (d + i_X)^2,$$
$$i_{[X,Y]} = \mathcal{L}_X \circ i_Y - i_Y \circ \mathcal{L}_X.$$

For a smooth mapping of manifolds $f : M \to N$ we denote by $f_* : TM \to TN$ and $f^* : T^*N \to T^*M$, respectively, the induced mappings on tangent and cotangent bundles. Other notations, including those traditional for classical mechanics, will be introduced in the main text.

1. Lagrangian Mechanics

1.1. Generalized coordinates. Classical mechanics describes systems of finitely many interacting *particles*[1]. A system is called *closed* if its particles do not interact with the outside material bodies. The position of a system in space is specified by positions of its particles and defines a point in a smooth, finite-dimensional manifold M, the *configuration space* of a system. Coordinates on M are called *generalized coordinates* of a system, and the dimension $n = \dim M$ is called the number of *degrees of freedom*[2].

The *state* of a system at any instant of time is described by a point $q \in M$ and by a tangent vector $v \in T_qM$ at this point. The basic principle of classical mechanics is the *Newton-Laplace determinacy principle* which asserts that a state of a system at a given instant completely determines its motion at all times t (in the future and in the past). The motion is described by the *classical trajectory* — a path $\gamma(t)$ in the configuration space M. In generalized coordinates $\gamma(t) = (q^1(t), \dots, q^n(t))$, and corresponding derivatives $\dot{q}^i = \dfrac{dq^i}{dt}$ are called *generalized velocities*. The Newton-Laplace principle is a fundamental experimental fact confirmed by our perception of everyday experiences. It implies that *generalized accelerations* $\ddot{q}^i = \dfrac{d^2q^i}{dt^2}$ are uniquely defined by generalized coordinates q^i and generalized velocities $\dot{q}^i$, so that classical trajectories satisfy a system of second order ordinary differential equations, called *equations of motion*. In the next section we formulate the most general principle governing the motion of mechanical systems.

1.2. The principle of the least action. A *Lagrangian system* on a configuration space M is defined by a smooth, real-valued function L on $TM \times \mathbb{R}$ — the direct product of a tangent bundle TM of M and the time axis[3] — called the *Lagrangian function* (or simply, *Lagrangian*). The motion of a

[1] A particle is a material body whose dimensions may be neglected in describing its motion.

[2] Systems with infinitely many degrees of freedom are described by classical field theory.

[3] It follows from the Newton-Laplace principle that L could depend only on generalized coordinates and velocities, and on time.

Lagrangian system (M, L) is described by the *principle of the least action in the configuration space* (or *Hamilton's principle*), formulated as follows.

Let

$$P(M)_{q_0,t_0}^{q_1,t_1} = \{\gamma : [t_0, t_1] \to M;\ \gamma(t_0) = q_0,\ \gamma(t_1) = q_1\}$$

be the space of smooth parametrized paths in M connecting points q_0 and q_1. The path space $P(M) = P(M)_{q_0,t_0}^{q_1,t_1}$ is an infinite-dimensional real Fréchet manifold, and the tangent space $T_\gamma P(M)$ to $P(M)$ at $\gamma \in P(M)$ consists of all smooth vector fields along the path γ in M which vanish at the endpoints q_0 and q_1. A smooth path Γ in $P(M)$, passing through $\gamma \in P(M)$, is called a *variation with fixed ends* of the path $\gamma(t)$ in M. A variation Γ is a family $\gamma_\varepsilon(t) = \Gamma(t, \varepsilon)$ of paths in M given by a smooth map

$$\Gamma : [t_0, t_1] \times [-\varepsilon_0, \varepsilon_0] \to M$$

such that $\Gamma(t, 0) = \gamma(t)$ for $t_0 \leq t \leq t_1$ and $\Gamma(t_0, \varepsilon) = q_0, \Gamma(t_1, \varepsilon) = q_1$ for $-\varepsilon_0 \leq \varepsilon \leq \varepsilon_0$. The tangent vector

$$\delta\gamma = \left.\frac{\partial \Gamma}{\partial \varepsilon}\right|_{\varepsilon=0} \in T_\gamma P(M)$$

corresponding to a variation $\gamma_\varepsilon(t)$ is traditionally called an *infinitesimal variation*. Explicitly,

$$\delta\gamma(t) = \Gamma_*(\tfrac{\partial}{\partial\varepsilon})(t, 0) \in T_{\gamma(t)}M, \quad t_0 \leq t \leq t_1,$$

where $\frac{\partial}{\partial\varepsilon}$ is a tangent vector to the interval $[-\varepsilon_0, \varepsilon_0]$ at 0. Finally, a tangential lift of a path $\gamma : [t_0, t_1] \to M$ is the path $\gamma' : [t_0, t_1] \to TM$ defined by $\gamma'(t) = \gamma_*(\frac{\partial}{\partial t}) \in T_{\gamma(t)}M$, $t_0 \leq t \leq t_1$, where $\frac{\partial}{\partial t}$ is a tangent vector to $[t_0, t_1]$ at t. In other words, $\gamma'(t)$ is the velocity vector of a path $\gamma(t)$ at time t.

Definition. The *action functional* $S : P(M) \to \mathbb{R}$ of a Lagrangian system (M, L) is defined by

$$S(\gamma) = \int_{t_0}^{t_1} L(\gamma'(t), t)dt.$$

Principle of the Least Action (Hamilton's principle)**.** A path $\gamma \in PM$ describes the motion of a Lagrangian system (M, L) between the position $q_0 \in M$ at time t_0 and the position $q_1 \in M$ at time t_1 if and only if it is a critical point of the action functional S,

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(\gamma_\varepsilon) = 0$$

for all variations $\gamma_\varepsilon(t)$ of $\gamma(t)$ with fixed ends.

The critical points of the action functional are called *extremals* and the principle of the least action states that a Lagrangian system (M, L)

moves along the extremals[4]. The extremals are characterized by equations of motion — a system of second order differential equations in local coordinates on TM. The equations of motion have the most elegant form for the following choice of local coordinates on TM.

Definition. Let (U, φ) be a coordinate chart on M with local coordinates $\boldsymbol{q} = (q^1, \dots, q^n)$. Coordinates

$$(\boldsymbol{q}, \boldsymbol{v}) = (q^1, \dots, q^n, v^1, \dots, v^n)$$

on a chart TU on TM, where $\boldsymbol{v} = (v^1, \dots, v^n)$ are coordinates in the fiber corresponding to the basis $\dfrac{\partial}{\partial q^1}, \dots, \dfrac{\partial}{\partial q^n}$ for T_qM, are called *standard coordinates.*

Standard coordinates are Cartesian coordinates on $\varphi_*(TU) \subset T\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$ and have the property that for $(q, v) \in TU$ and $f \in C^\infty(U)$,

$$v(f) = \sum_{i=1}^n v^i \frac{\partial f}{\partial q^i} = \boldsymbol{v} \frac{\partial f}{\partial \boldsymbol{q}}.$$

Let (U, φ) and (U', φ') be coordinate charts on M with the transition functions $F = (F^1, \dots, F^n) = \varphi' \circ \varphi^{-1} : \varphi(U \cap U') \to \varphi'(U \cap U')$, and let $(\boldsymbol{q}, \boldsymbol{v})$ and $(\boldsymbol{q}', \boldsymbol{v}')$, respectively, be the standard coordinates on TU and TU'. We have $\boldsymbol{q}' = F(\boldsymbol{q})$ and $\boldsymbol{v}' = F_*(\boldsymbol{q})\boldsymbol{v}$, where $F_*(\boldsymbol{q}) = \left\{ \dfrac{\partial F^i}{\partial q^j}(\boldsymbol{q}) \right\}_{i,j=1}^n$ is a matrix-valued function on $\varphi(U \cap U')$. Thus "vertical" coordinates $\boldsymbol{v} = (v^1, \dots, v^n)$ in the fibers of $TM \to M$ transform like components of a tangent vector on M under the change of coordinates on M.

The tangential lift $\gamma'(t)$ of a path $\gamma(t)$ in M in standard coordinates on TU is $(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)) = (q^1(t), \dots, q^n(t), \dot{q}^1(t), \dots, \dot{q}^n(t))$, where the dot stands for the time derivative, so that

$$L(\gamma'(t), t) = L(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t).$$

Following a centuries long tradition[5], we will usually denote standard coordinates by

$$(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (q^1, \dots, q^n, \dot{q}^1, \dots, \dot{q}^n),$$

where the dot *does not* stand for the time derivative. Since we only consider paths in TM that are tangential lifts of paths in M, there will be no confusion[6].

[4]The principle of the least action does not state that an extremal connecting points q_0 and q_1 is a minimum of S, nor that such an extremal is unique. It also does not state that any two points can be connected by an extremal.

[5]Used in all texts on classical mechanics and theoretical physics.

[6]We reserve the notation $(\boldsymbol{q}(t), \boldsymbol{v}(t))$ for general paths in TM.

Theorem 1.1. *The equations of motion of a Lagrangian system (M, L) in standard coordinates on TM are given by the Euler-Lagrange equations*

$$\frac{\partial L}{\partial \boldsymbol{q}}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t)\right) = 0.$$

Proof. Suppose first that an extremal $\gamma(t)$ lies in a coordinate chart U of M. Then a simple computation in standard coordinates, using integration by parts, gives

$$\begin{aligned}
0 &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(\gamma_\varepsilon) \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_{t_0}^{t_1} L\left(\boldsymbol{q}(t,\varepsilon), \dot{\boldsymbol{q}}(t,\varepsilon), t\right) dt \\
&= \sum_{i=1}^{n} \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i\right) dt \\
&= \sum_{i=1}^{n} \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\right) \delta q^i dt + \sum_{i=1}^{n} \left.\frac{\partial L}{\partial \dot{q}^i}\delta q^i\right|_{t_0}^{t_1}.
\end{aligned}$$

The second sum in the last line vanishes due to the property $\delta q^i(t_0) = \delta q^i(t_1) = 0,\ i = 1, \dots, n$. The first sum is zero for arbitrary smooth functions δq^i on the interval $[t_0, t_1]$ which vanish at the endpoints. This implies that for each term in the sum the integrand is identically zero,

$$\frac{\partial L}{\partial q^i}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t)\right) = 0, \quad i = 1, \dots, n.$$

Since the restriction of an extremal of the action functional S to a coordinate chart on M is again an extremal, each extremal in standard coordinates on TM satisfies Euler-Lagrange equations. □

Remark. In calculus of variations, the directional derivative of a functional S with respect to a tangent vector $V \in T_\gamma P(M)$ — the *Gato derivative* — is defined by

$$\delta_V S = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(\gamma_\varepsilon),$$

where γ_ε is a path in $P(M)$ with a tangent vector V at $\gamma_0 = \gamma$. The result of the above computation (when γ lies in a coordinate chart $U \subset M$) can be written as

$$\begin{aligned}
\delta_V S &= \int_{t_0}^{t_1} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\right) (\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t) v^i(t) dt \\
&= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial \boldsymbol{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) (\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t) \boldsymbol{v}(t) dt. \qquad (1.1)
\end{aligned}$$

Here $V(t) = \sum_{i=1}^{n} v^i(t)\frac{\partial}{\partial q^i}$ is a vector field along the path γ in M. Formula (1.1) is called the formula for the *first variation of the action with fixed ends*. The principle of the least action is a statement that $\delta_V S(\gamma) = 0$ for all $V \in T_\gamma P(M)$.

Remark. It is also convenient to consider a space $\widehat{P(M)} = \{\gamma : [t_0, t_1] \to M\}$ of all smooth parametrized paths in M. The tangent space $T_\gamma\widehat{P(M)}$ to $\widehat{P(M)}$ at $\gamma \in \widehat{P(M)}$ is the space of all smooth vector fields along the path γ in M (no condition at the endpoints). The computation in the proof of Theorem 1.1 yields the following formula for the *first variation of the action with free ends*:

$$\delta_V S = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial \boldsymbol{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)\boldsymbol{v}\,dt + \frac{\partial L}{\partial \dot{\boldsymbol{q}}}\boldsymbol{v}\Big|_{t_0}^{t_1}. \tag{1.2}$$

Problem 1.1. Show that the action functional is given by the evaluation of the 1-form Ldt on $TM \times \mathbb{R}$ over the 1-chain $\tilde{\gamma}$ on $TM \times \mathbb{R}$,

$$S(\gamma) = \int_{\tilde{\gamma}} Ldt,$$

where $\tilde{\gamma} = \{(\gamma'(t), t); t_0 \leq t \leq t_1\}$ and $Ldt\left(w, c\frac{\partial}{\partial t}\right) = cL(q, v)$, $w \in T_{(q,v)}TM$, $c \in \mathbb{R}$.

Problem 1.2. Let $f \in C^\infty(M)$. Show that Lagrangian systems (M, L) and $(M, L+df)$ (where df is a fibre-wise linear function on TM) have the same equations of motion.

Problem 1.3. Give examples of Lagrangian systems such that an extremal connecting two given points (i) is not a local minimum; (ii) is not unique; (iii) does not exist.

Problem 1.4. For γ an extremal of the action functional S, the *second variation* of S is defined by

$$\delta^2_{V_1V_2}S = \frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2}\Big|_{\varepsilon_1=\varepsilon_2=0} S(\gamma_{\varepsilon_1,\varepsilon_2}),$$

where $\gamma_{\varepsilon_1,\varepsilon_2}$ is a smooth two-parameter family of paths in M such that the paths $\gamma_{\varepsilon_1,0}$ and γ_{0,ε_2} in $P(M)$ at the point $\gamma_{0,0} = \gamma \in P(M)$ have tangent vectors V_1 and V_2, respectively. For a Lagrangian system (M, L) find the second variation of S and verify that for given V_1 and V_2 it does not depend on the choice of $\gamma_{\varepsilon_1,\varepsilon_2}$.

1.3. Examples of Lagrangian systems. To describe a mechanical phenomena it is necessary to choose a *frame of reference*. The properties of the *space-time* where the motion takes place depend on this choice. The space-time is characterized by the following postulates[7].

[7]Strictly speaking, these postulates are valid only in the non-relativistic limit of special relativity, when the speed of light in the vacuum is assumed to be infinite.

Newtonian Space-Time. The space is a three-dimensional affine Euclidean space E^3. A choice of the *origin* $0 \in E^3$ — a *reference point* — establishes the isomorphism $E^3 \simeq \mathbb{R}^3$, where the vector space $\mathbb{R}^3$ carries the Euclidean inner product and has a fixed orientation. The time is one-dimensional — a time axis $\mathbb{R}$ — and the space-time is a direct product $E^3 \times \mathbb{R}$. An *inertial* reference frame is a coordinate system with respect to the origin $0 \in E^3$, initial time t_0, and an orthonormal basis in $\mathbb{R}^3$. In an inertial frame the space is *homogeneous* and *isotropic* and the time is *homogeneous*. The laws of motion are invariant with respect to the transformations

$$\boldsymbol{r} \mapsto g \cdot \boldsymbol{r} + \boldsymbol{r}_0, \quad t \mapsto t + t_0,$$

where $\boldsymbol{r}, \boldsymbol{r}_0 \in \mathbb{R}^3$ and g is an orthogonal linear transformation in $\mathbb{R}^3$. The time in classical mechanics is *absolute*.

The Galilean group is the group of all affine transformations of $E^3 \times \mathbb{R}$ which preserve time intervals and which for every $t \in \mathbb{R}$ are isometries in E^3. Every Galilean transformation is a composition of rotation, space-time translation, and a transformation

$$\boldsymbol{r} \mapsto \boldsymbol{r} + \boldsymbol{v}t, \quad t \mapsto t,$$

where $\boldsymbol{v} \in \mathbb{R}^3$. Any two inertial frames are related by a Galilean transformation.

Galileo's Relativity Principle. The laws of motion are invariant with respect to the Galilean group.

These postulates impose restrictions on Lagrangians of mechanical systems. Thus it follows from the first postulate that the Lagrangian L of a closed system does not explicitly depend on time. Physical systems are described by special Lagrangians, in agreement with the experimental facts about the motion of material bodies.

Example 1.1 (Free particle)**.** The configuration space for a free particle is $M = \mathbb{R}^3$, and it can be deduced from Galileo's relativity principle that the Lagrangian for a free particle is

$$L = \tfrac{1}{2} m \dot{\boldsymbol{r}}^2.$$

Here $m > 0$[8] is the mass of a particle and $\dot{\boldsymbol{r}}^2 = |\dot{\boldsymbol{r}}|^2$ is the length square of the velocity vector $\dot{\boldsymbol{r}} \in T_{\boldsymbol{r}}\mathbb{R}^3 \simeq \mathbb{R}^3$. Euler-Lagrange equations give *Newton's law of inertia*,

$$\ddot{\boldsymbol{r}} = 0.$$

[8] Otherwise the action functional is not bounded from below.

Example 1.2 (Interacting particles)**.** A closed system of N interacting particles in $\mathbb{R}^3$ with masses $m_1, \dots, m_N$ is described by a configuration space

$$M = \mathbb{R}^{3N} = \underbrace{\mathbb{R}^3 \times \cdots \times \mathbb{R}^3}_{N}$$

with a position vector $\boldsymbol{r} = (\boldsymbol{r}_1, \dots, \boldsymbol{r}_N)$, where $\boldsymbol{r}_a \in \mathbb{R}^3$ is the position vector of the a-th particle, $a = 1, \dots, N$. It is found that the Lagrangian is given by

$$L = \sum_{a=1}^{N} \tfrac{1}{2} m_a \dot{\boldsymbol{r}}_a^2 - V(\boldsymbol{r}) = T - V,$$

where

$$T = \sum_{a=1}^{N} \tfrac{1}{2} m_a \dot{\boldsymbol{r}}_a^2$$

is called *kinetic energy* of a system and $V(\boldsymbol{r})$ is *potential energy*. The Euler-Lagrange equations give *Newton's equations*

$$m_a \ddot{\boldsymbol{r}}_a = \boldsymbol{F}_a,$$

where

$$\boldsymbol{F}_a = -\frac{\partial V}{\partial \boldsymbol{r}_a}$$

is the *force* on the a-th particle, $a = 1, \dots, N$. Forces of this form are called *conservative*. It follows from homogeneity of space that potential energy $V(\boldsymbol{r})$ of a closed system of N interacting particles with conservative forces depends only on relative positions of the particles, which leads to the equation

$$\sum_{a=1}^{N} \boldsymbol{F}_a = 0.$$

In particular, for a closed system of two particles $\boldsymbol{F}_1 + \boldsymbol{F}_2 = 0$, which is the equality of action and reaction forces, also called *Newton's third law*.

The potential energy of a closed system with only pair-wise interaction between the particles has the form

$$V(\boldsymbol{r}) = \sum_{1 \leq a < b \leq N} V_{ab}(\boldsymbol{r}_a - \boldsymbol{r}_b).$$

It follows from the isotropy of space that $V(\boldsymbol{r})$ depends only on relative distances between the particles, so that the Lagrangian of a closed system of N particles with pair-wise interaction has the form

$$L = \sum_{a=1}^{N} \tfrac{1}{2} m_a \dot{\boldsymbol{r}}_a^2 \; - \sum_{1 \leq a < b \leq N} V_{ab}(|\boldsymbol{r}_a - \boldsymbol{r}_b|).$$

If the potential energy $V(\boldsymbol{r})$ is a homogeneous function of degree ρ, $V(\lambda\boldsymbol{r}) = \lambda^\rho V(\boldsymbol{r})$, then the average values $\overline{T}$ and $\overline{V}$ of kinetic energy and potential energy over a closed trajectory are related by the *virial theorem*

$$2\overline{T} = \rho\overline{V}. \tag{1.3}$$

Indeed, let $\boldsymbol{r}(t)$ be a periodic trajectory with period $\tau > 0$, i.e., $\boldsymbol{r}(0) = \boldsymbol{r}(\tau)$, $\dot{\boldsymbol{r}}(0) = \dot{\boldsymbol{r}}(\tau)$. Using integration by parts, Newton's equations, and Euler's homogeneous function theorem, we get

$$\begin{aligned} 2\overline{T} &= \frac{1}{\tau}\int_0^\tau \sum_{a=1}^N m_a \dot{\boldsymbol{r}}_a^2 dt = -\frac{1}{\tau}\int_0^\tau \sum_{a=1}^N m_a \boldsymbol{r}_a \ddot{\boldsymbol{r}}_a dt \\ &= \frac{1}{\tau}\int_0^\tau \sum_{a=1}^N \boldsymbol{r}_a \frac{\partial V}{\partial \boldsymbol{r}_a} dt = \rho\overline{V}. \end{aligned}$$

Example 1.3 (Universal gravitation)**.** According to *Newton's law of gravitation*, the potential energy of the gravitational force between two particles with masses m_a and m_b is

$$V(\boldsymbol{r}_a - \boldsymbol{r}_b) = -G\frac{m_a m_b}{|\boldsymbol{r}_a - \boldsymbol{r}_b|},$$

where G is the gravitational constant. The configuration space of N particles with gravitational interaction is

$$M = \{(\boldsymbol{r}_1, \dots, \boldsymbol{r}_N) \in \mathbb{R}^{3N} : \boldsymbol{r}_a \neq \boldsymbol{r}_b \text{ for } a \neq b,\ a, b = 1, \dots, N\}.$$

Example 1.4 (Particle in an external potential field)**.** Here $M = \mathbb{R}^3$ and

$$L = \tfrac{1}{2}m\dot{\boldsymbol{r}}^2 - V(\boldsymbol{r}, t),$$

where potential energy can explicitly depend on time. Equations of motion are Newton's equations

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F} = -\frac{\partial V}{\partial \boldsymbol{r}}.$$

If $V = V(|\boldsymbol{r}|)$ is a function only of the distance $|\boldsymbol{r}|$, the potential field is called *central*.

Example 1.5 (Charged particle in electromagnetic field[9])**.** Consider a particle of charge e and mass m in $\mathbb{R}^3$ moving in a time-independent electromagnetic field with scalar and vector potentials $\varphi(\boldsymbol{r})$ and $\boldsymbol{A}(\boldsymbol{r}) = (A_1(\boldsymbol{r}), A_2(\boldsymbol{r}), A_3(\boldsymbol{r}))$. The Lagrangian has the form

$$L = \frac{m\dot{\boldsymbol{r}}^2}{2} + e\left(\frac{\dot{\boldsymbol{r}}\,\boldsymbol{A}}{c} - \varphi\right),$$

[9] This is a non-relativistic limit of an example in classical electrodynamics.

where c is the speed of light. The Euler-Lagrange equations give Newton's equations with the *Lorentz force*,

$$m\ddot{\boldsymbol{r}} = e\left(\boldsymbol{E} + \frac{\dot{\boldsymbol{r}}}{c} \times \boldsymbol{B}\right),$$

where $\times$ is the cross-product of vectors in $\mathbb{R}^3$, and

$$\boldsymbol{E} = -\frac{\partial\varphi}{\partial\boldsymbol{r}} \quad \text{and} \quad \boldsymbol{B} = \operatorname{curl}\boldsymbol{A}$$

are electric and magnetic[10] fields, respectively.

Example 1.6 (Small oscillations)**.** Consider a particle of mass m with n degrees of freedom moving in a potential field $V(\boldsymbol{q})$, and suppose that potential energy U has a minimum at $\boldsymbol{q} = 0$. Expanding $V(\boldsymbol{q})$ in Taylor series around 0 and keeping only quadratic terms, one obtains a Lagrangian system which describes small oscillations from equilibrium. Explicitly,

$$L = \tfrac{1}{2}m\dot{\boldsymbol{q}}^2 - V_0(\boldsymbol{q}),$$

where V_0 is a positive-definite quadratic form on $\mathbb{R}^n$ given by

$$V_0(\boldsymbol{q}) = \tfrac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 V}{\partial q^i \partial q^j}(0) q^i q^j.$$

Since every quadratic form can be diagonalized by an orthogonal transformation, we can assume from the very beginning that coordinates $\boldsymbol{q} = (q^1, \dots, q^n)$ are chosen so that $V_0(\boldsymbol{q})$ is diagonal and

$$L = \tfrac{1}{2}m\big(\dot{\boldsymbol{q}}^2 - \sum_{i=1}^{n} \omega_i^2 (q^i)^2\big), \tag{1.4}$$

where $\omega_1, \dots, \omega_n > 0$. Such coordinates $\boldsymbol{q}$ are called *normal coordinates.* In normal coordinates Euler-Lagrange equations take the form

$$\ddot{q}^i + \omega_i^2 q^i = 0, \quad i = 1, \dots, n,$$

and describe n decoupled (i.e., non-interacting) *harmonic oscillators* with *frequencies* $\omega_1, \dots, \omega_n$.

Example 1.7 (Free particle on a Riemannian manifold)**.** Let (M, ds^2) be a Riemannian manifold with the Riemannian metric ds^2. In local coordinates $x^1, \dots, x^n$ on M,

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu,$$

where following tradition we assume the summation over repeated indices. The Lagrangian of a free particle on M is

$$L(v) = \tfrac{1}{2}\langle v, v\rangle = \tfrac{1}{2}\|v\|^2,\ v \in TM,$$

[10]Notation $\boldsymbol{B} = \operatorname{rot}\boldsymbol{A}$ is also used.

where $\langle\ ,\ \rangle$ stands for the inner product in fibers of TM given by the Riemannian metric. The corresponding functional

$$S(\gamma) = \tfrac{1}{2}\int_{t_0}^{t_1} \|\gamma'(t)\|^2 dt = \tfrac{1}{2}\int_{t_0}^{t_1} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu dt$$

is called the action functional in Riemannian geometry. The Euler-Lagrange equations are

$$g_{\mu\nu}\ddot{x}^\mu + \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\dot{x}^\mu \dot{x}^\lambda = \frac{1}{2}\frac{\partial g_{\mu\lambda}}{\partial x^\nu}\dot{x}^\mu \dot{x}^\lambda,$$

and after multiplying by the inverse metric tensor $g^{\sigma\nu}$ and summation over ν they take the form

$$\ddot{x}^\sigma + \Gamma^\sigma_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0, \quad \sigma = 1, \dots, n,$$

where

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda}\left(\frac{\partial g_{\mu\lambda}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right)$$

are Christoffel's symbols. The Euler-Lagrange equations of a free particle moving on a Riemannian manifold are geodesic equations.

Let ∇ be the Levi-Civita connection — the metric connection in the tangent bundle TM — and let ∇_ξ be a covariant derivative with respect to the vector field $\xi \in \mathrm{Vect}(M)$. Explicitly,

$$(\nabla_\xi\, \eta)^\mu = \left(\frac{\partial \eta^\mu}{\partial x^\nu} + \Gamma^\mu_{\nu\lambda}\eta^\lambda\right)\xi^\nu, \quad \text{where} \quad \xi = \xi^\mu(x)\frac{\partial}{\partial x^\mu},\ \eta = \eta^\mu(x)\frac{\partial}{\partial x^\mu}.$$

For a path $\gamma(t) = (x^\mu(t))$ denote by $\nabla_{\dot{\gamma}}$ a covariant derivative along γ,

$$(\nabla_{\dot{\gamma}}\eta)^\mu(t) = \frac{d\eta^\mu(t)}{dt} + \Gamma^\mu_{\nu\lambda}(\gamma(t))\dot{x}^\nu(t)\eta^\lambda(t), \quad \text{where} \quad \eta = \eta^\mu(t)\frac{\partial}{\partial x^\mu}$$

is a vector field along γ. Formula (1.1) can now be written in an invariant form

$$\delta S = -\int_{t_0}^{t_1} \langle \nabla_{\dot{\gamma}}\dot{\gamma}, \delta\gamma\rangle dt,$$

which is known as the formula for the first variation of the action in Riemannian geometry.

Example 1.8 (The rigid body)**.** The configuration space of a rigid body in $\mathbb{R}^3$ with a fixed point is a Lie group $G = \mathrm{SO}(3)$ of orientation preserving orthogonal linear transformations in $\mathbb{R}^3$. Every left-invariant Riemannian metric $\langle\ ,\ \rangle$ on G defines a Lagrangian $L : TG \to \mathbb{R}$ by

$$L(v) = \tfrac{1}{2}\langle v, v\rangle, \quad v \in TG.$$

According to the previous example, equations of motion of a rigid body are geodesic equations on G with respect to the Riemannian metric $\langle\ ,\ \rangle$. Let $\mathfrak{g} = \mathfrak{so}(3)$ be the Lie algebra of G. A velocity vector $\dot{g} \in T_gG$ defines the *angular velocity of the body* by $\Omega = (L_{g^{-1}})_*\dot{g} \in \mathfrak{g}$, where $L_g : G \to G$ are left translations on G. In terms of angular velocity, the Lagrangian takes the form

$$L = \tfrac{1}{2}\langle \Omega, \Omega\rangle_e,$$

where $\langle\ ,\ \rangle_e$ is an inner product on $\mathfrak{g} = T_eG$ given by the Riemannian metric $\langle\ ,\ \rangle$. The Lie algebra $\mathfrak{g}$ — the Lie algebra of 3×3 skew-symmetric matrices — has the invariant inner product $\langle u, v\rangle_0 = -\frac{1}{2}\operatorname{Tr} uv$ (the Killing form), so that $\langle \Omega, \Omega\rangle_e = \langle \boldsymbol{A}\cdot\Omega, \Omega\rangle_0$ for some symmetric linear operator $\boldsymbol{A} : \mathfrak{g} \to \mathfrak{g}$ which is positive-definite with respect to the Killing form. Such a linear operator $\boldsymbol{A}$ is called the *inertia tensor* of the body. The *principal axes of inertia* of the body are orthonormal eigenvectors e_1, e_2, e_3 of $\boldsymbol{A}$; corresponding eigenvalues I_1, I_2, I_3 are called the *principal moments of inertia*. Setting $\Omega = \Omega_1 e_1 + \Omega_2 e_2 + \Omega_3 e_3$[11], we get

$$L = \tfrac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2).$$

In this parametrization, the Euler-Lagrange equations become *Euler's equations*

$$\begin{aligned} I_1\dot{\Omega}_1 &= (I_2 - I_3)\Omega_2\Omega_3,\\ I_2\dot{\Omega}_2 &= (I_3 - I_1)\Omega_1\Omega_3,\\ I_3\dot{\Omega}_3 &= (I_1 - I_2)\Omega_1\Omega_2. \end{aligned}$$

Euler's equations describe the rotation of a free rigid body around a fixed point. In the system of coordinates with axes which are the principal axes of inertia, principal moments of inertia of the body are I_1, I_2, I_3.

Problem 1.5. Determine the motion of a charged particle in a constant uniform magnetic field. Show that if the initial velocity $v_3 = 0$ in the z-axis (taken in the direction of the field, $\boldsymbol{B} = (0, 0, B)$), the trajectories are circles of radii $r = \dfrac{cmv_t}{eB}$ in a plane perpendicular to the field (the xy-plane), where $v_t = \sqrt{v_1^2 + v_2^2}$ is the initial velocity in the xy-plane. The centers (x_0, y_0) of circles are given by

$$x_0 = \frac{cmv_1}{eB} + x, \quad y_0 = -\frac{cmv_2}{eB} + y,$$

where (x, y) are points on a circle of radius r.

Problem 1.6. Show that the Euler-Lagrange equations for the Lagrangian $L(v) = \|v\|$, $v \in TM$, coincide with the geodesic equations written with respect to a constant multiple of the natural parameter.

Problem 1.7. Prove that for a particle in a potential field, discussed in Example 1.4, the second variation of the action functional, defined in Problem 1.4, is given by

$$\delta^2 S = \int_{t_0}^{t_1} \mathcal{J}(\delta_1\boldsymbol{r})\delta_2\boldsymbol{r}\,dt,$$

[11]This establishes the Lie algebra isomorphism $\mathfrak{g} \simeq \mathbb{R}^3$, where the Lie bracket in $\mathbb{R}^3$ is given by the cross-product.

where $\delta_1 \boldsymbol{r}, \delta_2 \boldsymbol{r} \in T_\gamma P\mathbb{R}^3$, $\gamma = \boldsymbol{r}(t)$ is the classical trajectory, $\mathcal{J} = -m\dfrac{d^2}{dt^2}I - \dfrac{\partial^2 V}{\partial \boldsymbol{r}^2}(t)$, I is the 3×3 identity matrix, and $\dfrac{\partial^2 V}{\partial \boldsymbol{r}^2}(t) = \left\{\dfrac{\partial^2 V}{\partial \boldsymbol{r}_a \partial \boldsymbol{r}_b}(\boldsymbol{r}(t))\right\}_{a,b=1}^{3}$. A second-order linear differential operator $\mathcal{J}$, acting on vector fields along γ, is called the *Jacobi operator*.

Problem 1.8. Find normal coordinates and frequencies for the Lagrangian system considered in Example 1.6 with $V_0(\boldsymbol{q}) = \frac{1}{2}a^2\sum_{i=1}^{n}(q^{i+1}-q^i)^2$, where $q^{n+1} = q^1$.

Problem 1.9. Prove that the second variation of the action functional in Riemannian geometry is given by

$$\delta^2 S = \int_{t_0}^{t_1} \langle \mathcal{J}(\delta_1\gamma), \delta_2\gamma\rangle dt.$$

Here $\delta_1\gamma, \delta_2\gamma \in T_\gamma PM$, $\mathcal{J} = -\nabla_{\dot\gamma}^2 - R(\dot\gamma, \cdot\,)\dot\gamma$ is the Jacobi operator, and R is a curvature operator — a fibre-wise linear mapping $R : TM \otimes TM \to \operatorname{End}(TM)$ of vector bundles, defined by $R(\xi,\eta) = \nabla_\eta\nabla_\xi - \nabla_\xi\nabla_\eta + \nabla_{[\xi,\eta]} : TM \to TM$, where $\xi, \eta \in \operatorname{Vect}(M)$.

Problem 1.10. Choosing the principal axes of inertia as a basis in $\mathbb{R}^3$, show that the Lie algebra isomorphism $\mathfrak{g} \simeq \mathbb{R}^3$ is given by $\mathfrak{g} \ni \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \mapsto (\Omega_1, \Omega_2, \Omega_3) \in \mathbb{R}^3$.

Problem 1.11. Show that for every symmetric $\boldsymbol{A} \in \operatorname{End}\mathfrak{g}$ there exists a symmetric 3×3 matrix A such that $\boldsymbol{A}\cdot\Omega = A\Omega + \Omega A$, and find A for diagonal $\boldsymbol{A}$.

Problem 1.12. Derive Euler's equations for a rigid body. (*Hint:* Use that $L = -\frac{1}{2}\operatorname{Tr} A\Omega^2$, where $\Omega = g^{-1}\dot g$ and $\delta\Omega = -g^{-1}\delta g\,\Omega + g^{-1}\delta\dot g$, and obtain the Euler-Lagrange equations in the matrix form $A\dot\Omega + \dot\Omega A = A\Omega^2 - \Omega^2 A$.)

1.4. Symmetries and Noether's theorem. To describe the motion of a mechanical system one needs to solve the corresponding Euler-Lagrange equations — a system of second order ordinary differential equations for the generalized coordinates. This could be a very difficult problem. Therefore of particular interest are those functions of generalized coordinates and velocities which remain constant during the motion.

Definition. A smooth function $I : TM \to \mathbb{R}$ is called the *integral of motion* (*first integral*, or *conservation law*) for a Lagrangian system (M, L) if

$$\frac{d}{dt}I(\gamma'(t)) = 0$$

for all extremals γ of the action functional.

Definition. The *energy* of a Lagrangian system (M, L) is a function E on $TM \times \mathbb{R}$ defined in standard coordinates on TM by

$$E(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \sum_{i=1}^{n} \dot q^i\,\frac{\partial L}{\partial \dot q^i}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) - L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t).$$

Lemma 1.1. *The energy* $E = \dot{\boldsymbol{q}}\,\dfrac{\partial L}{\partial \dot{\boldsymbol{q}}} - L$ *is a well-defined function on* $TM \times \mathbb{R}$.

Proof. Let (U, φ) and (U', φ') be coordinate charts on M with the transition functions $F = (F^1, \dots, F^n) = \varphi' \circ \varphi^{-1} : \varphi(U \cap U') \to \varphi'(U \cap U')$. Corresponding standard coordinates $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ and $(\boldsymbol{q}', \dot{\boldsymbol{q}}')$ are related by $\boldsymbol{q}' = F(\boldsymbol{q})$ and $\dot{\boldsymbol{q}}' = F_*(\boldsymbol{q})\dot{\boldsymbol{q}}$ (see Section 1.2). We have $d\boldsymbol{q}' = F_*(\boldsymbol{q})d\boldsymbol{q}$ and $d\dot{\boldsymbol{q}}' = G(\boldsymbol{q}, \dot{\boldsymbol{q}})d\boldsymbol{q} + F_*(\boldsymbol{q})d\dot{\boldsymbol{q}}$ (for some matrix-valued function $G(\boldsymbol{q}, \dot{\boldsymbol{q}})$), so that

$$\begin{aligned} dL &= \frac{\partial L}{\partial \boldsymbol{q}'}d\boldsymbol{q}' + \frac{\partial L}{\partial \dot{\boldsymbol{q}}'}d\dot{\boldsymbol{q}}' + \frac{\partial L}{\partial t}dt \\ &= \left(\frac{\partial L}{\partial \boldsymbol{q}'}F_*(\boldsymbol{q}) + \frac{\partial L}{\partial \dot{\boldsymbol{q}}'}G(\boldsymbol{q}, \dot{\boldsymbol{q}})\right)d\boldsymbol{q} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}'}F_*(\boldsymbol{q})d\dot{\boldsymbol{q}} + \frac{\partial L}{\partial t}dt \\ &= \frac{\partial L}{\partial \boldsymbol{q}}d\boldsymbol{q} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}}d\dot{\boldsymbol{q}} + \frac{\partial L}{\partial t}dt. \end{aligned}$$

Thus under a change of coordinates

$$\frac{\partial L}{\partial \dot{\boldsymbol{q}}'}\,F_*(\boldsymbol{q}) = \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \quad \text{and} \quad \dot{\boldsymbol{q}}'\,\frac{\partial L}{\partial \dot{\boldsymbol{q}}'} = \dot{\boldsymbol{q}}\frac{\partial L}{\partial \dot{\boldsymbol{q}}},$$

so that E is a well-defined function on TM. □

Corollary 1.2. *Under a change of local coordinates on* M*, components of* $\dfrac{\partial L}{\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \left(\dfrac{\partial L}{\partial \dot{q}^1}, \dots, \dfrac{\partial L}{\partial \dot{q}^n}\right)$ *transform like components of a* 1*-form on* M.

Proposition 1.1 (Conservation of energy)**.** *The energy of a closed system is an integral of motion.*

Proof. For an extremal γ set $E(t) = E(\gamma(t))$. We have, according to the Euler-Lagrange equations,

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)\dot{\boldsymbol{q}} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}}\ddot{\boldsymbol{q}} - \frac{\partial L}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} - \frac{\partial L}{\partial \dot{\boldsymbol{q}}}\ddot{\boldsymbol{q}} - \frac{\partial L}{\partial t} \\ &= \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial L}{\partial \boldsymbol{q}}\right)\dot{\boldsymbol{q}} - \frac{\partial L}{\partial t} = -\frac{\partial L}{\partial t}. \end{aligned}$$

Since for a closed system $\dfrac{\partial L}{\partial t} = 0$, the energy is conserved. □

Conservation of energy for a closed mechanical system is a fundamental law of physics which follows from the homogeneity of time. For a general closed system of N interacting particles considered in Example 1.2,

$$E = \sum_{a=1}^{N} m_a \dot{\boldsymbol{r}}_a^2 - L = \sum_{a=1}^{N} \tfrac{1}{2} m_a \dot{\boldsymbol{r}}_a^2 + V(\boldsymbol{r}).$$

In other words, the total energy $E = T + V$ is a sum of the kinetic energy and the potential energy.

Definition. A Lagrangian $L : TM \to \mathbb{R}$ is invariant with respect to the diffeomorphism $g : M \to M$ if $L(g_*(v)) = L(v)$ for all $v \in TM$. The diffeomorphism g is called the *symmetry* of a closed Lagrangian system (M, L). A Lie group G is the *symmetry group* of (M, L) (group of *continuous symmetries*) if there is a left G-action on M such that for every $g \in G$ the mapping $M \ni x \mapsto g \cdot x \in M$ is a symmetry.

Continuous symmetries give rise to conservation laws.

Theorem 1.3 (Noether)**.** *Suppose that a Lagrangian $L : TM \to \mathbb{R}$ is invariant under a one-parameter group $\{g_s\}_{s\in\mathbb{R}}$ of diffeomorphisms of M. Then the Lagrangian system (M, L) admits an integral of motion I, given in standard coordinates on TM by*

$$I(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}^i}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \left(\left.\frac{dg_s^i(\boldsymbol{q})}{ds}\right|_{s=0} \right) = \frac{\partial L}{\partial \dot{\boldsymbol{q}}}\boldsymbol{a},$$

where $X = \sum_{i=1}^{n} a^i(\boldsymbol{q}) \frac{\partial}{\partial q^i}$ is the vector field on M associated with the flow g_s. The integral of motion I is called the Noether integral.

Proof. It follows from Corollary 1.2 that I is a well-defined function on TM. Now differentiating $L((g_s)_*(\gamma'(t))) = L(\gamma'(t))$ with respect to s at $s = 0$ and using the Euler-Lagrange equations we get

$$0 = \frac{\partial L}{\partial \boldsymbol{q}}\boldsymbol{a} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}}\dot{\boldsymbol{a}} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)\boldsymbol{a} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}}\frac{d\boldsymbol{a}}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\boldsymbol{a}\right),$$

where $\boldsymbol{a}(t) = \big(a^1(\gamma(t)), \dots, a^n(\gamma(t))\big)$. $\square$

Remark. A vector field X on M is called an *infinitesimal symmetry* if the corresponding local flow g_s of X (defined for each $s \in \mathbb{R}$ on some $U_s \subseteq M$) is a symmetry: $L \circ (g_s)_* = L$ on U_s. Every vector field X on M lifts to a vector field X' on TM, defined by a local flow on TM induced from the corresponding local flow on M. In standard coordinates on TM,

$$X = \sum_{i=1}^{n} a^i(\boldsymbol{q}) \frac{\partial}{\partial q^i} \quad \text{and} \quad X' = \sum_{i=1}^{n} a^i(\boldsymbol{q}) \frac{\partial}{\partial q^i} + \sum_{i,j=1}^{n} \dot{q}^j \frac{\partial a^i}{\partial q^j}(\boldsymbol{q}) \frac{\partial}{\partial \dot{q}^i}. \tag{1.5}$$

It is easy to verify that X is an infinitesimal symmetry if and only if $dL(X') = 0$ on TM, which in standard coordinates has the form

$$\sum_{i=1}^{n} a^i(\boldsymbol{q}) \frac{\partial L}{\partial q^i} + \sum_{i,j=1}^{n} \dot{q}^j \frac{\partial a^i}{\partial q^j}(\boldsymbol{q}) \frac{\partial L}{\partial \dot{q}^i} = 0. \tag{1.6}$$

Remark. Noether's theorem generalizes to time-dependent Lagrangians $L : TM \times \mathbb{R} \to \mathbb{R}$. Namely, on the *extended configuration space* $M_1 = M \times \mathbb{R}$ define a time-independent Lagrangian L_1 by

$$L_1(\boldsymbol{q}, \tau, \dot{\boldsymbol{q}}, \dot{\tau}) = L\left(\boldsymbol{q}, \frac{\dot{\boldsymbol{q}}}{\dot{\tau}}, \tau\right)\dot{\tau},$$

where $(\boldsymbol{q}, \tau)$ are local coordinates on M_1 and $(\boldsymbol{q}, \tau, \dot{\boldsymbol{q}}, \dot{\tau})$ are standard coordinates on TM_1. The Noether integral I_1 for a closed system (M_1, L_1) defines an integral of motion I for a system (M, L) by the formula

$$I(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = I_1(\boldsymbol{q}, t, \dot{\boldsymbol{q}}, 1).$$

When the Lagrangian L does not depend on time, L_1 is invariant with respect to the one-parameter group of translations $\tau \mapsto \tau + s$, and the Noether integral $I_1 = \dfrac{\partial L_1}{\partial \dot{\tau}}$ gives $I = -E$.

Noether's theorem can be generalized as follows.

Proposition 1.2. *Suppose that for the Lagrangian $L : TM \to \mathbb{R}$ there exist a vector field X on M and a function K on TM such that for every path γ in M,*

$$dL(X')(\gamma(t)) = \frac{d}{dt}K(\gamma'(t)).$$

Then

$$I = \sum_{i=1}^{n} a^i(\boldsymbol{q})\,\frac{\partial L}{\partial \dot{q}^i}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - K(\boldsymbol{q}, \dot{\boldsymbol{q}})$$

is an integral of motion for the Lagrangian system (M, L).

Proof. Using Euler-Lagrange equations, we have along the extremal γ,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\boldsymbol{a}\right) = \frac{\partial L}{\partial \boldsymbol{q}}\boldsymbol{a} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}}\dot{\boldsymbol{a}} = \frac{dK}{dt}. \qquad \square$$

Example 1.9 (Conservation of momentum)**.** Let $M = V$ be a vector space, and suppose that a Lagrangian L is invariant with respect to a one-parameter group $g_s(q) = q + sv$, $v \in V$. According to Noether's theorem,

$$I = \sum_{i=1}^{n} v^i\,\frac{\partial L}{\partial \dot{q}^i}$$

is an integral of motion. Now let (M, L) be a closed Lagrangian system of N interacting particles considered in Example 1.2. We have $M = V = \mathbb{R}^{3N}$, and the Lagrangian L is invariant under simultaneous translation of coordinates $\boldsymbol{r}_a = (r_a^1, r_a^2, r_a^3)$ of all particles by the same vector $\boldsymbol{c} \in \mathbb{R}^3$. Thus $v = (\boldsymbol{c}, \dots, \boldsymbol{c}) \in \mathbb{R}^{3N}$ and for every $\boldsymbol{c} = (c^1, c^2, c^3) \in \mathbb{R}^3$,

$$I = \sum_{a=1}^{N} \left(c^1 \frac{\partial L}{\partial \dot{r}_a^1} + c^2 \frac{\partial L}{\partial \dot{r}_a^2} + c^3 \frac{\partial L}{\partial \dot{r}_a^3} \right) = c^1 P_1 + c^2 P_2 + c^3 P_3$$

is an integral of motion. The integrals of motion P_1, P_2, P_3 define the vector

$$\boldsymbol{P} = \sum_{a=1}^{N} \frac{\partial L}{\partial \dot{\boldsymbol{r}}_a} \in \mathbb{R}^3$$

(or rather a vector in the dual space to $\mathbb{R}^3$), called the *momentum* of the system. Explicitly,

$$\boldsymbol{P} = \sum_{a=1}^{N} m_a \dot{\boldsymbol{r}}_a,$$

so that the total momentum of a closed system is the sum of momenta of individual particles. Conservation of momentum is a fundamental physical law which reflects the homogeneity of space.

Traditionally, $p_i = \dfrac{\partial L}{\partial \dot{q}^i}$ are called *generalized momenta* corresponding to generalized coordinates q^i, and $F_i = \dfrac{\partial L}{\partial q^i}$ are called *generalized forces*. In these notations, the Euler-Lagrange equations have the same form

$$\dot{\boldsymbol{p}} = \boldsymbol{F}$$

as Newton's equations in Cartesian coordinates. Conservation of momentum implies Newton's third law.

Example 1.10 (Conservation of angular momentum)**.** Let $M = V$ be a vector space with Euclidean inner product. Let $G = \mathrm{SO}(V)$ be the connected Lie group of automorphisms of V preserving the inner product, and let $\mathfrak{g} = \mathfrak{so}(V)$ be the Lie algebra of G. Suppose that a Lagrangian L is invariant with respect to the action of a one-parameter subgroup $g_s(q) = e^{sx} \cdot q$ of G on V, where $x \in \mathfrak{g}$ and e^x is the exponential map. According to Noether's theorem,

$$I = \sum_{i=1}^{n} (x \cdot q)^i \, \frac{\partial L}{\partial \dot{q}^i}$$

is an integral of motion. Now let (M, L) be a closed Lagrangian system of N interacting particles considered in Example 1.2. We have $M = V = \mathbb{R}^{3N}$, and the Lagrangian L is invariant under a simultaneous rotation of coordinates $\boldsymbol{r}_a$ of all particles by the same orthogonal transformation in $\mathbb{R}^3$.

Thus $x = (u, \dots, u) \in \underbrace{\mathfrak{so}(3) \oplus \cdots \oplus \mathfrak{so}(3)}_{N}$, and for every $u \in \mathfrak{so}(3)$,

$$I = \sum_{a=1}^{N} \left((u \cdot \boldsymbol{r}_a)^1 \frac{\partial L}{\partial \dot{r}_a^1} + (u \cdot \boldsymbol{r}_a)^2 \frac{\partial L}{\partial \dot{r}_a^2} + (u \cdot \boldsymbol{r}_a)^3 \frac{\partial L}{\partial \dot{r}_a^3} \right)$$

is an integral of motion. Let $u = u^1 X_1 + u^2 X_2 + u^3 X_3$, where $X_1 = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{smallmatrix}\right)$, $X_2 = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{smallmatrix}\right)$, $X_3 = \left(\begin{smallmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ is the basis in $\mathfrak{so}(3) \simeq \mathbb{R}^3$ corresponding to the rotations about the vectors e_1, e_2, e_3 of the standard orthonormal basis in $\mathbb{R}^3$ (see Problem 1.10). We get

$$I = u^1 M_1 + u^2 M_2 + u^3 M_3,$$

where $\boldsymbol{M} = (M_1, M_2, M_3) \in \mathbb{R}^3$ (or rather a vector in the dual space to $\mathfrak{so}(3)$) is given by

$$\boldsymbol{M} = \sum_{a=1}^{N} \boldsymbol{r}_a \times \frac{\partial L}{\partial \dot{\boldsymbol{r}}_a}.$$

The vector $\boldsymbol{M}$ is called the *angular momentum* of the system. Explicitly,

$$\boldsymbol{M} = \sum_{a=1}^{N} \boldsymbol{r}_a \times m_a \dot{\boldsymbol{r}}_a,$$

so that the total angular momentum of a closed system is the sum of angular momenta of individual particles. Conservation of angular momentum is a fundamental physical law which reflects the isotropy of space.

Problem 1.13. Find how total momentum and total angular momentum transform under the Galilean transformations.

1.5. One-dimensional motion. The motion of systems with one degree of freedom is called one-dimensional. In terms of a Cartesian coordinate x on $M = \mathbb{R}$ the Lagrangian takes the form

$$L = \tfrac{1}{2} m \dot{x}^2 - V(x).$$

The conservation of energy

$$E = \frac{1}{2} m \dot{x}^2 + V(x)$$

allows us to solve the equation of motion in a closed form by separation of variables. We have

$$\frac{dx}{dt} = \sqrt{\frac{2}{m}(E - V(x))},$$

so that

$$t = \sqrt{\frac{m}{2}} \int \frac{dx}{\sqrt{E - V(x)}}.$$

The inverse function $x(t)$ is a general solution of Newton's equation

$$m\ddot{x} = -\frac{dV}{dx},$$

with two arbitrary constants, the energy E and the constant of integration.

Since kinetic energy is non-negative, for a given value of E the actual motion takes place in the region of $\mathbb{R}$ where $V(x) \leq E$. The points where $V(x) = E$ are called *turning points*. The motion which is confined between two turning points is called *finite*. The finite motion is periodic — the particle oscillates between the turning points x_1 and x_2 with the period

$$T(E) = \sqrt{2m} \int_{x_1}^{x_2} \frac{dx}{\sqrt{E - V(x)}}.$$

If the region $V(x) \leq E$ is unbounded, then the motion is called *infinite* and the particle eventually goes to infinity. The regions where $V(x) > E$ are forbidden.

On the phase plane with coordinates (x, y) Newton's equation reduces to the first order system

$$m\dot{x} = y, \quad \dot{y} = -\frac{dV}{dx}.$$

Trajectories correspond to the phase curves $(x(t), y(t))$, which lie on the level sets

$$\frac{y^2}{2m} + V(x) = E$$

of the energy function. The points $(x_0, 0)$, where x_0 is a critical point of the potential energy $V(x)$, correspond to the equilibrium solutions. The local minima correspond to the stable solutions and local maxima correspond to the unstable solutions. For the values of E which do not correspond to the equilibrium solutions the level sets are smooth curves. These curves are closed if the motion is finite.

The simplest non-trivial one-dimensional system, besides the free particle, is the harmonic oscillator with $V(x) = \frac{1}{2}kx^2$ $(k > 0)$, considered in Example 1.6. The general solution of the equation of motion is

$$x(t) = A\cos(\omega t + \alpha),$$

where A is the *amplitude*, $\omega = \sqrt{\dfrac{k}{m}}$ is the *frequency*, and α is the *phase* of a simple harmonic motion with the period $T = \dfrac{2\pi}{\omega}$. The energy is $E = \frac{1}{2}m\omega^2 A^2$ and the motion is finite with the same period T for $E > 0$.

Problem 1.14. Show that for $V(x) = -x^4$ there are phase curves which do not exist for all times. Prove that if $V(x) \geq 0$ for all x, then all phase curves exist for all times.

Problem 1.15. The simple pendulum is a Lagrangian system with $M = S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $L = \frac{1}{2}\dot{\theta}^2 + \cos\theta$. Find the period T of the pendulum as a function of the amplitude of the oscillations.

Problem 1.16. Suppose that the potential energy $V(x)$ is even, $V(0) = 0$, and $V(x)$ is a one-to-one monotonically increasing function for $x \geq 0$. Prove that the inverse function $x(V)$ and the period $T(E)$ are related by the Abel transform

$$T(E) = 2\sqrt{2m}\int_0^E \frac{dx}{dV}\frac{dV}{\sqrt{E-V}} \quad \text{and} \quad x(V) = \frac{1}{2\pi\sqrt{2m}}\int_0^V \frac{T(E)dE}{\sqrt{V-E}}.$$

1.6. The motion in a central field and the Kepler problem. The motion of a system of two interacting particles — the *two-body problem* — can also be solved completely. Namely, in this case (see Example 1.2) $M = \mathbb{R}^6$ and

$$L = \frac{m_1\dot{\boldsymbol{r}}_1^2}{2} + \frac{m_2\dot{\boldsymbol{r}}_2^2}{2} - V(|\boldsymbol{r}_1 - \boldsymbol{r}_2|).$$

Introducing on $\mathbb{R}^6$ new coordinates

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 \quad \text{and} \quad \boldsymbol{R} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2},$$

we get

$$L = \tfrac{1}{2}m\dot{\boldsymbol{R}}^2 + \tfrac{1}{2}\mu\dot{\boldsymbol{r}}^2 - V(|\boldsymbol{r}|),$$

where $m = m_1 + m_2$ is the *total mass* and $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ is the *reduced mass* of a two-body system. The Lagrangian L depends only on the velocity $\dot{\boldsymbol{R}}$ of the center of mass and not on its position $\boldsymbol{R}$. A generalized coordinate with this property is called *cyclic*. It follows from the Euler-Lagrange equations that generalized momentum corresponding to the cyclic coordinate is conserved. In our case it is a total momentum of the system,

$$\boldsymbol{P} = \frac{\partial L}{\partial \dot{\boldsymbol{R}}} = m\dot{\boldsymbol{R}},$$

so that the center of mass $\boldsymbol{R}$ moves uniformly. Thus in the frame of reference where $\boldsymbol{R} = 0$, the two-body problem reduces to the problem of a single particle of mass μ in the external central field $V(|\boldsymbol{r}|)$. In spherical coordinates in $\mathbb{R}^3$,

$$x = r\sin\vartheta\cos\varphi,\ y = r\sin\vartheta\sin\varphi,\ z = r\cos\vartheta,$$

where $0 \leq \vartheta < \pi$, $0 \leq \varphi < 2\pi$, its Lagrangian takes the form

$$L = \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\vartheta}^2 + r^2\sin^2\vartheta\,\dot{\varphi}^2) - V(r).$$

It follows from the conservation of angular momentum $\boldsymbol{M} = \mu\boldsymbol{r} \times \dot{\boldsymbol{r}}$ that during motion the position vector $\boldsymbol{r}$ lies in the plane P orthogonal to $\boldsymbol{M}$ in $\mathbb{R}^3$. Introducing polar coordinates (r, χ) in the plane P we get $\dot{\chi}^2 = \dot{\vartheta}^2 + \sin^2\vartheta\,\dot{\varphi}^2$, so that

$$L = \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\chi}^2) - V(r).$$

The coordinate χ is cyclic and its generalized momentum $\mu r^2\dot\chi$ coincides with $|\boldsymbol{M}|$ if $\dot\chi > 0$ and with $-|\boldsymbol{M}|$ if $\dot\chi < 0$. Denoting this quantity by M, we get the equation

$$\mu r^2\dot\chi = M, \tag{1.7}$$

which is equivalent to *Kepler's second law*[12]. Using (1.7) we get for the total energy

$$E = \tfrac{1}{2}\mu(\dot r^2 + r^2\dot\chi^2) + V(r) = \tfrac{1}{2}\mu\dot r^2 + V(r) + \frac{M^2}{2\mu r^2}. \tag{1.8}$$

Thus the radial motion reduces to a one-dimensional motion on the half-line $r > 0$ with the effective potential energy

$$V_{eff}(r) = V(r) + \frac{M^2}{2\mu r^2},$$

where the second term is called the *centrifugal energy*. As in the previous section, the solution is given by

$$t = \sqrt{\frac{\mu}{2}}\int\frac{dr}{\sqrt{E - V_{eff}(r)}}. \tag{1.9}$$

It follows from (1.7) that the angle χ is a monotonic function of t, given by another quadrature

$$\chi = \frac{M}{\sqrt{2\mu}}\int\frac{dr}{r^2\sqrt{E - V_{eff}(r)}}, \tag{1.10}$$

yielding an equation of the trajectory in polar coordinates.

The set $V_{eff}(r) \leq E$ is a union of annuli $0 \leq r_{min} \leq r \leq r_{max} \leq \infty$, and the motion is finite if $0 < r_{min} \leq r \leq r_{max} < \infty$. Though for a finite motion $r(t)$ oscillates between r_{min} and r_{max}, corresponding trajectories are not necessarily closed. The necessary and sufficient condition for a finite motion to have a closed trajectory is that the angle

$$\Delta\chi = \frac{M}{\sqrt{2\mu}}\int_{r_{min}}^{r_{max}}\frac{dr}{r^2\sqrt{E - V_{eff}(r)}}$$

is commensurable with 2π, i.e., $\Delta\chi = 2\pi\dfrac{m}{n}$ for some $m, n \in \mathbb{Z}$. If the angle $\Delta\chi$ is not commensurable with 2π, the orbit is everywhere dense in the annulus $r_{min} \leq r \leq r_{max}$. If

$$\lim_{r\to\infty} V_{eff}(r) = \lim_{r\to\infty} V(r) = V < \infty,$$

the motion is infinite for $E > V$ — the particle goes to ∞ with finite velocity $\sqrt{\frac{2}{\mu}(E - V)}$.

[12]It is the statement that *sectorial velocity* of a particle in a central field is constant.

A very important special case is when

$$V(r) = -\frac{\alpha}{r}.$$

It describes Newton's gravitational attraction ($\alpha > 0$) and Coulomb electrostatic interaction (either attractive or repulsive). First consider the case when $\alpha > 0$ — Kepler's problem. The effective potential energy is

$$V_{eff}(r) = -\frac{\alpha}{r} + \frac{M^2}{2\mu r^2}$$

and has the global minimum

$$V_0 = -\frac{\alpha^2\mu}{2M^2}$$

at $r_0 = \dfrac{M^2}{\alpha\mu}$. The motion is infinite for $E \geq 0$ and is finite for $V_0 \leq E < 0$. The explicit form of trajectories can be determined by an elementary integration in (1.10), which gives

$$\chi = \cos^{-1}\frac{\dfrac{M}{r} - \dfrac{M}{r_0}}{\sqrt{2\mu(E - V_0)}} + C.$$

Choosing a constant of integration $C = 0$ and introducing notation

$$p = r_0 \quad \text{and} \quad e = \sqrt{1 - \frac{E}{V_0}},$$

we get the equation of the orbit (trajectory)

$$\frac{p}{r} = 1 + e\cos\chi. \tag{1.11}$$

This is the equation of a conic section with one focus at the origin. Quantity $2p$ is called the *latus rectum* of the orbit, and e is called the *eccentricity*. The choice $C = 0$ is such that the point with $\chi = 0$ is the point nearest to the origin (called the *perihelion*). When $V_0 \leq E < 0$, the eccentricity $e < 1$ so that the orbit is the ellipse[13] with the major and minor semi-axes

$$a = \frac{p}{1 - e^2} = \frac{\alpha}{2|E|}, \quad b = \frac{p}{\sqrt{1 - e^2}} = \frac{|M|}{\sqrt{2\mu|E|}}. \tag{1.12}$$

Correspondingly, $r_{min} = \dfrac{p}{1+e}$, $r_{max} = \dfrac{p}{1-e}$, and the period T of elliptic orbit is given by

$$T = \pi\alpha\sqrt{\frac{\mu}{2|E|^3}}.$$

The last formula is *Kepler's third law.* When $E > 0$, the eccentricity $e > 1$ and the motion is infinite — the orbit is a hyperbola with the origin as

[13]The statement that planets have elliptic orbits with a focus at the Sun is *Kepler's first law.*

internal focus. When $E = 0$, the eccentricity $e = 1$ — the particle starts from rest at ∞ and the orbit is a parabola.

For the repulsive case $\alpha < 0$ the effective potential energy $V_{eff}(r)$ is always positive and decreases monotonically from ∞ to 0. The motion is always infinite and the trajectories are hyperbolas (parabola if $E = 0$)

$$\frac{p}{r} = -1 + e\cos\chi$$

with

$$p = \frac{M^2}{\alpha\mu} \quad \text{and} \quad e = \sqrt{1 + \frac{2EM^2}{\mu\alpha^2}}.$$

Kepler's problem is very special: for every $\alpha \in \mathbb{R}$ the Lagrangian system on $\mathbb{R}^3$ with

$$L = \tfrac{1}{2}\mu\dot{\boldsymbol{r}}^2 + \frac{\alpha}{r} \tag{1.13}$$

has three extra integrals of motion W_1, W_2, W_3 in addition to the components of the angular momentum $\boldsymbol{M}$. The corresponding vector $\boldsymbol{W} = (W_1, W_2, W_3)$, called the *Laplace-Runge-Lenz vector*, is given by

$$\boldsymbol{W} = \dot{\boldsymbol{r}} \times \boldsymbol{M} - \frac{\alpha\boldsymbol{r}}{r}. \tag{1.14}$$

Indeed, using equations of motion $\mu\ddot{\boldsymbol{r}} = -\dfrac{\alpha\boldsymbol{r}}{r^3}$ and conservation of the angular momentum $\boldsymbol{M} = \mu\boldsymbol{r} \times \dot{\boldsymbol{r}}$, we get

$$\begin{aligned}\dot{\boldsymbol{W}} &= \mu\ddot{\boldsymbol{r}} \times (\boldsymbol{r} \times \dot{\boldsymbol{r}}) - \frac{\alpha\dot{\boldsymbol{r}}}{r} + \frac{\alpha(\dot{\boldsymbol{r}}\cdot\boldsymbol{r})\boldsymbol{r}}{r^3} \\ &= (\mu\ddot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}})\boldsymbol{r} - (\mu\ddot{\boldsymbol{r}}\cdot\boldsymbol{r})\dot{\boldsymbol{r}} - \frac{\alpha\dot{\boldsymbol{r}}}{r} + \frac{\alpha(\dot{\boldsymbol{r}}\cdot\boldsymbol{r})\boldsymbol{r}}{r^3} \\ &= 0.\end{aligned}$$

Using $\mu(\dot{\boldsymbol{r}} \times \boldsymbol{M})\cdot\boldsymbol{r} = \boldsymbol{M}^2$ and the identity $(\boldsymbol{a}\times\boldsymbol{b})^2 = \boldsymbol{a}^2\,\boldsymbol{b}^2 - (\boldsymbol{a}\cdot\boldsymbol{b})^2$, we get

$$\boldsymbol{W}^2 = \alpha^2 + \frac{2\boldsymbol{M}^2E}{\mu} \tag{1.15}$$

where

$$E = \frac{\boldsymbol{p}^2}{2\mu} - \frac{\alpha}{r}$$

is the energy corresponding to the Lagrangian (1.13). The fact that all orbits are conic sections follows from this extra symmetry of Kepler's problem.

Problem 1.17. Prove all the statements made in this section.

Problem 1.18. Show that if

$$\lim_{r \to 0} V_{eff}(r) = -\infty,$$

then there are orbits with $r_{min} = 0$ — "fall" of the particle to the center.

Problem 1.19. Prove that all finite trajectories in the central field are closed only when

$$V(r) = kr^2, \;\; k > 0, \quad \text{and} \quad V(r) = -\frac{\alpha}{r}, \;\; \alpha > 0.$$

Problem 1.20. Find parametric equations for orbits in Kepler's problem.

Problem 1.21. Prove that the Laplace-Runge-Lenz vector $\boldsymbol{W}$ points in the direction of the major axis of the orbit and that $|\boldsymbol{W}| = \alpha e$, where e is the eccentricity of the orbit.

Problem 1.22. Using the conservation of the Laplace-Runge-Lenz vector, prove that trajectories in Kepler's problem with $E < 0$ are ellipses. (*Hint:* Evaluate $\boldsymbol{W} \cdot \boldsymbol{r}$ and use the result of the previous problem.)

1.7. Legendre transform. The equations of motion of a Lagrangian system (M, L) in standard coordinates associated with a coordinate chart $U \subset M$ are the Euler-Lagrange equations. In expanded form, they are given by the following system of second order ordinary differential equations:

$$\begin{aligned}\frac{\partial L}{\partial q^i}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}(\boldsymbol{q}, \dot{\boldsymbol{q}})\right) \\ &= \sum_{j=1}^{n}\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}(\boldsymbol{q}, \dot{\boldsymbol{q}})\, \ddot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}(\boldsymbol{q}, \dot{\boldsymbol{q}})\, \dot{q}^j\right), \quad i = 1, \dots, n.\end{aligned}$$

In order for this system to be solvable for the highest derivatives for all initial conditions in TU, the symmetric $n \times n$ matrix

$$H_L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \left\{\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}(\boldsymbol{q}, \dot{\boldsymbol{q}})\right\}_{i,j=1}^{n}$$

should be invertible on TU.

Definition. A Lagrangian system (M, L) is called *non-degenerate* if for every coordinate chart U on M the matrix $H_L(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is invertible on TU.

Remark. Note that the $n \times n$ matrix H_L is a Hessian of the Lagrangian function L for vertical directions on TM. Under the change of standard coordinates $\boldsymbol{q}' = F(\boldsymbol{q})$ and $\boldsymbol{q}' = F_*(\boldsymbol{q})\boldsymbol{v}$ (see Section 1.2) it has the transformation law

$$H_L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = F_*(\boldsymbol{q})^T H_L(\boldsymbol{q}', \dot{\boldsymbol{q}}') F_*(\boldsymbol{q}),$$

where $F_*(\boldsymbol{q})^T$ is the transposed matrix, so that the condition $\det H_L \neq 0$ does not depend on the choice of standard coordinates.

For an invariant formulation, consider the 1-form θ_L, defined in standard coordinates associated with a coordinate chart $U \subset M$ by

$$\theta_L = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}^i} dq^i = \frac{\partial L}{\partial \dot{\boldsymbol{q}}} d\boldsymbol{q}.$$

It follows from Corollary 1.2 that θ_L is a well-defined 1-form on TM.

Lemma 1.2. *A Lagrangian system (M, L) is non-degenerate if and only if the 2-form $d\theta_L$ on TM is non-degenerate.*

Proof. In standard coordinates,

$$d\theta_L = \sum_{i,j=1}^{n} \left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} d\dot{q}^j \wedge dq^i + \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} dq^j \wedge dq^i \right),$$

and it is easy to see, by considering the $2n$-form $d\theta_L^n = \underbrace{d\theta_L \wedge \cdots \wedge d\theta_L}_{n}$, that the 2-form $d\theta_L$ is non-degenerate if and only if the matrix H_L is non-degenerate. $\square$

Remark. Using the 1-form θ_L, the Noether integral I in Theorem 1.3 can be written as

$$I = i_{X'}(\theta_L), \tag{1.16}$$

where X' is a lift to TM of a vector field X on M given by (1.5). It also immediately follows from (1.6) that if X is an infinitesimal symmetry, then

$$\mathcal{L}_{X'}(\theta_L) = 0. \tag{1.17}$$

Definition. Let (U, φ) be a coordinate chart on M. Coordinates

$$(\boldsymbol{p}, \boldsymbol{q}) = (p_1, \ldots, p_n, q^1, \ldots, q^n)$$

on the chart $T^*U \simeq \mathbb{R}^n \times U$ on the cotangent bundle T^*M are called *standard coordinates*[14] if for $(p, q) \in T^*U$ and $f \in C^\infty(U)$

$$p_i(df) = \frac{\partial f}{\partial q^i}, \quad i = 1, \ldots, n.$$

Equivalently, standard coordinates on T^*U are uniquely characterized by the condition that $\boldsymbol{p} = (p_1, \ldots, p_n)$ are coordinates in the fiber corresponding to the basis $dq^1, \ldots, dq^n$ for T_q^*M, dual to the basis $\dfrac{\partial}{\partial q^1}, \ldots, \dfrac{\partial}{\partial q^n}$ for T_qM.

[14] Following tradition, the first n coordinates parametrize the fiber of T^*U and the last n coordinates parametrize the base.

Definition. The 1-form θ on T^*M, defined in standard coordinates by

$$\theta = \sum_{i=1}^{n} p_i dq^i = \boldsymbol{p} d\boldsymbol{q},$$

is called *Liouville's canonical* 1*-form.*

Corollary 1.2 shows that θ is a well-defined 1-form on T^*M. Clearly, the 1-form θ also admits an invariant definition

$$\theta(u) = p(\pi_*(u)), \quad \text{where} \quad u \in T_{(p,q)}T^*M,$$

and $\pi : T^*M \to M$ is the canonical projection.

Definition. A fibre-wise mapping $\tau_L : TM \to T^*M$ is called a *Legendre transform* associated with the Lagrangian L if

$$\theta_L = \tau_L^*(\theta).$$

In standard coordinates the Legendre transform is given by

$$\tau_L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{p}, \boldsymbol{q}), \quad \text{where} \quad \boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}, \dot{\boldsymbol{q}}).$$

The mapping τ_L is a local diffeomorphism if and only if the Lagrangian L is non-degenerate.

Definition. Suppose that the Legendre transform $\tau_L : TM \to T^*M$ is a diffeomorphism. The *Hamiltonian* function $H : T^*M \to \mathbb{R}$, associated with the Lagrangian $L : TM \to \mathbb{R}$, is defined by

$$H \circ \tau_L = E_L = \dot{\boldsymbol{q}} \frac{\partial L}{\partial \dot{\boldsymbol{q}}} - L.$$

In standard coordinates,

$$H(\boldsymbol{p}, \boldsymbol{q}) = \left.(\boldsymbol{p}\dot{\boldsymbol{q}} - L(\boldsymbol{q}, \dot{\boldsymbol{q}}))\right|_{\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}}},$$

where $\dot{\boldsymbol{q}}$ is a function of $\boldsymbol{p}$ and $\boldsymbol{q}$ defined by the equation $\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ through the implicit function theorem. The cotangent bundle T^*M is called the *phase space* of the Lagrangian system (M, L). It turns out that on the phase space the equations of motion take a very simple and symmetric form.

Theorem 1.4. *Suppose that the Legendre transform $\tau_L : TM \to T^*M$ is a diffeomorphism. Then the Euler-Lagrange equations in standard coordinates on TM,*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \dots, n,$$

are equivalent to the following system of first order differential equations in standard coordinates on T^*M*:*

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad \dot{q}^i = \frac{\partial H}{\partial p_i}, \quad i = 1, \ldots, n.$$

Proof. We have

$$\begin{aligned} dH &= \frac{\partial H}{\partial \boldsymbol{p}} d\boldsymbol{p} + \frac{\partial H}{\partial \boldsymbol{q}} d\boldsymbol{q} \\ &= \left(\boldsymbol{p} d\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}} d\boldsymbol{p} - \frac{\partial L}{\partial \boldsymbol{q}} d\boldsymbol{q} - \frac{\partial L}{\partial \dot{\boldsymbol{q}}} d\dot{\boldsymbol{q}} \right) \Big|_{\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}}} \\ &= \left(\dot{\boldsymbol{q}} d\boldsymbol{p} - \frac{\partial L}{\partial \boldsymbol{q}} d\boldsymbol{q} \right) \Big|_{\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}}} . \end{aligned}$$

Thus under the Legendre transform,

$$\dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}} \quad \text{and} \quad \dot{\boldsymbol{p}} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\boldsymbol{q}}} = \frac{\partial L}{\partial \boldsymbol{q}} = -\frac{\partial H}{\partial \boldsymbol{q}}. \qquad \square$$

Corresponding first order differential equations on T^*M are called *Hamilton's equations* (*canonical equations*).

Corollary 1.5. *The Hamiltonian* H *is constant on the solutions of Hamilton's equations.*

Proof. For $H(t) = H(\boldsymbol{p}(t), \boldsymbol{q}(t))$ we have

$$\frac{dH}{dt} = \frac{\partial H}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} + \frac{\partial H}{\partial \boldsymbol{p}} \dot{\boldsymbol{p}} = \frac{\partial H}{\partial \boldsymbol{q}} \frac{\partial H}{\partial \boldsymbol{p}} - \frac{\partial H}{\partial \boldsymbol{p}} \frac{\partial H}{\partial \boldsymbol{q}} = 0. \qquad \square$$

For the Lagrangian

$$L = \frac{m\dot{\boldsymbol{r}}^2}{2} - V(\boldsymbol{r}) = T - V, \quad \boldsymbol{r} \in \mathbb{R}^3,$$

of a particle of mass m in a potential field $V(\boldsymbol{r})$, considered in Example 1.4, we have

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{r}}} = m\dot{\boldsymbol{r}}.$$

Thus the Legendre transform $\tau_L : T\mathbb{R}^3 \to T^*\mathbb{R}^3$ is a global diffeomorphism, linear on the fibers, and

$$H(\boldsymbol{p}, \boldsymbol{r}) = (\boldsymbol{p}\dot{\boldsymbol{r}} - L)|_{\dot{\boldsymbol{r}} = \frac{\boldsymbol{p}}{m}} = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{r}) = T + V.$$

Hamilton's equations

$$\dot{\boldsymbol{r}} = \frac{\partial H}{\partial \boldsymbol{p}} = \frac{\boldsymbol{p}}{m},$$
$$\dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{r}} = -\frac{\partial V}{\partial \boldsymbol{r}}$$

are equivalent to Newton's equations with the force $\boldsymbol{F} = -\dfrac{\partial V}{\partial \boldsymbol{r}}$.

For the Lagrangian system describing small oscillators, considered in Example 1.6, we have $\boldsymbol{p} = m\dot{\boldsymbol{q}}$, and using normal coordinates we get

$$H(\boldsymbol{p},\boldsymbol{q}) = (\boldsymbol{p}\dot{\boldsymbol{q}} - L(\boldsymbol{q},\dot{\boldsymbol{q}}))\big|_{\dot{\boldsymbol{q}}=\frac{\boldsymbol{p}}{m}} = \frac{\boldsymbol{p}^2}{2m} + V_0(\boldsymbol{q}) = \frac{1}{2m}\Big(\boldsymbol{p}^2 + m^2\sum_{i=1}^{n}\omega_i^2(q^i)^2\Big).$$

Similarly, for the system of N interacting particles, considered in Example 1.2, we have $\boldsymbol{p} = (\boldsymbol{p}_1,\dots,\boldsymbol{p}_N)$, where

$$\boldsymbol{p}_a = \frac{\partial L}{\partial \dot{\boldsymbol{r}}_a} = m_a\dot{\boldsymbol{r}}_a, \quad a = 1,\dots,N.$$

The Legendre transform $\tau_L : T\mathbb{R}^{3N} \to T^*\mathbb{R}^{3N}$ is a global diffeomorphism, linear on the fibers, and

$$H(\boldsymbol{p},\boldsymbol{r}) = (\boldsymbol{p}\dot{\boldsymbol{r}} - L)\big|_{\dot{\boldsymbol{r}}=\frac{\boldsymbol{p}}{m}} = \sum_{a=1}^{N}\frac{\boldsymbol{p}_a^2}{2m_a} + V(\boldsymbol{r}) = T + V.$$

In particular, for a closed system with pair-wise interaction,

$$H(\boldsymbol{p},\boldsymbol{r}) = \sum_{a=1}^{N}\frac{\boldsymbol{p}_a^2}{2m_a} + \sum_{1\le a<b\le N} V_{ab}(\boldsymbol{r}_a - \boldsymbol{r}_b).$$

In general, consider the Lagrangian

$$L = \sum_{i,j=1}^{n}\tfrac{1}{2}a_{ij}(\boldsymbol{q})\dot{q}^i\dot{q}^j - V(\boldsymbol{q}),\ \ \boldsymbol{q}\in\mathbb{R}^n,$$

where $A(\boldsymbol{q}) = \{a_{ij}(\boldsymbol{q})\}_{i,j=1}^n$ is a symmetric $n\times n$ matrix. We have

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = \sum_{j=1}^{n}a_{ij}(\boldsymbol{q})\dot{q}^j, \quad i = 1,\dots,n,$$

and the Legendre transform is a global diffeomorphism, linear on the fibers, if and only if the matrix $A(\boldsymbol{q})$ is non-degenerate for all $\boldsymbol{q}\in\mathbb{R}^n$. In this case,

$$H(\boldsymbol{p},\boldsymbol{q}) = (\boldsymbol{p}\dot{\boldsymbol{q}} - L(\boldsymbol{q},\dot{\boldsymbol{q}}))\big|_{\boldsymbol{p}=\frac{\partial L}{\partial \dot{\boldsymbol{q}}}} = \sum_{i,j=1}^{n}\tfrac{1}{2}a^{ij}(\boldsymbol{q})p_ip_j + V(\boldsymbol{q}),$$

where $\{a^{ij}(\boldsymbol{q})\}_{i,j=1}^n = A^{-1}(\boldsymbol{q})$ is the inverse matrix.

Problem 1.23 (Second tangent bundle)**.** Let $\pi : TM \to M$ be the canonical projection and let $T_{\mathrm{V}}(TM)$ be the *vertical tangent bundle* of TM along the fibers of π — the kernel of the bundle mapping $\pi_* : T(TM) \to TM$. Prove that there is a natural bundle isomorphism $i : TM \simeq T_{\mathrm{V}}(TM)$.

Problem 1.24 (Invariant definition of the 1-form θ_L)**.** Show that $\theta_L(v) = dL((i \circ \pi_*)v)$, where $v \in T(TM)$.

Problem 1.25. Give an invariant proof of (1.17).

Problem 1.26. Prove that the path $\gamma(t)$ in M is a trajectory for the Lagrangian system (M, L) if and only if

$$i_{\dot{\gamma}'(t)}(d\theta_L) + dE_L(\gamma'(t)) = 0,$$

where $\dot{\gamma}'(t)$ is the velocity vector of the path $\gamma'(t)$ in TM.

Problem 1.27. Show that for a charged particle in an electromagnetic field, considered in Example 1.5,

$$\boldsymbol{p} = m\dot{\boldsymbol{r}} + \frac{e}{c}\boldsymbol{A} \quad \text{and} \quad H(\boldsymbol{p}, \boldsymbol{r}) = \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2 + e\varphi(\boldsymbol{r}).$$

Problem 1.28. Suppose that for a Lagrangian system $(\mathbb{R}^n, L)$ the Legendre transform τ_L is a diffeomorphism and let H be the corresponding Hamiltonian. Prove that for fixed $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ the function $\boldsymbol{p}\dot{\boldsymbol{q}} - H(\boldsymbol{p}, \boldsymbol{q})$ has a single critical point at $\boldsymbol{p} = \dfrac{\partial L}{\partial \dot{\boldsymbol{q}}}$.

Problem 1.29. Give an example of a non-degenerate Lagrangian system (M, L) such that the Legendre transform $\tau_L : TM \to T^*M$ is one-to-one but not onto.

2. Hamiltonian Mechanics

2.1. Hamilton's equations. With every function $H : T^*M \to \mathbb{R}$ on the phase space T^*M there are associated Hamilton's equations — a first-order system of ordinary differential equations, which in the standard coordinates on T^*U has the form

$$\dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}}, \quad \dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}. \tag{2.1}$$

The corresponding vector field X_H on T^*U,

$$X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} \right) = \frac{\partial H}{\partial \boldsymbol{p}}\frac{\partial}{\partial \boldsymbol{q}} - \frac{\partial H}{\partial \boldsymbol{q}}\frac{\partial}{\partial \boldsymbol{p}},$$

gives rise to a well-defined vector field X_H on T^*M, called the *Hamiltonian vector field*. Suppose now that the vector field X_H on T^*M is complete, i.e., its integral curves exist for all times. The corresponding one-parameter group $\{g_t\}_{t\in\mathbb{R}}$ of diffeomorphisms of T^*M generated by X_H is called the *Hamiltonian phase flow*. It is defined by $g_t(p, q) = (p(t), q(t))$, where $p(t)$, $q(t)$ is a solution of Hamilton's equations satisfying $p(0) = p$, $q(0) = q$.

Liouville's canonical 1-form θ on T^*M defines a 2-form $\omega = d\theta$. In standard coordinates on T^*M it is given by

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq^i = d\boldsymbol{p} \wedge d\boldsymbol{q},$$

and is a non-degenerate 2-form. The form ω is called the *canonical symplectic form* on T^*M. The symplectic form ω defines an isomorphism $J : T^*(T^*M) \to T(T^*M)$ between tangent and cotangent bundles to T^*M. For every $(p,q) \in T^*M$ the linear mapping $J^{-1} : T_{(p,q)}T^*M \to T^*_{(p,q)}T^*M$ is given by

$$\omega(u_1, u_2) = J^{-1}(u_2)(u_1), \quad u_1, u_2 \in T_{(p,q)}T^*M.$$

The mapping J induces the isomorphism between the infinite-dimensional vector spaces $\mathcal{A}^1(T^*M)$ and $\mathrm{Vect}(T^*M)$, which is linear over $C^\infty(T^*M)$. If ϑ is a 1-form on T^*M, then the corresponding vector field $J(\vartheta)$ on T^*M satisfies

$$\omega(X, J(\vartheta)) = \vartheta(X), \quad X \in \mathrm{Vect}(T^*M),$$

and $J^{-1}(X) = -i_X\omega$. In particular, in standard coordinates,

$$J(d\boldsymbol{p}) = \frac{\partial}{\partial \boldsymbol{q}} \quad \text{and} \quad J(d\boldsymbol{q}) = -\frac{\partial}{\partial \boldsymbol{p}},$$

so that $X_H = J(dH)$.

Theorem 2.1. *The Hamiltonian phase flow on T^*M preserves the canonical symplectic form.*

Proof. We need to prove that $(g_t)^*\omega = \omega$. Since g_t is a one-parameter group of diffeomorphisms, it is sufficient to show that

$$\frac{d}{dt}(g_t)^*\omega\bigg|_{t=0} = \mathcal{L}_{X_H}\omega = 0,$$

where $\mathcal{L}_{X_H}$ is the Lie derivative along the vector field X_H. Since for every vector field X,

$$\mathcal{L}_X(df) = d(X(f)),$$

we compute

$$\mathcal{L}_{X_H}(dp_i) = -d\left(\frac{\partial H}{\partial q^i}\right) \quad \text{and} \quad \mathcal{L}_{X_H}(dq^i) = d\left(\frac{\partial H}{\partial p_i}\right),$$

so that

$$\begin{aligned}\mathcal{L}_{X_H}\omega &= \sum_{i=1}^{n}\left(\mathcal{L}_{X_H}(dp_i) \wedge dq^i + dp_i \wedge \mathcal{L}_{X_H}(dq^i)\right) \\ &= \sum_{i=1}^{n}\left(-d\left(\frac{\partial H}{\partial q^i}\right) \wedge dq^i + dp_i \wedge d\left(\frac{\partial H}{\partial p_i}\right)\right) = -d(dH) = 0. \quad \square\end{aligned}$$

Corollary 2.2. $\mathcal{L}_{X_H}(\theta) = d(-H + \theta(X_H))$, *where* θ *is Liouville's canonical* 1*-form.*

The canonical symplectic form ω on T^*M defines the volume form $\dfrac{\omega^n}{n!} = \dfrac{1}{n!}\underbrace{\omega\wedge\cdots\wedge\omega}_{n}$ on T^*M, called *Liouville's volume form.*

Corollary 2.3 (Liouville's theorem)**.** *The Hamiltonian phase flow on* T^*M *preserves Liouville's volume form.*

The restriction of the symplectic form ω on T^*M to the configuration space M is 0. Generalizing this property, we get the following notion.

Definition. A submanifold $\mathscr{L}$ of the phase space T^*M is called a *Lagrangian submanifold* if $\dim\mathscr{L} = \dim M$ and $\omega|_{\mathscr{L}} = 0$.

It follows from Theorem 2.1 that the image of a Lagrangian submanifold under the Hamiltonian phase flow is a Lagrangian submanifold.

Problem 2.1. Verify that X_H is a well-defined vector field on T^*M.

Problem 2.2. Show that if all level sets of the Hamiltonian H are compact submanifolds of T^*M, then the Hamiltonian vector field X_H is complete.

Problem 2.3. Let $\pi : T^*M \to M$ be the canonical projection, and let $\mathscr{L}$ be a Lagrangian submanifold. Show that if the mapping $\pi|_{\mathscr{L}} : \mathscr{L} \to M$ is a diffeomorphism, then $\mathscr{L}$ is a graph of a smooth function on M. Give examples when for some $t > 0$ the corresponding projection of $g_t(\mathscr{L})$ onto M is no longer a diffeomorphism.

2.2. The action functional in the phase space. With every function H on the phase space T^*M there is an associated 1-form

$$\theta - H dt = \boldsymbol{p}d\boldsymbol{q} - H dt$$

on the *extended phase space* $T^*M \times \mathbb{R}$, called the *Poincaré-Cartan form.* Let $\gamma : [t_0, t_1] \to T^*M$ be a smooth parametrized path in T^*M such that $\pi(\gamma(t_0)) = q_0$ and $\pi(\gamma(t_1)) = q_1$, where $\pi : T^*M \to M$ is the canonical projection. By definition, the lift of a path γ to the extended phase space $T^*M \times \mathbb{R}$ is a path $\sigma : [t_0, t_1] \to T^*M \times \mathbb{R}$ given by $\sigma(t) = (\gamma(t), t)$, and a path σ in $T^*M \times \mathbb{R}$ is called an *admissible* path if it is a lift of a path γ in T^*M. The space of admissible paths in $T^*M\times\mathbb{R}$ is denoted by $\tilde{P}(T^*M)^{q_1,t_1}_{q_0,t_0}$. A variation of an admissible path σ is a smooth family of admissible paths σ_ε, where $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ and $\sigma_0 = \sigma$, and the corresponding infinitesimal variation is

$$\delta\sigma = \left.\frac{\partial\sigma_\varepsilon}{\partial\varepsilon}\right|_{\varepsilon=0} \in T_\sigma\tilde{P}(T^*M)^{q_1,t_1}_{q_0,t_0}$$

(cf. Section 1.2). The principle of the least action in the phase space is the following statement.

Theorem 2.4 (Poincaré)**.** *The admissible path σ in $T^*M\times\mathbb{R}$ is an extremal for the action functional*

$$S(\sigma)=\int_\sigma(\boldsymbol{p}d\boldsymbol{q}-Hdt)=\int_{t_0}^{t_1}(\boldsymbol{p}\dot{\boldsymbol{q}}-H)dt$$

if and only if it is a lift of a path $\gamma(t)=(\boldsymbol{p}(t),\boldsymbol{q}(t))$ in T^*M, where $\boldsymbol{p}(t)$ and $\boldsymbol{q}(t)$ satisfy canonical Hamilton's equations

$$\dot{\boldsymbol{p}}=-\frac{\partial H}{\partial\boldsymbol{q}},\quad \dot{\boldsymbol{q}}=\frac{\partial H}{\partial\boldsymbol{p}}.$$

Proof. As in the proof of Theorem 1.1, for an admissible family $\sigma_\varepsilon(t)=(\boldsymbol{p}(t,\varepsilon),\boldsymbol{q}(t,\varepsilon),t)$ we compute using integration by parts,

$$\begin{aligned}\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}S(\sigma_\varepsilon)&=\sum_{i=1}^n\int_{t_0}^{t_1}\left(\dot{q}^i\delta p_i-\dot{p}_i\delta q^i-\frac{\partial H}{\partial q^i}\delta q^i-\frac{\partial H}{\partial p_i}\delta p_i\right)dt\\&+\sum_{i=1}^n p_i\,\delta q^i\Big|_{t_0}^{t_1}.\end{aligned}$$

Since $\delta\boldsymbol{q}(t_0)=\delta\boldsymbol{q}(t_1)=0$, the path σ is critical if and only if $\boldsymbol{p}(t)$ and $\boldsymbol{q}(t)$ satisfy canonical Hamilton's equations (2.1). □

Remark. For a Lagrangian system (M,L), every path $\gamma(t)=(\boldsymbol{q}(t))$ in the configuration space M connecting points q_0 and q_1 defines an admissible path $\hat{\gamma}(t)=(\boldsymbol{p}(t),\boldsymbol{q}(t),t)$ in the phase space T^*M by setting $\boldsymbol{p}=\dfrac{\partial L}{\partial\dot{\boldsymbol{q}}}$. If the Legendre transform $\tau_L:TM\to T^*M$ is a diffeomorphism, then

$$S(\hat{\gamma})=\int_{t_0}^{t_1}(\boldsymbol{p}\dot{\boldsymbol{q}}-H)dt=\int_{t_0}^{t_1}L(\gamma'(t),t)dt.$$

Thus the principle of the least action in a configuration space — Hamilton's principle — follows from the principle of the least action in a phase space. In fact, in this case the two principles are equivalent (see Problem 1.28).

From Corollary 1.5 we immediately get the following result.

Corollary 2.5. *Solutions of canonical Hamilton's equations lying on the hypersurface $H(\boldsymbol{p},\boldsymbol{q})=E$ are extremals of the functional $\int_\sigma\boldsymbol{p}d\boldsymbol{q}$ in the class of admissible paths σ lying on this hypersurface.*

Corollary 2.6 (Maupertuis' principle)**.** *The trajectory $\gamma=(\boldsymbol{q}(\tau))$ of a closed Lagrangian system (M,L) connecting points q_0 and q_1 and having energy E is the extremal of the functional*

$$\int_\gamma\boldsymbol{p}d\boldsymbol{q}=\int_\gamma\frac{\partial L}{\partial\dot{\boldsymbol{q}}}(\boldsymbol{q}(\tau),\dot{\boldsymbol{q}}(\tau))\dot{\boldsymbol{q}}(\tau)d\tau$$

on the space of all paths in the configuration space M connecting points q_0 and q_1 and parametrized such that $H(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}(\tau), \boldsymbol{q}(\tau)) = E$.

The functional

$$S_0(\gamma) = \int_\gamma \boldsymbol{p} d\boldsymbol{q}$$

is called the *abbreviated action*[15].

Proof. Every path $\gamma = \boldsymbol{q}(\tau)$, parametrized such that $H(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}, \boldsymbol{q}) = E$, lifts to an admissible path $\sigma = (\frac{\partial L}{\partial \dot{\boldsymbol{q}}}(\tau), \boldsymbol{q}(\tau), \tau)$, $a \leq \tau \leq b$, lying on the hypersurface $H(\boldsymbol{p}, \boldsymbol{q}) = E$. □

Problem 2.4 (Jacobi)**.** On a Riemannian manifold (M, ds^2) consider a Lagrangian system with $L(q, v) = \frac{1}{2}\|v\|^2 - V(q)$. Let $E > V(q)$ for all $q \in M$. Show that the trajectories of a closed Lagrangian system (M, L) with total energy E are geodesics for the Riemannian metric $d\hat{s}^2 = (E - V(q))ds^2$ on M.

2.3. The action as a function of coordinates. Consider a non-degenerate Lagrangian system (M, L) and denote by $\gamma(t; q_0, v_0)$ the solution of Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} - \frac{\partial L}{\partial \boldsymbol{q}} = 0$$

with the initial conditions $\gamma(t_0) = q_0 \in M$ and $\dot{\gamma}(t_0) = v_0 \in T_{q_0}M$. Suppose that there exist a neighborhood $V_0 \subset T_{v_0}M$ of v_0 and $t_1 > t_0$ such that for all $v \in V_0$ the extremals $\gamma(t; q_0, v)$, which start at time t_0 at q_0, do not intersect in the extended configuration space $M \times \mathbb{R}$ for times $t_0 < t < t_1$. Such extremals are said to form a *central field* which includes the extremal $\gamma_0(t) = \gamma(t; q_0, v_0)$. The existence of the central field of extremals is equivalent to the condition that for every $t_0 < t < t_1$ there is a neighborhood $U_t \subset M$ of $\gamma_0(t) \in M$ such that the mapping

$$V_0 \ni v \mapsto q(t) = \gamma(t; q_0, v) \in U_t \tag{2.2}$$

is a diffeomorphism. Basic theorems in the theory of ordinary differential equations guarantee that for t_1 sufficiently close to t_0 every extremal $\gamma(t)$ for $t_0 < t < t_1$ can be included into the central field. In standard coordinates the mapping (2.2) is given by $\dot{\boldsymbol{q}} \mapsto \boldsymbol{q}(t) = \gamma(t; \boldsymbol{q}_0, \dot{\boldsymbol{q}})$.

For the central field of extremals $\gamma(t; \boldsymbol{q}_0, \dot{\boldsymbol{q}})$, $t_0 < t < t_1$, we define the *action as a function of coordinates and time* (or, *classical action*) by

$$S(\boldsymbol{q}, t; \boldsymbol{q}_0, t_0) = \int_{t_0}^{t} L(\gamma'(\tau))d\tau,$$

[15]The accurate formulation of Maupertuis' principle is due to Euler and Lagrange.

where $\gamma(\tau)$ is the extremal from the central field that connects $\boldsymbol{q}_0$ and $\boldsymbol{q}$. For given $\boldsymbol{q}_0$ and t_0, the classical action is defined for $t \in (t_0, t_1)$ and $\boldsymbol{q} \in \bigcup_{t_0<t<t_1} U_t$. For a fixed energy E,

$$S(\boldsymbol{q}, t; \boldsymbol{q}_0, t_0) = S_0(\boldsymbol{q}, t; \boldsymbol{q}_0, t_0) - E(t - t_0), \tag{2.3}$$

where S_0 is the abbreviated action from the previous section.

Theorem 2.7. *The differential of the classical action $S(\boldsymbol{q}, t)$ with fixed initial point is given by*

$$dS = \boldsymbol{p} d\boldsymbol{q} - H dt,$$

where $\boldsymbol{p} = \dfrac{\partial L}{\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ and $H = \boldsymbol{p}\dot{\boldsymbol{q}} - L(\boldsymbol{q}, \dot{\boldsymbol{q}})$ are determined by the velocity $\dot{\boldsymbol{q}}$ of the extremal $\gamma(\tau)$ at time t.

Proof. Let $\boldsymbol{q}_\varepsilon$ be a path in M passing through $\boldsymbol{q}$ at $\varepsilon = 0$ with the tangent vector $\boldsymbol{v} \in T_{\boldsymbol{q}} M \simeq \mathbb{R}^n$, and for ε small enough let $\gamma_\varepsilon(\tau)$ be the family of extremals from the central field satisfying $\gamma_\varepsilon(t_0) = \boldsymbol{q}_0$ and $\gamma_\varepsilon(t) = \boldsymbol{q}_\varepsilon$. For the infinitesimal variation $\delta\gamma$ we have $\delta\gamma(t_0) = 0$ and $\delta\gamma(t) = \boldsymbol{v}$, and for fixed t we get from the formula for variation with the free ends (1.2) that

$$dS(\boldsymbol{v}) = \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \boldsymbol{v}.$$

This shows that $\dfrac{\partial S}{\partial \boldsymbol{q}} = \boldsymbol{p}$. Setting $\boldsymbol{q}(t) = \gamma(t)$, we obtain

$$\frac{d}{dt} S(\boldsymbol{q}(t), t) = \frac{\partial S}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} + \frac{\partial S}{\partial t} = L,$$

so that $\dfrac{\partial S}{\partial t} = L - \boldsymbol{p}\dot{\boldsymbol{q}} = -H$. $\square$

Corollary 2.8. *The classical action satisfies the following nonlinear partial differential equation*

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial \boldsymbol{q}}, \boldsymbol{q}\right) = 0. \tag{2.4}$$

This equation is called the *Hamilton-Jacobi equation*. Hamilton's equations (2.1) can be used for solving the Cauchy problem

$$S(\boldsymbol{q}, t)|_{t=0} = s(\boldsymbol{q}), \quad s \in C^\infty(M), \tag{2.5}$$

for Hamilton-Jacobi equation (2.4) by the method of characteristics. Namely, assume the existence of the Hamiltonian phase flow g_t on T^*M and consider the Lagrangian submanifold

$$\mathscr{L} = \left\{ (\boldsymbol{p}, \boldsymbol{q}) \in T^*M : \boldsymbol{p} = \frac{\partial s(\boldsymbol{q})}{\partial \boldsymbol{q}} \right\},$$

a graph of the 1-form ds on M — a section of the cotangent bundle $\pi : T^*M \to M$. The mapping $\pi|_{\mathscr{L}}$ is one-to-one and for sufficiently small t the restriction of the projection π to the Lagrangian submanifold $\mathscr{L}_t = g_t(\mathscr{L})$ remains to be one-to-one. In other words, there is $t_1 > 0$ such that for all $0 \leq t < t_1$ the mapping $\pi_t = \pi \circ g_t \circ (\pi|_{\mathscr{L}})^{-1} : M \to M$ is a diffeomorphism, and the extremals $\gamma(\tau, \boldsymbol{q}_0, \dot{\boldsymbol{q}}_0)$ in the extended configuration space $M \times \mathbb{R}$, where $\dot{\boldsymbol{q}}_0 = \dfrac{\partial H}{\partial \boldsymbol{p}}(\boldsymbol{p}_0, \boldsymbol{q}_0)$ and $(\boldsymbol{p}_0, \boldsymbol{q}_0) \in \mathscr{L}$, do not intersect. Such extremals are called the *characteristics* of the Hamilton-Jacobi equation.

Proposition 2.1. *For $0 \leq t < t_1$ the solution $S(\boldsymbol{q}, t)$ to the Cauchy problem* (2.4)–(2.5) *is given by*

$$S(\boldsymbol{q}, t) = s(\boldsymbol{q}_0) + \int_0^t L(\gamma'(\tau))d\tau.$$

Here $\gamma(\tau)$ is the characteristic with $\gamma(t) = \boldsymbol{q}$ and with the starting point $\boldsymbol{q}_0 = \gamma(0)$ which is uniquely determined by $\boldsymbol{q}$.

Proof. As in the proof of Theorem 2.7 we use formula (1.2), where now $\boldsymbol{q}_0$ depends on $\boldsymbol{q}$, and obtain

$$\frac{\partial S}{\partial \boldsymbol{q}}(\boldsymbol{q}) = \frac{\partial s}{\partial \boldsymbol{q}_0}(\boldsymbol{q}_0)\frac{\partial \boldsymbol{q}_0}{\partial \boldsymbol{q}} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \frac{\partial L}{\partial \dot{\boldsymbol{q}}_0}(\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0)\frac{\partial \boldsymbol{q}_0}{\partial \boldsymbol{q}} = \boldsymbol{p},$$

since $\dfrac{\partial s}{\partial \boldsymbol{q}_0}(\boldsymbol{q}_0) = \boldsymbol{p}_0 = \dfrac{\partial L}{\partial \dot{\boldsymbol{q}}_0}(\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0)$. Setting $\boldsymbol{q}(t) = \gamma(t)$ we get

$$\frac{d}{dt}S(\boldsymbol{q}(t), t) = \frac{\partial S}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} + \frac{\partial S}{\partial t} = L(\boldsymbol{q}, \dot{\boldsymbol{q}}),$$

so that

$$\frac{\partial S}{\partial t} = -H(\boldsymbol{p}, \boldsymbol{q}),$$

and S satisfies the Hamilton-Jacobi equation. $\square$

We can also consider the action $S(\boldsymbol{q}, t; \boldsymbol{q}_0, t_0)$ as a function of both variables $\boldsymbol{q}$ and $\boldsymbol{q}_0$. The analog of Theorem 2.7 is the following statement.

Proposition 2.2. *The differential of the classical action as a function of initial and final points is given by*

$$dS = \boldsymbol{p}d\boldsymbol{q} - \boldsymbol{p}_0 d\boldsymbol{q}_0 - H(\boldsymbol{p}, \boldsymbol{q})dt + H(\boldsymbol{p}_0, \boldsymbol{q}_0)dt_0.$$

Problem 2.5. Prove that the solution to the Cauchy problem for the Hamilton-Jacobi equation is unique.

2.4. Classical observables and Poisson bracket. Smooth real-valued functions on the phase space T^*M are called *classical observables*. The vector space $C^\infty(T^*M)$ is an $\mathbb{R}$-algebra — an associative algebra over $\mathbb{R}$ with a unit given by the constant function 1, and with a multiplication given by the point-wise product of functions. The commutative algebra $C^\infty(T^*M)$ is called the *algebra of classical observables*. Assuming that the Hamiltonian phase flow g_t exists for all times, the time evolution of every observable $f \in C^\infty(T^*M)$ is given by

$$f_t(p,q) = f(g_t(p,q)) = f(p(t),q(t)), \quad (p,q) \in TM.$$

Equivalently, the time evolution is described by the differential equation

$$\begin{aligned}\frac{df_t}{dt} &= \left.\frac{df_{s+t}}{ds}\right|_{s=0} = \left.\frac{d(f_t \circ g_s)}{ds}\right|_{s=0} = X_H(f_t) \\ &= \sum_{i=1}^{n}\left(\frac{\partial H}{\partial p_i}\frac{\partial f_t}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial f_t}{\partial p_i}\right) = \frac{\partial H}{\partial \boldsymbol{p}}\frac{\partial f_t}{\partial \boldsymbol{q}} - \frac{\partial H}{\partial \boldsymbol{q}}\frac{\partial f_t}{\partial \boldsymbol{p}},\end{aligned}$$

called Hamilton's equation for classical observables. Setting

$$\{f,g\} = X_f(g) = \frac{\partial f}{\partial \boldsymbol{p}}\frac{\partial g}{\partial \boldsymbol{q}} - \frac{\partial f}{\partial \boldsymbol{q}}\frac{\partial g}{\partial \boldsymbol{p}}, \quad f,g \in C^\infty(T^*M), \tag{2.6}$$

we can rewrite Hamilton's equation in the concise form

$$\frac{df}{dt} = \{H, f\}, \tag{2.7}$$

where it is understood that (2.7) is a differential equation for a family of functions f_t on T^*M with the initial condition $f_t(p,q)|_{t=0} = f(p,q)$. The properties of the bilinear mapping

$$\{\ ,\ \} : C^\infty(T^*M) \times C^\infty(T^*M) \to C^\infty(T^*M)$$

are summarized below.

Theorem 2.9. *The mapping $\{\ ,\ \}$ satisfies the following properties.*

(i) *(Relation with the symplectic form)*

$$\{f,g\} = \omega(J(df), J(dg)) = \omega(X_f, X_g).$$

(ii) *(Skew-symmetry)*

$$\{f,g\} = -\{g,f\}.$$

(iii) *(Leibniz rule)*

$$\{fg,h\} = f\{g,h\} + g\{f,h\}.$$

(iv) *(Jacobi identity)*

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$$

for all $f,g,h \in C^\infty(T^*M)$.

Proof. Property (i) immediately follows from the definitions of ω and J in Section 2.1. Properties (ii)-(iii) are obvious. The Jacobi identity could be verified by a direct computation using (2.6), or by the following elegant argument. Observe that $\{f,g\}$ is a bilinear form in the first partial derivatives of f and g, and every term in the left-hand side of the Jacobi identity is a linear homogenous function of second partial derivatives of f, g, and h. Now the only terms in the Jacobi identity which could actually contain second partial derivatives of a function h are the following:

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} = (X_f X_g - X_g X_f)(h).$$

However, this expression does not contain second partial derivatives of h since it is a commutator of two differential operators of the first order which is again a differential operator of the first order! □

The observable $\{f,g\}$ is called the *canonical Poisson bracket* of the observables f and g. The Poisson bracket map $\{\ ,\ \} : C^\infty(T^*M)\times C^\infty(T^*M) \to C^\infty(T^*M)$ turns the algebra of classical observables $C^\infty(T^*M)$ into a Lie algebra with a Lie bracket given by the Poisson bracket. It has an important property that the Lie bracket is a bi-derivation with respect to the multiplication in $C^\infty(T^*M)$. The algebra of classical observables $C^\infty(T^*M)$ is an example of the *Poisson algebra* — a commutative algebra over $\mathbb{R}$ carrying a structure of a Lie algebra with the property that the Lie bracket is a derivation with respect to the algebra product.

In Lagrangian mechanics, a function I on TM is an integral of motion for the Lagrangian system (M, L) if it is constant along the trajectories. In Hamiltonian mechanics, an observable I — a function on the phase space T^*M — is called an integral of motion (first integral) for Hamilton's equations (2.1) if it is constant along the Hamiltonian phase flow. According to (2.7), this is equivalent to the condition

$$\{H, I\} = 0.$$

It is said that the observables H and I are *in involution* (*Poisson commute*).

2.5. Canonical transformations and generating functions.

Definition. A diffeomorphism g of the phase space T^*M is called a *canonical transformation*, if it preserves the canonical symplectic form ω on T^*M, i.e., $g^*(\omega) = \omega$. By Theorem 2.1, the Hamiltonian phase flow g_t is a one-parameter group of canonical transformations.

Proposition 2.3. *Canonical transformations preserve Hamilton's equations.*

Proof. From $g^*(\omega) = \omega$ it follows that the mapping $J : T^*(T^*M) \to T(T^*M)$ satisfies

$$g_* \circ J \circ g^* = J. \tag{2.8}$$

Indeed, for all $X, Y \in \mathrm{Vect}(M)$ we have[16]

$$\omega(X,Y) = g^*(\omega)(X,Y) = \omega(g_*(X), g_*(Y)) \circ g,$$

so that for every 1-form ϑ on M,

$$\omega(X, J(g^*(\vartheta))) = g^*(\vartheta)(X) = \vartheta(g_*(X)) \circ g = \omega(g_*(X), J(\vartheta)) \circ g,$$

which gives $g_*(J(g^*(\vartheta))) = J(\vartheta)$. Using (2.8), we get

$$g_*(X_H) = g_*(J(dH)) = J((g^*)^{-1}(dH)) = X_K,$$

where $K = H \circ g^{-1}$. Thus the canonical transformation g maps trajectories of the Hamiltonian vector field X_H into the trajectories of the Hamiltonian vector field X_K. $\square$

Remark. In classical terms, Proposition 2.3 means that canonical Hamilton's equations

$$\dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}}(\boldsymbol{p}, \boldsymbol{q}), \quad \dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}(\boldsymbol{p}, \boldsymbol{q})$$

in new coordinates $(\boldsymbol{P}, \boldsymbol{Q}) = g(\boldsymbol{p}, \boldsymbol{q})$ continue to have the canonical form

$$\dot{\boldsymbol{P}} = -\frac{\partial K}{\partial \boldsymbol{Q}}(\boldsymbol{P}, \boldsymbol{Q}), \quad \dot{\boldsymbol{Q}} = \frac{\partial K}{\partial \boldsymbol{P}}(\boldsymbol{P}, \boldsymbol{Q})$$

with the old Hamiltonian function $K(\boldsymbol{P}, \boldsymbol{Q}) = H(\boldsymbol{p}, \boldsymbol{q})$.

Consider now the classical case $M = \mathbb{R}^n$. For a canonical transformation $(\boldsymbol{P}, \boldsymbol{Q}) = g(\boldsymbol{p}, \boldsymbol{q})$ set $\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{p}, \boldsymbol{q})$ and $\boldsymbol{Q} = \boldsymbol{Q}(\boldsymbol{p}, \boldsymbol{q})$. Since $d\boldsymbol{P} \wedge d\boldsymbol{Q} = d\boldsymbol{p} \wedge d\boldsymbol{q}$ on $T^*M \simeq \mathbb{R}^{2n}$, the 1-form $\boldsymbol{p}d\boldsymbol{q} - \boldsymbol{P}d\boldsymbol{Q}$ — the difference between the canonical Liouville 1-form and its pullback by the mapping g — is closed. From the Poincaré lemma it follows that there exists a function $F(\boldsymbol{p}, \boldsymbol{q})$ on $\mathbb{R}^{2n}$ such that

$$\boldsymbol{p}d\boldsymbol{q} - \boldsymbol{P}d\boldsymbol{Q} = dF(\boldsymbol{p}, \boldsymbol{q}). \tag{2.9}$$

Now assume that at some point $(\boldsymbol{p}_0, \boldsymbol{q}_0)$ the $n \times n$ matrix $\dfrac{\partial \boldsymbol{P}}{\partial \boldsymbol{p}} = \left\{\dfrac{\partial P_i}{\partial p_j}\right\}_{i,j=1}^n$ is non-degenerate. By the inverse function theorem, there exists a neighborhood U of $(\boldsymbol{p}_0, \boldsymbol{q}_0)$ in $\mathbb{R}^{2n}$ for which the functions $\boldsymbol{P}, \boldsymbol{q}$ are coordinate functions. The function

$$S(\boldsymbol{P}, \boldsymbol{q}) = F(\boldsymbol{p}, \boldsymbol{q}) + \boldsymbol{P}\boldsymbol{Q}$$

[16]Since g is a diffeomorphism, g_*X is a well-defined vector field on M.

is called a *generating function* of the canonical transformation g in U. It follows from (2.9) that in new coordinates $\boldsymbol{P}, \boldsymbol{q}$ on U,

$$\boldsymbol{p} = \frac{\partial S}{\partial \boldsymbol{q}}(\boldsymbol{P}, \boldsymbol{q}) \quad \text{and} \quad \boldsymbol{Q} = \frac{\partial S}{\partial \boldsymbol{P}}(\boldsymbol{P}, \boldsymbol{q}).$$

The converse statement below easily follows from the implicit function theorem.

Proposition 2.4. *Let $S(\boldsymbol{P}, \boldsymbol{q})$ be a function in some neighborhood U of a point $(\boldsymbol{P}_0, \boldsymbol{q}_0) \in \mathbb{R}^{2n}$ such that the $n \times n$ matrix*

$$\frac{\partial^2 S}{\partial \boldsymbol{P} \partial \boldsymbol{q}}(\boldsymbol{P}_0, \boldsymbol{q}_0) = \left\{ \frac{\partial^2 S}{\partial P_i \partial q^j}(\boldsymbol{P}_0, \boldsymbol{q}_0) \right\}_{i,j=1}^{n}$$

is non-degenerate. Then S is a generating function of a local (i.e., defined in some neighborhood of $(\boldsymbol{P}_0, \boldsymbol{q}_0)$ in $\mathbb{R}^{2n}$) canonical transformation.

Suppose there is a canonical transformation $(\boldsymbol{P}, \boldsymbol{Q}) = g(\boldsymbol{p}, \boldsymbol{q})$ such that $H(\boldsymbol{p}, \boldsymbol{q}) = K(\boldsymbol{P})$ for some function K. Then in the new coordinates Hamilton's equations take the form

$$\dot{\boldsymbol{P}} = 0, \quad \dot{\boldsymbol{Q}} = \frac{\partial K}{\partial \boldsymbol{P}}, \tag{2.10}$$

and are trivially integrated:

$$\boldsymbol{P}(t) = \boldsymbol{P}(0), \quad \boldsymbol{Q}(t) = \boldsymbol{Q}(0) + t\frac{\partial K}{\partial \boldsymbol{P}}(\boldsymbol{P}(0)).$$

Assuming that the matrix $\dfrac{\partial \boldsymbol{P}}{\partial \boldsymbol{p}}$ is non-degenerate, the generating function $S(\boldsymbol{P}, \boldsymbol{q})$ satisfies the differential equation

$$H\Big(\frac{\partial S}{\partial \boldsymbol{q}}(\boldsymbol{P}, \boldsymbol{q}), \boldsymbol{q}\Big) = K(\boldsymbol{P}), \tag{2.11}$$

where after the differentiation one should substitute $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{P}, \boldsymbol{Q})$, defined by the canonical transformation g^{-1}. The differential equation (2.11) for fixed $\boldsymbol{P}$, as it follows from (2.3), coincides with the Hamilton-Jacobi equation for the abbreviated action $S_0 = S - Et$ where $E = K(\boldsymbol{P})$,

$$H\Big(\frac{\partial S_0}{\partial \boldsymbol{q}}(\boldsymbol{P}, \boldsymbol{q}), \boldsymbol{q}\Big) = E.$$

Theorem 2.10 (Jacobi)**.** *Suppose that there is a function $S(\boldsymbol{P}, \boldsymbol{q})$ which depends on n parameters $\boldsymbol{P} = (P_1, \dots, P_n)$, satisfies the Hamilton-Jacobi equation* (2.11) *for some function $K(\boldsymbol{P})$, and has the property that the $n \times n$ matrix $\dfrac{\partial^2 S}{\partial \boldsymbol{P} \partial \boldsymbol{q}}$ is non-degenerate. Then Hamilton's equations*

$$\dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}}, \quad \dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}$$

can be solved explicitly, and the functions $\boldsymbol{P}(\boldsymbol{p},\boldsymbol{q}) = (P_1(\boldsymbol{p},\boldsymbol{q}),\dots,P_n(\boldsymbol{p},\boldsymbol{q}))$, *defined by the equations* $\boldsymbol{p} = \dfrac{\partial S}{\partial \boldsymbol{q}}(\boldsymbol{P},\boldsymbol{q})$, *are integrals of motion in involution.*

Proof. Set $\boldsymbol{p} = \dfrac{\partial S}{\partial \boldsymbol{q}}(\boldsymbol{P},\boldsymbol{q})$ and $\boldsymbol{Q} = \dfrac{\partial S}{\partial \boldsymbol{P}}(\boldsymbol{P},\boldsymbol{q})$. By the inverse function theorem, $g(\boldsymbol{p},\boldsymbol{q}) = (\boldsymbol{P},\boldsymbol{Q})$ is a local canonical transformation with the generating function S. It follows from (2.11) that $H(\boldsymbol{p}(\boldsymbol{P},\boldsymbol{Q}),\boldsymbol{q}(\boldsymbol{P},\boldsymbol{Q})) = K(\boldsymbol{P})$, so that Hamilton's equations take the form (2.10). Since $\omega = d\boldsymbol{P}\wedge d\boldsymbol{Q}$, integrals of motion $P_1(\boldsymbol{p},\boldsymbol{q}),\dots,P_n(\boldsymbol{p},\boldsymbol{q})$ are in involution. $\square$

The solution of the Hamilton-Jacobi equation satisfying conditions in Theorem 2.10 is called the *complete integral*. At first glance it seems that solving the Hamilton-Jacobi equation, which is a nonlinear partial differential equation, is a more difficult problem then solving Hamilton's equations, which is a system of ordinary differential equations. It is quite remarkable that for many problems of classical mechanics one can find the complete integral of the Hamilton-Jacobi equation by the method of separation of variables. By Theorem 2.10, this solves the corresponding Hamilton's equations.

Problem 2.6. Find the generating function for the identity transformation $\boldsymbol{P} = \boldsymbol{p}, \boldsymbol{Q} = \boldsymbol{q}$.

Problem 2.7. Prove Proposition 2.4.

Problem 2.8. Suppose that the canonical transformation $g(\boldsymbol{p},\boldsymbol{q}) = (\boldsymbol{P},\boldsymbol{Q})$ is such that locally $(\boldsymbol{Q},\boldsymbol{q})$ can be considered as new coordinates (canonical transformations with this property are called *free*). Prove that $S_1(\boldsymbol{Q},\boldsymbol{q}) = F(\boldsymbol{p},\boldsymbol{q})$, also called a generating function, satisfies

$$\boldsymbol{p} = \frac{\partial S_1}{\partial \boldsymbol{q}} \quad \text{and} \quad \boldsymbol{P} = -\frac{\partial S_1}{\partial \boldsymbol{Q}}.$$

Problem 2.9. Find the complete integral for the case of a particle in $\mathbb{R}^3$ moving in a central field.

2.6. Symplectic manifolds. The notion of a symplectic manifold is a generalization of the example of a cotangent bundle T^*M.

Definition. A non-degenerate, closed 2-form ω on a manifold $\mathscr{M}$ is called a *symplectic form*, and the pair $(\mathscr{M},\omega)$ is called a *symplectic manifold*.

Since a symplectic form ω is non-degenerate, a symplectic manifold $\mathscr{M}$ is necessarily even-dimensional, $\dim\mathscr{M} = 2n$. The nowhere vanishing $2n$-form ω^n defines a canonical orientation on $\mathscr{M}$, and as in the case $\mathscr{M} = T^*M$, $\dfrac{\omega^n}{n!}$ is called Liouville's volume form. We also have the general notion of a Lagrangian submanifold.

Definition. A submanifold $\mathscr{L}$ of a symplectic manifold $(\mathscr{M}, \omega)$ is called a *Lagrangian submanifold*, if $\dim \mathscr{L} = \frac{1}{2}\dim \mathscr{M}$ and the restriction of the symplectic form ω to $\mathscr{L}$ is 0.

Symplectic manifolds form a category. A morphism between $(\mathscr{M}_1, \omega_1)$ and $(\mathscr{M}_2, \omega_2)$, also called a *symplectomorphism*, is a mapping $f : \mathscr{M}_1 \to \mathscr{M}_2$ such that $\omega_1 = f^*(\omega_2)$. When $\mathscr{M}_1 = \mathscr{M}_2$ and $\omega_1 = \omega_2$, the notion of a symplectomorphism generalizes the notion of a canonical transformation. The direct product of symplectic manifolds $(\mathscr{M}_1, \omega_1)$ and $(\mathscr{M}_2, \omega_2)$ is a symplectic manifold

$$(\mathscr{M}_1 \times \mathscr{M}_2, \pi_1^*(\omega_1) + \pi_2^*(\omega_2)),$$

where π_1 and π_2 are, respectively, projections of $\mathscr{M}_1 \times \mathscr{M}_2$ onto the first and second factors in the Cartesian product.

Besides cotangent bundles, another important class of symplectic manifolds is given by Kähler manifolds[17]. Recall that a complex manifold $\mathscr{M}$ is a Kähler manifold if it carries the Hermitian metric whose imaginary part is a closed $(1,1)$-form. In local complex coordinates $\boldsymbol{z} = (z^1, \dots, z^n)$ on $\mathscr{M}$ the Hermitian metric is written as

$$h = \sum_{\alpha,\beta=1}^{n} h_{\alpha\bar{\beta}}(\boldsymbol{z}, \bar{\boldsymbol{z}}) dz^\alpha \otimes d\bar{z}^\beta.$$

Correspondingly,

$$g = \operatorname{Re} h = \frac{1}{2} \sum_{\alpha,\beta=1}^{n} h_{\alpha\bar{\beta}}(\boldsymbol{z}, \bar{\boldsymbol{z}})(dz^\alpha \otimes d\bar{z}^\beta + d\bar{z}^\beta \otimes dz^\alpha)$$

is the Riemannian metric on $\mathscr{M}$ and

$$\omega = -\operatorname{Im} h = \frac{i}{2} \sum_{\alpha,\beta=1}^{n} h_{\alpha\bar{\beta}}(\boldsymbol{z}, \bar{\boldsymbol{z}}) dz^\alpha \wedge d\bar{z}^\beta$$

is the symplectic form on $\mathscr{M}$ (considered as a $2n$-dimensional real manifold).

The simplest compact Kähler manifold is $\mathbb{C}P^1 \simeq S^2$ with the symplectic form given by the area 2-form of the Hermitian metric of Gaussian curvature 1 — the round metric on the 2-sphere. In terms of the local coordinate z associated with the stereographic projection $\mathbb{C}P^1 \simeq \mathbb{C} \cup \{\infty\}$,

$$\omega = 2i \, \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}.$$

Similarly, the natural symplectic form on the complex projective space $\mathbb{C}P^n$ is the symplectic form of the Fubini-Study metric. By pull-back, it defines symplectic forms on complex projective varieties.

[17] Needless to say, not every symplectic manifold admits a complex structure, not to mention a Kähler structure.

The simplest non-compact Kähler manifold is the n-dimensional complex vector space $\mathbb{C}^n$ with the standard Hermitian metric. In complex coordinates $\boldsymbol{z} = (z^1, \dots, z^n)$ on $\mathbb{C}^n$ it is given by

$$h = d\boldsymbol{z} \otimes d\bar{\boldsymbol{z}} = \sum_{\alpha=1}^{n} dz^\alpha \otimes d\bar{z}^\alpha.$$

In terms of real coordinates $(\boldsymbol{x}, \boldsymbol{y}) = (x^1, \dots, x^n, y^1, \dots, y^n)$ on $\mathbb{R}^{2n} \simeq \mathbb{C}^n$, where $\boldsymbol{z} = \boldsymbol{x} + i\boldsymbol{y}$, the corresponding symplectic form $\omega = -\operatorname{Im} h$ has the canonical form

$$\omega = \frac{i}{2} d\boldsymbol{z} \wedge d\bar{\boldsymbol{z}} = \sum_{\alpha=1}^{n} dx^\alpha \wedge dy^\alpha = d\boldsymbol{x} \wedge d\boldsymbol{y}.$$

This example naturally leads to the following definition.

Definition. A *symplectic vector space* is a pair (V, ω), where V is a vector space over $\mathbb{R}$ and ω is a non-degenerate, skew-symmetric bilinear form on V.

It follows from basic linear algebra that every symplectic vector space V has a *symplectic basis* — a basis $e^1, \dots, e^n, f_1, \dots, f_n$ of V, where $2n = \dim V$, such that

$$\omega(e^i, e^j) = \omega(f_i, f_j) = 0 \quad \text{and} \quad \omega(e^i, f_j) = \delta^i_j, \;\; i, j = 1, \dots, n.$$

In coordinates $(\boldsymbol{p}, \boldsymbol{q}) = (p_1, \dots, p_n, q^1, \dots, q^n)$ corresponding to this basis, $V \simeq \mathbb{R}^{2n}$ and

$$\omega = d\boldsymbol{p} \wedge d\boldsymbol{q} = \sum_{i=1}^{n} dp_i \wedge dq^i.$$

Thus every symplectic vector space is isomorphic to a direct product of the phase planes $\mathbb{R}^2$ with the canonical symplectic form $dp \wedge dq$. Introducing complex coordinates $\boldsymbol{z} = \boldsymbol{p} + i\boldsymbol{q}$, we get the isomorphism $V \simeq \mathbb{C}^n$, so that every symplectic vector space admits a Kähler structure.

It is a basic fact of symplectic geometry that every symplectic manifold is locally isomorphic to a symplectic vector space.

Theorem 2.11 (Darboux' theorem). *Let $(\mathscr{M}, \omega)$ be a $2n$-dimensional symplectic manifold. For every point $x \in \mathscr{M}$ there is a neighborhood U of x with local coordinates $(\boldsymbol{p}, \boldsymbol{q}) = (p_1, \dots, p_n, q^1, \dots, q^n)$ such that on U*

$$\omega = d\boldsymbol{p} \wedge d\boldsymbol{q} = \sum_{i=1}^{n} dp_i \wedge dq^i.$$

Coordinates $\boldsymbol{p}, \boldsymbol{q}$ are called *canonical coordinates* (*Darboux coordinates*). The proof proceeds by induction on n with the two main steps stated as Problems 2.13 and 2.14.

A non-degenerate 2-form ω for every $x \in \mathscr{M}$ defines an isomorphism $J : T_x^*\mathscr{M} \to T_x\mathscr{M}$ by

$$\omega(u_1, u_2) = J^{-1}(u_2)(u_1),\ u_1, u_2 \in T_x\mathscr{M}.$$

Explicitly, for every $X \in \mathrm{Vect}(\mathscr{M})$ and $\vartheta \in \mathcal{A}^1(\mathscr{M})$ we have

$$\omega(X, J(\vartheta)) = \vartheta(X) \quad \text{and} \quad J^{-1}(X) = -i_X(\omega)$$

(cf. Section 2.1). In local coordinates $\boldsymbol{x} = (x^1, \dots, x^{2n})$ for the coordinate chart (U, φ) on $\mathscr{M}$, the 2-form ω is given by

$$\omega = \tfrac{1}{2} \sum_{i,j=1}^{2n} \omega_{ij}(\boldsymbol{x})\, dx^i \wedge dx^j,$$

where $\{\omega_{ij}(\boldsymbol{x})\}_{i,j=1}^{2n}$ is a non-degenerate, skew-symmetric matrix-valued function on $\varphi(U)$. Denoting the inverse matrix by $\{\omega^{ij}(\boldsymbol{x})\}_{i,j=1}^{2n}$, we have

$$J(dx^i) = -\sum_{j=1}^{2n} \omega^{ij}(\boldsymbol{x}) \frac{\partial}{\partial x^j}, \quad i = 1, \dots, 2n.$$

Definition. A *Hamiltonian system* is a pair consisting of a symplectic manifold $(\mathscr{M}, \omega)$, called a *phase space*, and a smooth real-valued function H on $\mathscr{M}$, called a *Hamiltonian.* The motion of points on the phase space is described by the vector field

$$X_H = J(dH),$$

called a *Hamiltonian vector field.*

The trajectories of a Hamiltonian system $((\mathscr{M}, \omega), H)$ are the integral curves of a Hamiltonian vector field X_H on $\mathscr{M}$. In canonical coordinates $(\boldsymbol{p}, \boldsymbol{q})$ they are described by the canonical Hamilton's equations (2.1),

$$\dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}}, \quad \dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}.$$

Suppose now that the Hamiltonian vector field X_H on $\mathscr{M}$ is complete. The *Hamiltonian phase flow* on $\mathscr{M}$ associated with a Hamiltonian H is a one-parameter group $\{g_t\}_{t\in\mathbb{R}}$ of diffeomorphisms of $\mathscr{M}$ generated by X_H. The following statement generalizes Theorem 2.1.

Theorem 2.12. *The Hamiltonian phase flow preserves the symplectic form.*

Proof. It is sufficient to show that $\mathcal{L}_{X_H}\omega = 0$. Using Cartan's formula

$$\mathcal{L}_X = i_X \circ d + d \circ i_X$$

and $d\omega = 0$, we get for every $X \in \mathrm{Vect}(\mathscr{M})$,

$$\mathcal{L}_X\omega = (d \circ i_X)(\omega).$$

Since $i_X(\omega)(Y) = \omega(X,Y)$, we have for $X = X_H$ and every $Y \in \mathrm{Vect}(\mathscr{M})$ that

$$i_{X_H}(\omega)(Y) = \omega(J(dH), Y) = -dH(Y).$$

Thus $i_{X_H}(\omega) = -dH$, and the statement follows from $d^2 = 0$. □

Corollary 2.13. *A vector field X on $\mathscr{M}$ is a Hamiltonian vector field if and only if the* 1*-form $i_X(\omega)$ is exact.*

Definition. A vector field X on a symplectic manifold $(\mathscr{M}, \omega)$ is called a *symplectic vector field* if the 1-form $i_X(\omega)$ is closed, which is equivalent to $\mathscr{L}_X\omega = 0$.

The commutative algebra $C^\infty(\mathscr{M})$, with a multiplication given by the point-wise product of functions, is called the *algebra of classical observables.* Assuming that the Hamiltonian phase flow g_t exists for all times, the time evolution of every observable $f \in C^\infty(\mathscr{M})$ is given by

$$f_t(x) = f(g_t(x)), \quad x \in \mathscr{M},$$

and is described by the differential equation

$$\frac{df_t}{dt} = X_H(f_t)$$

— Hamilton's equation for classical observables. Hamilton's equations for observables on $\mathscr{M}$ have the same form as Hamilton's equations on $\mathscr{M} = T^*M$, considered in Section 2.3. Since

$$X_H(f) = df(X_H) = \omega(X_H, J(df)) = \omega(X_H, X_f),$$

we have the following.

Definition. A Poisson bracket on the algebra $C^\infty(\mathscr{M})$ of classical observables on a symplectic manifold $(\mathscr{M}, \omega)$ is a bilinear mapping $\{\ ,\ \} : C^\infty(\mathscr{M}) \times C^\infty(\mathscr{M}) \to C^\infty(\mathscr{M})$, defined by

$$\{f, g\} = \omega(X_f, X_g), \quad f, g \in C^\infty(\mathscr{M}).$$

Now Hamilton's equation takes the concise form

$$\frac{df}{dt} = \{H, f\}, \tag{2.12}$$

understood as a differential equation for a family of functions f_t on $\mathscr{M}$ with the initial condition $f_t|_{t=0} = f$. In local coordinates $\boldsymbol{x} = (x^1, \dots, x^{2n})$ on $\mathscr{M}$,

$$\{f, g\}(\boldsymbol{x}) = -\sum_{i,j=1}^{2n} \omega^{ij}(\boldsymbol{x}) \frac{\partial f(\boldsymbol{x})}{\partial x^i} \frac{\partial g(\boldsymbol{x})}{\partial x^j}.$$

Theorem 2.14. *The Poisson bracket $\{\ ,\ \}$ on a symplectic manifold $(\mathscr{M}, \omega)$ is skew-symmetric and satisfies Leibniz rule and the Jacobi identity.*

Proof. The first two properties are obvious. It follows from the definition of a Poisson bracket and the formula

$$[X_f, X_g](h) = (X_g X_f - X_f X_g)(h) = \{g, \{f, h\}\} - \{f, \{g, h\}\}$$

that the Jacobi identity is equivalent to the property

$$[X_f, X_g] = X_{\{f,g\}}. \tag{2.13}$$

Let X and Y be symplectic vector fields. Using Cartan's formulas we get

$$\begin{aligned} i_{[X,Y]}(\omega) &= \mathcal{L}_X(i_Y(\omega)) - i_Y(\mathcal{L}_X(\omega)) \\ &= d(i_X \circ i_Y(\omega)) + i_X d(i_Y(\omega)) \\ &= d(\omega(Y, X)) = i_Z(\omega), \end{aligned}$$

where Z is a Hamiltonian vector field corresponding to $\omega(X, Y) \in C^\infty(\mathscr{M})$. Since the 2-form ω is non-degenerate, this implies $[X, Y] = Z$, so that setting $X = X_f, Y = X_g$ and using $\{f, g\} = \omega(X_f, X_g)$, we get (2.13). □

From (2.13) we immediately get the following result.

Corollary 2.15. *The subspace* $\mathrm{Ham}(\mathscr{M})$ *of Hamiltonian vector fields on* $\mathscr{M}$ *is a Lie subalgebra of* $\mathrm{Vect}(\mathscr{M})$. *The mapping* $C^\infty(\mathscr{M}) \to \mathrm{Ham}(\mathscr{M})$, *given by* $f \mapsto X_f$, *is a Lie algebra homomorphism with the kernel consisting of locally constant functions on* $\mathscr{M}$.

As in the case $\mathscr{M} = T^*M$ (see Section 2.4), an observable I — a function on the phase space $\mathscr{M}$ — is called an integral of motion (first integral) for the Hamiltonian system $((\mathscr{M}, \omega), H)$ if it is constant along the Hamiltonian phase flow. According to (2.12), this is equivalent to the condition

$$\{H, I\} = 0. \tag{2.14}$$

It is said that the observables H and I are *in involution* (*Poisson commute*). From the Jacobi identity for the Poisson bracket we get the following result.

Corollary 2.16 (Poisson's theorem)**.** *The Poisson bracket of two integrals of motion is an integral of motion.*

Proof. If $\{H, I_1\} = \{H, I_2\} = 0$, then

$$\{H, \{I_1, I_2\}\} = \{\{H, I_1\}, I_2\} - \{\{H, I_2\}, I_1\} = 0. \qquad \square$$

It follows from Poisson's theorem that integrals of motion form a Lie algebra and, by (2.13), corresponding Hamiltonian vector fields form a Lie subalgebra in $\mathrm{Vect}(\mathscr{M})$. Since $\{I, H\} = dH(X_I) = 0$, the vector fields X_I are tangent to submanifolds $\mathscr{M}_E = \{x \in \mathscr{M} : H(x) = E\}$ — the level sets of the Hamiltonian H. This defines a Lie algebra of integrals of motion for the Hamiltonian system $((\mathscr{M}, \omega), H)$ at the level set $\mathscr{M}_E$.

Let G be a finite-dimensional Lie group that acts on a connected symplectic manifold $(\mathscr{M},\omega)$ by symplectomorphisms. The Lie algebra $\mathfrak{g}$ of G acts on $\mathscr{M}$ by vector fields

$$X_\xi(f)(x) = \left.\frac{d}{ds}\right|_{s=0} f(e^{-s\xi}\cdot x),$$

and the linear mapping $\mathfrak{g} \ni \xi \mapsto X_\xi \in \mathrm{Vect}(\mathscr{M})$ is a homomorphism of Lie algebras,

$$[X_\xi, X_\eta] = X_{[\xi,\eta]}, \quad \xi,\eta \in \mathfrak{g}.$$

The G-action is called a *Hamiltonian action* if X_ξ are Hamiltonian vector fields, i.e., for every $\xi \in \mathfrak{g}$ there is $\Phi_\xi \in C^\infty(\mathscr{M})$, defined up to an additive constant, such that $X_\xi = X_{\Phi_\xi} = J(d\Phi_\xi)$. It is called a *Poisson action* if there is a choice of functions Φ_ξ such that the linear mapping $\Phi : \mathfrak{g} \to C^\infty(\mathscr{M})$ is a homomorphism of Lie algebras,

$$\{\Phi_\xi, \Phi_\eta\} = \Phi_{[\xi,\eta]}, \quad \xi,\eta \in \mathfrak{g}. \tag{2.15}$$

Definition. A Lie group G is a *symmetry group* of the Hamiltonian system $((\mathscr{M},\omega),H)$ if there is a Hamiltonian action of G on $\mathscr{M}$ such that

$$H(g\cdot x) = H(x), \quad g \in G,\ x \in \mathscr{M}.$$

Theorem 2.17 (Noether theorem with symmetries)**.** *If G is a symmetry group of the Hamiltonian system $((\mathscr{M},\omega),H)$, then the functions Φ_ξ, $\xi \in \mathfrak{g}$, are the integrals of motion. If the action of G is Poisson, the integrals of motion satisfy* (2.15).

Proof. By definition of the Hamiltonian action, for every $\xi \in \mathfrak{g}$,

$$0 = X_\xi(H) = X_{\Phi_\xi}(H) = \{\Phi_\xi, H\}. \qquad \square$$

Corollary 2.18. *Let (M,L) be a Lagrangian system such that the Legendre transform $\tau_L : TM \to T^*M$ is a diffeomorphism. Then if a Lie group G is a symmetry of (M,L), then G is a symmetry group of the corresponding Hamiltonian system $((T^*M,\omega), H = E_L \circ \tau_L^{-1})$, and the corresponding G-action on T^*M is Poisson. In particular, $\Phi_\xi = -I_\xi \circ \tau_L^{-1}$, where I_ξ are Noether integrals of motion for the one-parameter subgroups of G generated by $\xi \in \mathfrak{g}$.*

Proof. Let X be the vector field associated with the one-parameter subgroup $\{e^{s\xi}\}_{s\in\mathbb{R}}$ of diffeomorphisms of M, used in Theorem 1.3, and let X' be its lift to TM. We have[18]

$$X_\xi = -(\tau_L)_*(X'), \tag{2.16}$$

[18] The negative sign reflects the difference in definitions of X and X_ξ.

and it follows from (1.16) that $\Phi_\xi = i_{X_\xi}(\theta) = \theta(X_\xi)$, where θ is the canonical Liouville 1-form on T^*M. From Cartan's formula and (1.17) we get

$$d\Phi_\xi = d(i_{X_\xi}(\theta)) = -i_{X_\xi}(d\theta) + \mathcal{L}_{X_\xi}(\theta) = -i_{X_\xi}(\omega),$$

so that

$$J(d\Phi_\xi) = -J(i_{X_\xi}(\omega)) = X_\xi,$$

and the G-action is Hamiltonian. Using (1.17) and another Cartan's formula, we obtain

$$\begin{aligned}\Phi_{[\xi,\eta]} &= i_{[X_\xi,X_\eta]}(\theta) = \mathcal{L}_{X_\xi}(i_{X_\eta}(\theta)) + i_{X_\eta}(\mathcal{L}_{X_\xi}(\theta)) \\ &= X_\xi(\Phi_\eta) = \{\Phi_\xi, \Phi_\eta\}.\end{aligned}$$ □

Example 2.1. The Lagrangian

$$L = \tfrac{1}{2}m\dot{\boldsymbol{r}}^2 - V(r)$$

for a particle in $\mathbb{R}^3$ moving in a central field (see Section 1.6) is invariant with respect to the action of the group SO(3) of orthogonal transformations of the Euclidean space $\mathbb{R}^3$. Let u_1, u_2, u_3 be a basis for the Lie algebra $\mathfrak{so}(3)$ corresponding to the rotations with the axes given by the vectors of the standard basis e_1, e_2, e_3 for $\mathbb{R}^3$ (see Example 1.10 in Section 1.4). These generators satisfy the commutation relations

$$[u_i, u_j] = \varepsilon_{ijk}u_k,$$

where $i, j, k = 1, 2, 3$, and ε_{ijk} is a totally anti-symmetric tensor, $\varepsilon_{123} = 1$. Corresponding Noether integrals of motion are given by $\Phi_{u_i} = -M_i$, where

$$\begin{aligned}M_1 &= (\boldsymbol{r}\times\boldsymbol{p})_1 = r_2p_3 - r_3p_2, \\ M_2 &= (\boldsymbol{r}\times\boldsymbol{p})_2 = r_3p_1 - r_1p_3, \\ M_3 &= (\boldsymbol{r}\times\boldsymbol{p})_3 = r_1p_2 - r_2p_1\end{aligned}$$

are components of the angular momentum vector $\boldsymbol{M} = \boldsymbol{r}\times\boldsymbol{p}$. (Here it is convenient to lower the indices of the coordinates r_i by the Euclidean metric on $\mathbb{R}^3$.) For the Hamiltonian

$$H = \frac{\boldsymbol{p}^2}{2m} + V(r)$$

we have

$$\{H, M_i\} = 0.$$

According to Theorem 2.17 and Corollary 2.18, Poisson brackets of the components of the angular momentum satisfy

$$\{M_i, M_j\} = -\varepsilon_{ijk}M_k,$$

which is also easy to verify directly using (2.6),

$$\{f, g\}(\boldsymbol{p}, \boldsymbol{r}) = \frac{\partial f}{\partial \boldsymbol{p}}\frac{\partial g}{\partial \boldsymbol{r}} - \frac{\partial f}{\partial \boldsymbol{r}}\frac{\partial g}{\partial \boldsymbol{p}}.$$

Example 2.2 (Kepler's problem)**.** For every $\alpha \in \mathbb{R}$ the Lagrangian system on $\mathbb{R}^3$ with

$$L = \tfrac{1}{2}m\dot{\boldsymbol{r}}^2 + \frac{\alpha}{r}$$

has three extra integrals of motion — the components W_1, W_2, W_3 of the Laplace-Runge-Lenz vector, given by

$$\boldsymbol{W} = \frac{\boldsymbol{p}}{m} \times \boldsymbol{M} - \frac{\alpha\boldsymbol{r}}{r}$$

(see Section 1.6). Using Poisson brackets from the previous example, together with $\{r_i, M_j\} = -\varepsilon_{ijk}r_k$ and $\{p_i, M_j\} = -\varepsilon_{ijk}p_k$, we get by a straightforward computation,

$$\{W_i, M_j\} = -\varepsilon_{ijk}W_k \quad \text{and} \quad \{W_i, W_j\} = \frac{2H}{m}\varepsilon_{ijk}M_k,$$

where $H = \dfrac{\boldsymbol{p}^2}{2m} - \dfrac{\alpha}{r}$ is the Hamiltonian of Kepler's problem.

The Hamiltonian system $((\mathscr{M}, \omega), H)$, $\dim \mathscr{M} = 2n$, is called *completely integrable* if it has n independent integrals of motion $F_1 = H, \dots, F_n$ in involution. The former condition means that $dF_1(x), \dots, dF_n(x) \in T_x^*\mathscr{M}$ are linearly independent for almost all $x \in \mathscr{M}$. Hamiltonian systems with one degree of freedom such that dH has only finitely many zeros are completely integrable. Complete separation of variables in the Hamilton-Jacobi equation (see Section 2.5) provides other examples of completely integrable Hamiltonian systems.

Let $((\mathscr{M}, \omega), H)$ be a completely integrable Hamiltonian system. Suppose that the level set $\mathscr{M}_f = \{x \in \mathscr{M} : F_1(x) = f_1, \dots, F_n(x) = f_n\}$ is compact and tangent vectors $JdF_1, \dots, JdF_n$ are linearly independent for all $x \in \mathscr{M}_f$. Then by the Liouville-Arnold theorem, in a neighborhood of $\mathscr{M}_f$ there exist so-called *action-angle variables*: coordinates $\boldsymbol{I} = (I_1, \dots, I_n) \in \mathbb{R}_+^n = (\mathbb{R}_{>0})^n$ and $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_n) \in T^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ such that $\omega = d\boldsymbol{I} \wedge d\boldsymbol{\varphi}$ and $H = H(I_1, \dots, I_n)$. According to Hamilton's equations,

$$\dot{I}_i = 0 \quad \text{and} \quad \dot{\varphi}_i = \omega_i = \frac{\partial H}{\partial I_i}, \quad i = 1, \dots, n,$$

so that action variables are constants, and angle variables change uniformly, $\varphi_i(t) = \varphi_i(0) + \omega_i t$, $i = 1, \dots, n$. The classical motion is almost-periodic with the frequencies $\omega_1, \dots, \omega_n$.

Problem 2.10. Show that a symplectic manifold $(\mathscr{M}, \omega)$ admits an *almost complex structure*: a bundle map $\mathscr{J} : T\mathscr{M} \to T\mathscr{M}$ such that $\mathscr{J}^2 = -\mathrm{id}$.

Problem 2.11. Give an example of a symplectic manifold which admits a complex structure but not a Kähler structure.

Problem 2.12 (Coadjont orbits)**.** Let G be a finite-dimensional Lie group, let $\mathfrak{g}$ be its Lie algebra, and let $\mathfrak{g}^*$ be the dual vector space to $\mathfrak{g}$. For $u \in \mathfrak{g}^*$ let $\mathscr{M} = \mathcal{O}_u$ be the orbit of u under the coadjoint action of G on $\mathfrak{g}^*$. Show that the formula

$$\omega(u_1, u_2) = u([x_1, x_2]),$$

where $u_1 = \mathrm{ad}^* x_1(u), u_2 = \mathrm{ad}^* x_2(u) \in \mathcal{O}_u$, and ad^* stands for the coadjoint action of a Lie algebra $\mathfrak{g}$ on $\mathfrak{g}^*$, gives rise to a well-defined 2-form on $\mathscr{M}$, which is closed and non-degenerate. (The 2-form ω is called the *Kirillov-Kostant* symplectic form.)

Problem 2.13. Let $(\mathscr{M}, \omega)$ be a symplectic manifold. For $x \in \mathscr{M}$ choose a function q^1 on $\mathscr{M}$ such that $q^1(x) = 0$ and dq^1 does not vanish at x, and set $X = -X_{q^1}$. Show that there is a neighborhood U of $x \in \mathscr{M}$ and a function p_1 on U such that $X(q^1) = 1$ on U, and there exist coordinates $p_1, q^1, z^1, \dots, z^{2n-2}$ on U such that

$$X = \frac{\partial}{\partial p_1} \quad \text{and} \quad Y = X_{p_1} = \frac{\partial}{\partial q^1}.$$

Problem 2.14. Continuing Problem 2.13, show that the 2-form $\omega - dp_1 \wedge dq^1$ on U depends only on coordinates $z^1, \dots, z^{2n-2}$ and is non-degenerate.

Problem 2.15. Do the computation in Example 2.2 and show that the Lie algebra of the integrals $M_1, M_2, M_3, W_1, W_2, W_3$ in Kepler's problem at $H(\boldsymbol{p}, \boldsymbol{r}) = E$ is isomorphic to the Lie algebra $\mathfrak{so}(4)$, if $E < 0$, to the Euclidean Lie algebra $\mathfrak{e}(3)$, if $E = 0$, and to the Lie algebra $\mathfrak{so}(1, 3)$, if $E > 0$.

Problem 2.16. Find the action-angle variables for a particle with one degree of freedom, when the potential $V(x)$ is a convex function on $\mathbb{R}$ satisfying $\lim_{|x|\to\infty} V(x) = \infty$. (*Hint:* Define $I = \oint p dx$, where integration goes over the closed orbit with $H(p, x) = E$.)

Problem 2.17. Show that a Hamiltonian system describing a particle in $\mathbb{R}^3$ moving in a central field is completely integrable, and find the action-angle variables.

Problem 2.18 (Symplectic quotients)**.** For a Poisson action of a Lie group G on a symplectic manifold $(\mathscr{M}, \omega)$, define the *moment map* $P : \mathscr{M} \to \mathfrak{g}^*$ by

$$P(x)(\xi) = \Phi_\xi(x), \ \xi \in \mathfrak{g}, \ x \in \mathscr{M},$$

where $\mathfrak{g}$ is the Lie algebra of G. For every $p \in \mathfrak{g}^*$ such that a stabilizer G_p of p acts freely and properly on $\mathscr{M}_p = P^{-1}(p)$ (such p is called *the regular value* of the moment map), the quotient $M_p = G_p \backslash \mathscr{M}_p$ is called a *reduced phase space*. Show that M_p is a symplectic manifold with the symplectic form uniquely characterized by the condition that its pull-back to $\mathscr{M}_p$ coincides with the restriction to $\mathscr{M}_p$ of the symplectic form ω.

2.7. Poisson manifolds. The notion of a Poisson manifold generalizes the notion of a symplectic manifold.

Definition. A *Poisson manifold* is a manifold $\mathscr{M}$ equipped with a *Poisson structure* — a skew-symmetric bilinear mapping

$$\{\ ,\ \} : C^\infty(\mathscr{M}) \times C^\infty(\mathscr{M}) \to C^\infty(\mathscr{M})$$

which satisfies the Leibniz rule and Jacobi identity.

Equivalently, $\mathscr{M}$ is a Poisson manifold if the algebra $\mathcal{A} = C^\infty(\mathscr{M})$ of classical observables is a Poisson algebra — a Lie algebra such that the Lie bracket is a bi-derivation with respect to the multiplication in $\mathcal{A}$ (a point-wise product of functions). It follows from the derivation property that in local coordinates $\boldsymbol{x} = (x^1, \dots, x^N)$ on $\mathscr{M}$, the Poisson bracket has the form

$$\{f, g\}(\boldsymbol{x}) = \sum_{i,j=1}^{N} \eta^{ij}(\boldsymbol{x}) \frac{\partial f(\boldsymbol{x})}{\partial x^i} \frac{\partial g(\boldsymbol{x})}{\partial x^j}.$$

The 2-tensor $\eta^{ij}(\boldsymbol{x})$, called a *Poisson tensor*, defines a global section η of the vector bundle $T\mathscr{M} \wedge T\mathscr{M}$ over $\mathscr{M}$.

The evolution of classical observables on a Poisson manifold is given by Hamilton's equations, which have the same form as (2.12),

$$\frac{df}{dt} = X_H(f) = \{H, f\}.$$

The phase flow g_t for a complete Hamiltonian vector field $X_H = \{H, \cdot\}$ defines the *evolution operator* $U_t : \mathcal{A} \to \mathcal{A}$ by

$$U_t(f)(x) = f(g_t(x)), \;\; f \in \mathcal{A}.$$

Theorem 2.19. *Suppose that every Hamiltonian vector field on a Poisson manifold $(\mathscr{M}, \{\;,\;\})$ is complete. Then for every $H \in \mathcal{A}$, the corresponding evolution operator U_t is an automorphism of the Poisson algebra $\mathcal{A}$, i.e.,*

$$U_t(\{f, g\}) = \{U_t(f), U_t(g)\} \quad \textit{for all} \quad f, g \in \mathcal{A}. \tag{2.17}$$

Conversely, if a skew-symmetric bilinear mapping $\{\;,\;\} : C^\infty(\mathscr{M}) \times C^\infty(\mathscr{M}) \to C^\infty(\mathscr{M})$ is such that $X_H = \{H, \cdot\}$ are complete vector fields for all $H \in \mathcal{A}$, and corresponding evolution operators U_t satisfy (2.17)*, then $(\mathscr{M}, \{\;,\;\})$ is a Poisson manifold.*

Proof. Let $f_t = U_t(f)$, $g_t = U_t(g)$, and[19] $h_t = U_t(\{f, g\})$. By definition,

$$\frac{d}{dt}\{f_t, g_t\} = \{\{H, f_t\}, g_t\} + \{f_t, \{H, g_t\}\} \quad \text{and} \quad \frac{dh_t}{dt} = \{H, h_t\}.$$

If $(\mathscr{M}, \{\;,\;\})$ is a Poisson manifold, then it follows from the Jacobi identity that

$$\{\{H, f_t\}, g_t\} + \{f_t, \{H, g_t\}\} = \{H, \{f_t, g_t\}\},$$

so that h_t and $\{f_t, g_t\}$ satisfy the same differential equation (2.12). Since these functions coincide at $t = 0$, (2.17) follows from the uniqueness theorem for the ordinary differential equations.

Conversely, we get the Jacobi identity for the functions f, g, and H by differentiating (2.17) with respect to t at $t = 0$. □

[19] Here g_t is not the phase flow!

Corollary 2.20. *A global section η of $T\mathscr{M} \wedge T\mathscr{M}$ is a Poisson tensor if and only if*

$$\mathcal{L}_{X_f}\eta = 0 \quad \textit{for all} \quad f \in \mathcal{A}.$$

Definition. The *center* of a Poisson algebra $\mathcal{A}$ is

$$\mathcal{Z}(\mathcal{A}) = \{f \in \mathcal{A} : \{f, g\} = 0 \quad \text{for all} \quad g \in \mathcal{A}\}.$$

A Poisson manifold $(\mathscr{M}, \{\ ,\ \})$ is called *non-degenerate* if the center of a Poisson algebra of classical observables $\mathcal{A} = C^\infty(\mathscr{M})$ consists only of locally constant functions ($\mathcal{Z}(\mathcal{A}) = \mathbb{R}$ for connected $\mathscr{M}$).

Equivalently, a Poisson manifold $(\mathscr{M}, \{\ ,\ \})$ is non-degenerate if the Poisson tensor η is non-degenerate everywhere on $\mathscr{M}$, so that $\mathscr{M}$ is necessarily an even-dimensional manifold. A non-degenerate Poisson tensor for every $x \in \mathscr{M}$ defines an isomorphism $J : T_x^*\mathscr{M} \to T_x\mathscr{M}$ by

$$\eta(u_1, u_2) = u_2(J(u_1)),\ u_1, u_2 \in T_x^*\mathscr{M}.$$

In local coordinates $\boldsymbol{x} = (x^1, \dots, x^N)$ for the coordinate chart (U, φ) on $\mathscr{M}$, we have

$$J(dx^i) = \sum_{j=1}^{N} \eta^{ij}(\boldsymbol{x}) \frac{\partial}{\partial x^j}, \quad i = 1, \dots, N.$$

Poisson manifolds form a category. A morphism between $(\mathscr{M}_1, \{\ ,\ \}_1)$ and $(\mathscr{M}_2, \{\ ,\ \}_2)$ is a mapping $\varphi : \mathscr{M}_1 \to \mathscr{M}_2$ of smooth manifolds such that

$$\{f \circ \varphi, g \circ \varphi\}_1 = \{f, g\}_2 \circ \varphi \quad \text{for all} \quad f, g \in C^\infty(\mathscr{M}_2).$$

A direct product of Poisson manifolds $(\mathscr{M}_1, \{\ ,\ \}_1)$ and $(\mathscr{M}_2, \{\ ,\ \}_2)$ is a Poisson manifold $(\mathscr{M}_1 \times \mathscr{M}_2, \{\ ,\ \})$ defined by the property that natural projection maps $\pi_1 : \mathscr{M}_1 \times \mathscr{M}_2 \to \mathscr{M}_1$ and $\pi_2 : \mathscr{M}_1 \times \mathscr{M}_2 \to \mathscr{M}_2$ are Poisson mappings. For $f \in C^\infty(\mathscr{M}_1 \times \mathscr{M}_2)$ and $(x_1, x_2) \in \mathscr{M}_1 \times \mathscr{M}_2$ denote, respectively, by $f^{(1)}_{x_2}$ and $f^{(2)}_{x_1}$ restrictions of f to $\mathscr{M} \times \{x_2\}$ and $\{x_1\} \times \mathscr{M}_2$. Then for $f, g \in C^\infty(\mathscr{M}_1 \times \mathscr{M}_2)$,

$$\{f, g\}(x_1, x_2) = \{f^{(1)}_{x_2}, g^{(1)}_{x_2}\}_1(x_1) + \{f^{(2)}_{x_1}, g^{(2)}_{x_1}\}_2(x_2).$$

Non-degenerate Poisson manifolds form a subcategory of the category of Poisson manifolds.

Theorem 2.21. *The category of symplectic manifolds is (anti-) isomorphic to the category of non-degenerate Poisson manifolds.*

Proof. According to Theorem 2.14, every symplectic manifold carries a Poisson structure. Its non-degeneracy follows from the non-degeneracy of a symplectic form. Conversely, let $(\mathscr{M}, \{\ ,\ \})$ be a non-degenerate Poisson manifold. Define the 2-form ω on $\mathscr{M}$ by

$$\omega(X, Y) = J^{-1}(Y)(X), \quad X, Y \in \mathrm{Vect}(\mathscr{M}),$$

where the isomorphism $J : T^*\mathscr{M} \to T\mathscr{M}$ is defined by the Poisson tensor η. In local coordinates $\boldsymbol{x} = (x^1, \dots, x^N)$ on $\mathscr{M}$,

$$\omega = -\sum_{1\leq i<j\leq N} \eta_{ij}(\boldsymbol{x})\, dx^i \wedge dx^j,$$

where $\{\eta_{ij}(\boldsymbol{x})\}_{i,j=1}^N$ is the inverse matrix to $\{\eta^{ij}(\boldsymbol{x})\}_{i,j=1}^N$. The 2-form ω is skew-symmetric and non-degenerate. For every $f \in \mathcal{A}$ let $X_f = \{f, \cdot\}$ be the corresponding vector field on $\mathscr{M}$. The Jacobi identity for the Poisson bracket $\{\ ,\ \}$ is equivalent to $\mathcal{L}_{X_f}\eta = 0$ for every $f \in \mathcal{A}$, so that

$$\mathcal{L}_{X_f}\omega = 0.$$

Since $X_f = Jdf$, we have $\omega(X, Jdf) = df(X)$ for every $X \in \mathrm{Vect}(\mathscr{M})$, so that

$$\omega(X_f, X_g) = \{f, g\}.$$

By Cartan's formula,

$$\begin{aligned} d\omega(X,Y,Z) = \tfrac{1}{3}\,(&\mathcal{L}_X\omega(Y,Z) - \mathcal{L}_Y\omega(X,Z) + \mathcal{L}_Z\omega(X,Y) \\ &-\omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X))\,, \end{aligned}$$

where $X, Y, Z \in \mathrm{Vect}(\mathscr{M})$. Now setting $X = X_f, Y = X_g, Z = X_h$, we get

$$\begin{aligned} d\omega(X_f, X_g, X_h) &= \tfrac{1}{3}\left(\omega(X_h, [X_f, X_g]) + \omega(X_f, [X_g, X_h]) + \omega(X_g, [X_h, X_f])\right) \\ &= \tfrac{1}{3}\left(\omega(X_h, X_{\{f,g\}}) + \omega(X_f, X_{\{g,h\}}) + \omega(X_g, X_{\{h,f\}})\right) \\ &= \tfrac{1}{3}\left(\{h,\{f,g\}\} + \{f,\{g,h\}\} + \{g,\{h,f\}\}\right) \\ &= 0. \end{aligned}$$

The exact 1-forms df, $f \in \mathcal{A}$, generate the vector space of 1-forms $\mathcal{A}^1(\mathscr{M})$ as a module over $\mathcal{A}$, so that Hamiltonian vector fields $X_f = Jdf$ generate the vector space $\mathrm{Vect}(\mathscr{M})$ as a module over $\mathcal{A}$. Thus $d\omega = 0$ and $(\mathscr{M}, \omega)$ is a symplectic manifold associated with the Poisson manifold $(\mathscr{M}, \{\ ,\ \})$. It follows from the definitions that Poisson mappings of non-degenerate Poisson manifolds correspond to symplectomorphisms of associated symplectic manifolds. □

Remark. One can also prove this theorem by a straightforward computation in local coordinates $\boldsymbol{x} = (x^1, \dots, x^N)$ on $\mathscr{M}$. Just observe that the condition

$$\frac{\partial\eta_{ij}(\boldsymbol{x})}{\partial x^l} + \frac{\partial\eta_{jl}(\boldsymbol{x})}{\partial x^i} + \frac{\partial\eta_{li}(\boldsymbol{x})}{\partial x^j} = 0, \quad i, j, l = 1, \dots, N,$$

which is a coordinate form of $d\omega = 0$, follows from the condition

$$\sum_{j=1}^N \left(\eta^{ij}(\boldsymbol{x})\frac{\partial\eta^{kl}(\boldsymbol{x})}{\partial x^j} + \eta^{lj}(\boldsymbol{x})\frac{\partial\eta^{ik}(\boldsymbol{x})}{\partial x^j} + \eta^{kj}(\boldsymbol{x})\frac{\partial\eta^{li}(\boldsymbol{x})}{\partial x^j}\right) = 0,$$

which is a coordinate form of the Jacobi identity, by multiplying it three times by the inverse matrix $\eta_{ij}(\boldsymbol{x})$ using

$$\sum_{p=1}^{N}\left(\eta^{ip}(\boldsymbol{x})\frac{\partial\eta_{pk}(\boldsymbol{x})}{\partial x^j}+\frac{\partial\eta^{ip}(\boldsymbol{x})}{\partial x^j}\eta_{pk}(\boldsymbol{x})\right)=0.$$

Remark. Let $\mathscr{M}=T^*\mathbb{R}^n$ with the Poisson bracket $\{\ ,\ \}$ given by the canonical symplectic form $\omega=d\boldsymbol{p}\wedge d\boldsymbol{q}$, where $(\boldsymbol{p},\boldsymbol{q})=(p_1,\dots,p_n,q^1,\dots,q^n)$ are coordinate functions on $T^*\mathbb{R}^n$. The non-degeneracy of the Poisson manifold $(T^*\mathbb{R}^n,\{\ ,\ \})$ can be formulated as the property that the only observable $f\in C^\infty(T^*\mathbb{R}^n)$ satisfying

$$\{f,p_1\}=\cdots=\{f,p_n\}=0,\quad \{f,q^1\}=\cdots=\{f,q^n\}=0$$

is $f(\boldsymbol{p},\boldsymbol{q})=\text{const}$.

Problem 2.19 (Dual space to a Lie algebra)**.** Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with a Lie bracket $[\ ,\ ]$, and let $\mathfrak{g}^*$ be its dual space. For $f,g\in C^\infty(\mathfrak{g}^*)$ define

$$\{f,g\}(u)=u\left([df,dg]\right),$$

where $u\in\mathfrak{g}^*$ and $T_u^*\mathfrak{g}^*\simeq\mathfrak{g}$. Prove that $\{\ ,\ \}$ is a Poisson bracket. (It was introduced by Sophus Lie and is called a *linear*, or *Lie-Poisson* bracket.) Show that this bracket is degenerate and determine the center of $\mathcal{A}=C^\infty(\mathfrak{g}^*)$.

Problem 2.20. A Poisson bracket $\{\ ,\ \}$ on $\mathscr{M}$ restricts to a Poisson bracket $\{\ ,\ \}_0$ on a submanifold $\mathscr{N}$ if the inclusion $\imath:\mathscr{N}\to\mathscr{M}$ is a Poisson mapping. Show that the Lie-Poisson bracket on $\mathfrak{g}^*$ restricts to a non-degenerate Poisson bracket on a coadjoint orbit, associated with the Kirillov-Kostant symplectic form.

Problem 2.21 (Lie-Poisson groups)**.** A finite-dimensional Lie group is called a *Lie-Poisson group* if it has a structure of a Poisson manifold $(G,\{\ ,\ \})$ such that the group multiplication $G\times G\to G$ is a Poisson mapping, where $G\times G$ is a direct product of Poisson manifolds. Using a basis $\partial_1,\dots,\partial_n$ of left-invariant vector fields on G corresponding to a basis $x_1,\dots,x_n$ of the Lie algebra $\mathfrak{g}$, the Poisson bracket $\{\ ,\ \}$ can be written as

$$\{f_1,f_2\}(g)=\sum_{i,j=1}^{n}\eta^{ij}(g)\partial_i f_1\partial_j f_2,$$

where the 2-tensor $\eta^{ij}(g)$ defines a mapping $\eta:G\to\Lambda^2\mathfrak{g}$ by $\eta(g)=\sum_{i,j=1}^n\eta^{ij}(g)x_i\otimes x_j$. Show that the bracket $\{\ ,\ \}$ equips G with a Lie-Poisson structure if and only if the following conditions are satisfied: (i) for all $g\in G$,

$$\xi^{ijk}(g)=\sum_{l=1}^{n}\left(\eta^{il}(g)\partial_l\eta^{jk}(g)+\eta^{jl}(g)\partial_l\eta^{ki}(g)+\eta^{kl}(g)\partial_l\eta^{ij}(g)\right)$$
$$+\sum_{l,p=1}^{n}\left(c_{lp}^{i}\eta^{pj}(g)\eta^{kl}(g)+c_{lp}^{j}\eta^{pk}(g)\eta^{il}(g)+c_{lp}^{k}\eta^{pi}(g)\eta^{jl}(g)\right)=0,$$

where $[x_i,x_j]=\sum_{k=1}^n c_{ij}^k x_k$; (ii) the mapping η is a group 1-cocycle with the adjoint action on $\Lambda^2\mathfrak{g}$, i.e., $\eta(g_1g_2)=\mathrm{Ad}^{-1}g_2\cdot\eta(g_1)+\eta(g_2),\ g_1,g_2\in G$.

Problem 2.22. Show that the second condition in the previous problem trivially holds when η is a coboundary, $\eta(g) = -r + \mathrm{Ad}^{-1} g \cdot r$ for some $r = \sum_{i,j=1}^n r^{ij} x_i \otimes x_j \in \Lambda^2\mathfrak{g}$, and then the first condition is satisfied if and only if the element

$$\xi(r) = [r_{12}, r_{13} + r_{23}] + [r_{13}, r_{23}] \in \Lambda^3\mathfrak{g}$$

is invariant under the adjoint action of $\mathfrak{g}$ on $\Lambda^3\mathfrak{g}$. Here $r_{12} = \sum_{i,j=1}^n r^{ij} x_i \otimes x_j \otimes 1$, $r_{13} = \sum_{i,j=1}^n r^{ij} x_i \otimes 1 \otimes x_j$, and $r_{23} = \sum_{i,j=1}^n r^{ij} 1 \otimes x_i \otimes x_j$ are corresponding elements in the universal enveloping algebra $U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$. In particular, G is a Lie-Poisson group if $\xi(r) = 0$, which is called the *classical Yang-Baxter equation.*

Problem 2.23. Suppose that $r = \sum_{i,j=1}^n r^{ij} x_i \otimes x_j \in \Lambda^2\mathfrak{g}$ is such that the matrix $\{r^{ij}\}$ is non-degenerate, and let $\{r_{ij}\}$ be the inverse matrix. Show that r satisfies the classical Yang-Baxter equation if and only if the map $c : \Lambda^2\mathfrak{g} \to \mathbb{C}$, defined by $c(x, y) = \sum_{i,j=1}^n r_{ij} u^i v^j$, where $x = \sum_{i=1}^n u^i x_i$, $y = \sum_{i=1}^n v^i x_i$, is a non-degenerate Lie algebra 2-cocycle, i.e., it satisfies

$$c(x, [y, z]) + c(z, [x, y]) + c(y, [z, x]) = 0, \quad x, y, z \in \mathfrak{g}.$$

2.8. Hamilton's and Liouville's representations. To complete the formulation of classical mechanics, we need to describe the process of *measurement.* In physics, by a measurement of a classical system one understands the result of a physical experiment which gives numerical values for classical observables. The experiment consists of creating certain conditions for the system, and it is always assumed that these conditions can be repeated over and over. The conditions of the experiment define a *state* of the system if repeating these conditions results in probability distributions for the values of all observables of the system.

Mathematically, a state μ on the algebra $\mathcal{A} = C^\infty(\mathscr{M})$ of classical observables on the phase space $\mathscr{M}$ is the assignment

$$\mathcal{A} \ni f \mapsto \mu_f \in \mathscr{P}(\mathbb{R}),$$

where $\mathscr{P}(\mathbb{R})$ is a set of probability measures on $\mathbb{R}$ — Borel measures on $\mathbb{R}$ such that the total measure of $\mathbb{R}$ is 1. For every Borel subset $E \subseteq \mathbb{R}$ the quantity $0 \leq \mu_f(E) \leq 1$ is a probability that in the state μ the value of the observable f belongs to E. By definition, the *expectation value* of an observable f in the state μ is given by the Lebesgue-Stieltjes integral

$$\mathsf{E}_\mu(f) = \int_{-\infty}^{\infty} \lambda d\mu_f(\lambda),$$

where $\mu_f(\lambda) = \mu_f((-\infty, \lambda))$ is a distribution function of the measure $d\mu_f$. The correspondence $f \mapsto \mu_f$ should satisfy the following natural properties.

S1. $|\mathsf{E}_\mu(f)| < \infty$ for $f \in \mathcal{A}_0$ — the subalgebra of bounded observables.

S2. $\mathsf{E}_\mu(1) = 1$, where 1 is the unit in $\mathcal{A}$.

S3. For all $a, b \in \mathbb{R}$ and $f, g \in \mathcal{A}$,

$$\mathsf{E}_\mu(af + bg) = a\mathsf{E}_\mu(f) + b\mathsf{E}_\mu(g),$$

if both $\mathsf{E}_\mu(f)$ and $\mathsf{E}_\mu(g)$ exist.

S4. If $f_1 = \varphi \circ f_2$ with smooth $\varphi : \mathbb{R} \to \mathbb{R}$, then for every Borel subset $E \subseteq \mathbb{R}$,

$$\mu_{f_1}(E) = \mu_{f_2}(\varphi^{-1}(E)).$$

It follows from property **S4** and the definition of the Lebesgue-Stieltjes integral that

$$\mathsf{E}_\mu(\varphi(f)) = \int_{-\infty}^{\infty} \varphi(\lambda) d\mu_f(\lambda). \tag{2.18}$$

In particular, $\mathsf{E}_\mu(f^2) \geq 0$ for all $f \in \mathcal{A}$, so that the states define normalized, positive, linear functionals on the subalgebra $\mathcal{A}_0$.

Assuming that the functional E_μ extends to the space of bounded, piecewise continuous functions on $\mathscr{M}$, and satisfies (2.18) for measurable functions φ, one can recover the distribution function from the expectation values by the formula

$$\mu_f(\lambda) = \mathsf{E}_\mu\left(\theta(\lambda - f)\right),$$

where $\theta(x)$ is the Heavyside step function,

$$\theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Indeed, let χ be the characteristic function of the interval $(-\infty, \lambda) \subset \mathbb{R}$. Using (2.18) and the definition of the Lebesgue-Stieltjes integral we get

$$\mathsf{E}_\mu(\theta(\lambda - f)) = \int_{-\infty}^{\infty} \chi(s) d\mu_f(s) = \mu_f((-\infty, \lambda)) = \mu_f(\lambda).$$

Every probability measure $d\mu$ on $\mathscr{M}$ defines the state on $\mathcal{A}$ by assigning[20] to every observable f a probability measure $\mu_f = f_*(\mu)$ on $\mathbb{R}$ — a push-forward of the measure $d\mu$ on $\mathscr{M}$ by the mapping $f : \mathscr{M} \to \mathbb{R}$. It is defined by $\mu_f(E) = \mu(f^{-1}(E))$ for every Borel subset $E \subseteq \mathbb{R}$, and has the distribution function

$$\mu_f(\lambda) = \mu(f^{-1}(-\infty, \lambda)) = \int_{\mathscr{M}_\lambda(f)} d\mu,$$

where $\mathscr{M}_\lambda(f) = \{x \in \mathscr{M} : f(x) < \lambda\}$. It follows from the Fubini theorem that

$$E_\mu(f) = \int_{-\infty}^{\infty} \lambda d\mu_f(\lambda) = \int_{\mathscr{M}} f d\mu. \tag{2.19}$$

[20] There should be no confusion in denoting the state and the measure by μ.

It turns out that probability measures on $\mathscr{M}$ are essentially the only examples of states. Namely, for a locally compact topological space $\mathscr{M}$ the Riesz-Markov theorem asserts that for every positive, linear functional l on the space $C_{\mathrm{c}}(\mathscr{M})$ of continuous functions on $\mathscr{M}$ with compact support, there is a unique regular Borel measure $d\mu$ on $\mathscr{M}$ such that

$$l(f) = \int_{\mathscr{M}} f d\mu \quad \text{for all} \quad f \in C_{\mathrm{c}}(\mathscr{M}).$$

This leads to the following definition of states in classical mechanics.

Definition. The set of states $\mathcal{S}$ for a Hamiltonian system with the phase space $\mathscr{M}$ is the convex set $\mathscr{P}(\mathscr{M})$ of all probability measures on $\mathscr{M}$. The states corresponding to Dirac measures $d\mu_x$ supported at points $x \in \mathscr{M}$ are called *pure states*, and the phase space $\mathscr{M}$ is also called the *space of states*[21]. All other states are called *mixed states*. A process of measurement in classical mechanics is the correspondence

$$\mathcal{A} \times \mathcal{S} \ni (f, \mu) \mapsto \mu_f = f_*(\mu) \in \mathscr{P}(\mathbb{R}),$$

which to every observable $f \in \mathcal{A}$ and state $\mu \in \mathcal{S}$ assigns a probability measure μ_f on $\mathbb{R}$ — a push-forward of the measure $d\mu$ on $\mathscr{M}$ by f. For every Borel subset $E \subseteq \mathbb{R}$ the quantity $0 \leq \mu_f(E) \leq 1$ is the probability that for a system in the state μ the result of a measurement of the observable f is in the set E. The expectation value of an observable f in a state μ is given by (2.19).

In physics, pure states are characterized by having the property that a measurement of every observable always gives a well-defined result. Mathematically this can be expressed as follows. Let

$$\sigma_\mu^2(f) = \mathsf{E}_\mu\left((f - \mathsf{E}_\mu(f))^2\right) = \mathsf{E}_\mu(f^2) - \mathsf{E}_\mu(f)^2 \geq 0$$

be the *variance* of the observable f in the state μ.

Lemma 2.1. *Pure states are the only states in which every observable has zero variance.*

Proof. It follows from the Cauchy-Bunyakovski-Schwarz inequality that $\sigma_\mu^2(f) = 0$ if only if f is constant on the support of a probability measure $d\mu$. □

In particular, a *mixture* of pure states $d\mu_x$ and $d\mu_y$, $x, y \in \mathscr{M}$, is a mixed state with

$$d\mu = \alpha\, d\mu_x + (1-\alpha) d\mu_y, \quad 0 < \alpha < 1,$$

so that $\sigma_\mu^2(f) > 0$ for every observable f such that $f(x) \neq f(y)$.

[21]The space of pure states, to be precise.

For a system consisting of few interacting particles (say, a motion of planets in celestial mechanics) it is possible to measure all coordinates and momenta, so one considers only pure states. Mixed states necessarily appear for *macroscopic* systems, when it is impossible to measure all coordinates and momenta[22].

Remark. As a topological space, the space of states $\mathscr{M}$ can be reconstructed from the algebra $\mathcal{A}$ of classical observables. Namely, suppose for simplicity that $\mathscr{M}$ is compact. Then the $\mathbb{C}$-algebra $\mathcal{C} = C(\mathscr{M})$ of complex-valued continuous functions on $\mathscr{M}$ — the completion of the complexification of the $\mathbb{R}$-algebra $\mathcal{A}$ of classical observables with respect to the sup-norm — is a commutative $\mathbb{C}^*$-algebra. This means that $C(\mathscr{M})$ is a Banach space with respect to the norm $\|f\| = \sup_{x\in\mathscr{M}} |f(x)|$, has a $\mathbb{C}$-algebra structure (associative algebra over $\mathbb{C}$ with a unit) given by the point-wise product of functions such that $\|f \cdot g\| \leq \|f\|\|g\|$, and has a complex anti-linear anti-involution: a map $^* : \mathcal{C} \to \mathcal{C}$ given by the complex conjugation $f^*(x) = \overline{f(x)}$ and satisfying $\|f \cdot f^*\| = \|f\|^2$. Then the Gelfand-Naimark theorem asserts that every commutative $\mathbb{C}^*$-algebra $\mathcal{C}$ is isomorphic to the algebra $C(\mathscr{M})$ of continuous functions on its spectrum — the set of maximal ideals of $\mathcal{C}$ — a compact topological space with the topology induced by the weak topology on $\mathcal{C}^*$, the dual space of $\mathcal{C}$.

We conclude our exposition of classical mechanics by presenting two equivalent ways of describing the dynamics — the time evolution of a Hamiltonian system $((\mathscr{M}, \{\ ,\ \}), H)$ with the algebra of observables $\mathcal{A} = C^\infty(\mathscr{M})$ and the set of states $\mathcal{S} = \mathscr{P}(\mathscr{M})$. In addition, we assume that the Hamiltonian phase flow g_t exists for all times, and that the phase space $\mathscr{M}$ carries a volume form dx invariant under the phase flow[23].

Hamilton's Description of Dynamics. States do not depend on time, and time evolution of observables is given by Hamilton's equations of motion,

$$\frac{d\mu}{dt} = 0, \quad \mu \in \mathcal{S}, \quad \text{and} \quad \frac{df}{dt} = \{H, f\}, \quad f \in \mathcal{A}.$$

The expectation value of an observable f in the state μ at time t is given by

$$\mathsf{E}_\mu(f_t) = \int_{\mathscr{M}} f \circ g_t \, d\mu = \int_{\mathscr{M}} f(g_t(x))\rho(x)dx,$$

where $\rho(x) = \dfrac{d\mu}{dx}$ is the Radon-Nikodim derivative. In particular, the expectation value of f in the pure state $d\mu_x$ corresponding to the point $x \in \mathscr{M}$ is $f(g_t(x))$. Hamilton's picture is commonly used for mechanical systems consisting of few interacting particles.

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[22]Typically, a macroscopic system consists of $N \sim 10^{23}$ molecules. Macroscopic systems are studied in classical statistical mechanics.

[23]It is Liouville's volume form when the Poisson structure on $\mathscr{M}$ is non-degenerate.

Liouville's Description of Dynamics. The observables do not depend on time

$$\frac{df}{dt} = 0, \quad f \in \mathcal{A},$$

and states $d\mu(x) = \rho(x)dx$ satisfy Liouville's equation

$$\frac{d\rho}{dt} = -\{H, \rho\}, \quad \rho(x)dx \in \mathcal{S}.$$

Here the Radon-Nikodim derivative $\rho(x) = \dfrac{d\mu}{dx}$ and Liouville's equation are understood in the distributional sense. The expectation value of an observable f in the state μ at time t is given by

$$\mathsf{E}_{\mu_t}(f) = \int_{\mathscr{M}} f(x)\rho(g_{-t}(x))dx.$$

Liouville's picture, where states are described by the distribution functions $\rho(x)$ — positive distributions on $\mathscr{M}$ corresponding to probability measures $\rho(x)dx$ — is commonly used in statistical mechanics. The equality

$$\mathsf{E}_{\mu}(f_t) = \mathsf{E}_{\mu_t}(f) \quad \text{for all} \quad f \in \mathcal{A}, \ \ \mu \in \mathcal{S},$$

which follows from the invariance of the volume form dx and the change of variables, expresses the equivalence between Liouville's and Hamilton's descriptions of the dynamics.

3. Notes and references

Classical references are the textbooks [**Arn89**] and [**LL76**], which are written, respectively, from mathematics and physics perspectives. The elegance of [**LL76**] is supplemented by the attention to detail in [**Gol80**], another physics classic. A brief overview of Hamiltonian formalism necessary for quantum mechanics can be found in [**FY80**]. The treatise [**AM78**] and the encyclopaedia surveys [**AG90**], [**AKN97**] provide a comprehensive exposition of classical mechanics, including the history of the subject, and the references to classical works and recent contributions. The textbook [**Ste83**], monographs [**DFN84**], [**DFN85**], and lecture notes [**Bry95**] contain all the necessary material from differential geometry and the theory of Lie groups, as well as references to other sources. In addition, the lectures [**God69**] also provide an introduction to differential geometry and classical mechanics. In particular, [**God69**] and [**Bry95**] discuss the role the second tangent bundle plays in Lagrangian mechanics (see also the monograph [**YI73**] and [**Cra83**]). For a brief review of integration theory, including the Riesz-Markov theorem, see [**RS80**]; for the proof of the Gelfand-Naimark theorem and more details on $\mathbb{C}^*$-algebras, see [**Str05**] and references therein.

Our exposition follows the traditional outline in [**LL76**] and [**Arn89**], which starts with the Lagrangian formalism and introduces Hamiltonian formalism through the Legendre transform. As in [**Arn89**], we made special emphasis on precise mathematical formulations. Having graduate students and research mathematicians as a main audience, we have the advantage to use freely the calculus of differential forms and vector fields on smooth manifolds. This differs from the presentation in [**Arn89**], which is oriented at undergraduate students and needs to introduce this material in the main text. Since the goal of this chapter was to present only those basics of classical mechanics which are fundamental for the formulation of quantum mechanics, we have omitted many important topics, including mechanical–geometrical optics analogy, theory of oscillations, rotation of a rigid body, perturbation theory, etc. The interested reader can find this material in [**LL76**] and [**Arn89**] and in the above-mentioned monographs. Completely integrable Hamiltonian systems were also only briefly mentioned at the end of Section 2.6. We refer the reader to [**AKN97**] and references therein for a comprehensive exposition, and to the monograph [**FT07**] for the so-called Lax pair method in the theory of integrable systems, especially for the case of infinitely many degrees of freedom.

In Section 2.7, following [**FY80**], [**DFN84**], [**DFN85**], we discussed Poisson manifolds and Poisson algebras. These notions, usually not emphasized in standard exposition of classical mechanics, are fundamental for understanding the meaning of quantization — a passage from classical mechanics to quantum mechanics. We also have included in Sections 1.6 and 2.7 the treatment of the Laplace-Runge-Lenz vector, whose components are extra integrals of motions for the Kepler problem[24]. Though briefly mentioned in [**LL76**], the Laplace-Runge-Lenz vector does not actually appear in many textbooks, with the exception of [**Gol80**] and [**DFN84**]. In Section 2.7, following [**FY80**], we also included Theorem 2.19, which clarifies the meaning of the Jacobi identity, and presented Hamilton's and Liouville's descriptions of the dynamics.

Most of the problems in this chapter are fairly standard and are taken from various sources, mainly from [**Arn89**], [**LL76**], [**Bry95**], [**DFN84**], and [**DFN85**]. Other problems indicate interesting relations with representation theory and symplectic geometry. Thus Problems 2.12 and 2.20 introduce the reader to the orbit method [**Kir04**], and Problem 2.18 — to the method of symplectic reduction (see [**Arn89**], [**Bry95**] and references therein). Problems 2.21 – 2.23 introduce the reader to the theory of Lie-Poisson groups (see [**Dri86**], [**Dri87**], [**STS85**], and [**Tak90**] for an elementary exposition).

[24] We will see in Chapter 3 that these extra integrals are responsible for the hidden SO(4) symmetry of the hydrogen atom.

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