1. Let $H$ be the Hamiltonian of a harmonic oscillator,

$$H = \frac{P^2}{2} + \frac{\omega^2 Q^2}{2}.$$ 

Using Heisenberg’s uncertainty principle, prove that for every state $\psi \in D(H)$,

$$\langle H | P \psi \rangle \geq \frac{\hbar \omega}{2}.$$ 

2. Let $\mathcal{H} = L^2(\mathbb{R})$ and $\mathcal{D}$ be the Hilbert space of entire functions with the inner product

$$(f, g) = \frac{1}{\pi} \iint_{\mathbb{C}} f(z) \overline{g(z)} e^{-|z|^2} d^2 z.$$ 

Prove that the linear operator

$$f(z) = (U \psi)(z) = \int_{-\infty}^{\infty} U(z, q) f(q) dq,$$ 

where

$$U(z, q) = \sqrt{\frac{4}{\pi}} e^{-\frac{q^2}{2} - \frac{x^2}{2} + \sqrt{2} q x},$$ 

is a one-to-one mapping of $\mathcal{H}$ onto $\mathcal{D}$, and that the inverse operator is given by

$$\psi(q) = \lim_{R \to \infty} \frac{1}{2\pi} \iint_{|z| \leq R} \overline{U(z, q)} f(z) e^{-|z|^2} d^2 z,$$ 

where the limit is understood in strong topology on $\mathcal{H}$. 

3. Let $\mathcal{F}$ be the Fourier transform on $L^2(\mathbb{R})$. Find the eigenvalues and eigenfunctions of $\mathcal{F}$. 

4. (a) Verify that the linear map

$$f \mapsto K_1(q, q') = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} f(p, q) e^{i p (q - q')} \frac{dp}{\hbar}$$ 

defines the so-called $qp$-quantization: to every polynomial $f(p, q) = \sum_{m,n} c_{mn} p^m q^n$ it assigns an operator $f(P, Q) = \sum_{m,n} c_{mn} Q^n P^m$. 

(b) Verify that the linear map
\[ f \mapsto K_2(q, q') = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} f(p, q') e^{ip(q-q')/\hbar} \, dp \]
defines the so-called pq-quantization: to every polynomial \( f(p, q) = \sum_{m,n} c_{mn} p^m q^n \) it assigns an operator \( f(P, Q) = \sum_{m,n} c_{mn} P^m Q^n \).

(c) Verify that the Weyl quantization is a symmetric quantization: to every polynomial \( f(p, q) = \sum_{m,n} c_{mn} p^m q^n \) it assigns an operator \( f(P, Q) = \sum_{m,n} c_{mn} \text{Sym}(P^m Q^n) \), where by definition,
\[
(uP + vQ)^n = \sum_{i+j=n} \frac{n!}{i! j!} u^i v^j \text{Sym}(P^i Q^j).
\]

5. (a) Let \( H \) be the one-dimensional Schrödinger operator with \( m = \frac{1}{2} \),
\[ H = -\hbar^2 \frac{d^2}{dq^2} + v(q) \]
where \( v(q) \) is continuous bounded from below function on \( \mathbb{R} \). Prove that all eigenvalues of \( H \) — a self-adjoint operator on \( L^2(\mathbb{R}) \), are simple.

(b) Find the discrete spectrum and the corresponding eigenfunctions for the case
\[ v(q) = -\frac{2\alpha^2 \hbar^2}{\cosh^2 \alpha q}, \quad \alpha > 0. \]