III.5. Spectral theorem

III.5.1 Four forms of the spectral theorem

Let $A \in \mathcal{L}(\mathcal{H})$, $A = A^*$. 

**Theorem (continuous functional calculus)**

There exists a *-homomorphism

$$\varphi : C(\sigma(A)) \to \mathcal{L}(\mathcal{H})$$

such that

(a) $\|\varphi(f)\| = \|f\|_{\infty}$.

(b) If $f(\lambda) = \lambda$, then $\varphi(f) = A$.

(c) If $Ax = \lambda x$, $x \in \mathcal{H}$, then $\varphi(f)x = f(\lambda)x$.

(d) If $f \geq 0$, then $\varphi(f) \geq 0$.

(e) $\sigma(\varphi(f)) = f(\sigma(A))$

$$= \{ f(\lambda), \lambda \in \sigma(A) \} .$$

**Notation:** $\varphi(f) = f(A)$.

**In a nutshell:** $C(\sigma(A)) \approx$ the $C^*$-algebra generated by $A$.

Let $B(\mathbb{K})$ be the algebra of bounded Borel functions on $\mathbb{K}$ (see Section I.3).
**Thm 2** (measurable functional calculus)

\[ \exists ! \, \ast - \text{homomorphism} \]
\[ \hat{\varphi} : \mathcal{B}(\mathcal{R}) \to L(\mathcal{H}) \]

s.t.
(a) \[ \| \hat{\varphi}(\varphi) \| = \| \varphi \| \] (need to be explained what \( a.e. \) means: to the spectral measures determined by \( A \))
(b) - (d) as in Thm 1
(c) If \( f_n \to f \) pointwise and \( \| f_n \|_\infty \) are bounded, then \( \hat{\varphi}(f_n) \to \varphi(f) \) strongly.
(d) If \( AB = BA, \ B \in L(\mathcal{H}) \), then \( \varphi(f)B = B\varphi(f) \).

**Thm 3** (multiplication operator form)

Let \( \mathcal{H} \) be separable. Then \( \exists \) measures \( \mu_n \) on \( \sigma(A) \), \( n = 1, 2, \ldots, N \quad (N \in \mathbb{N} \cup \{\infty\}) \) and a unitary operator
\[ U : \mathcal{H} \to \bigoplus_{i=1}^{\infty} L^2(\sigma(A), d\mu_i) \]
such that
\[ (U A U^{-1} f)_i(\lambda) = \lambda f_i(\lambda), \ i = 1, \ldots, N \]
\[ \varphi = \bigoplus_{i=1}^{N} f_i \]
(spectral representation of \( A \)).
Def A family \( \{ P_\Omega \} \) is a bounded projective-valued measure on Borel sets \( \mathcal{B} \) if

(i) each \( P_\Omega \in \mathcal{L}(\mathcal{H}) \) is an orthogonal projector

(ii) \( P_\emptyset = 0 \), \( P_{(-a,a)} = I \) for some \( a > 0 \).

(iii) If \( \Omega = \bigcup_{n=1}^N \Omega_n \) (disjoint), then

\[
P_\Omega = \lim_{N \to \infty} \sum_{n=1}^N P_{\Omega_n}
\]

(iv) \( P_{\Omega_1} P_{\Omega_2} = P_{\Omega_1 \cap \Omega_2} \).

(Actually, (iv) follows from (i) and (iii):

\[
\Omega_1 \cup \Omega_2 = \Omega_1' \cup \Omega_1 \cap \Omega_2' \cup \Omega_2
\]

and \( P_1 + P_2 \) is projector \( \iff P_1 P_2 + P_2 P_1 = 0 \),

so \( P_{\Omega_1} \) anti-commutes with \( P_{\Omega_1 \cap \Omega_2} \).
but if $P_1P_2 + P_2P_1 = 0$, then $P_1P_2 = 0$.
Indeed,
\[ P_1P_2 = -P_2P_1, \]
so
\[ P_2P_1P_2 = -P_2P_1 = P_1P_2, \]
or
\[ (I - P_2)P_1P_2 = 0. \]
but
\[ P_2P_1P_2 = -P_2P_1P_2, \]
so also
\[ P_2P_1P_2 = 0; \]
Thus
\[ (P_{\Omega_1} + P_{\Omega_1 \cap \Omega_2})(P_{\Omega_2} + P_{\Omega_1 \cap \Omega_2}) = P_{\Omega_1 \cap \Omega_2}, \]
Set $P_\lambda = P_{(-\infty, \lambda]}$.

Every projection-valued measure defines a family of ordinary measures on $\mathbb{R}$:
\[ M_x(\Omega) = (P(\Omega)x, x). \]

### 5.2. Integration theory

For every bounded Borel $f$ and $x \in H$
\[ f \in B(\mathbb{R}) \]
set
\[ \int f(\lambda) \, d\mu_x(\lambda). \]
By polarization, we get
\[ \int f(\lambda) \, d(\mathcal{P}_\lambda x, y) \]
as Lebesgue–Stieltjes integral. By Riesz lemma, \( \exists T \) s.t.
\[ (Tx, y) = \int f(\lambda) \, d(\mathcal{P}_\lambda x, y) \]
and
\[ \| T \| \leq \sup |f| = \| f \|_\infty . \]

We set
\[ T = \int f(\lambda) \, d\mathcal{P}_\lambda \]

If \( f \in \text{BC}(\mathbb{R}) \), this integral converges uniformly.

Thm 4 (projection-valued measure form
- von Neumann)
\( \exists ! \) bounded projection-valued measure \( \mathcal{P}_\lambda \) s.t.
\[ A = \int \lambda \, d\mathcal{P}_\lambda , \]
\[ f(A) = \int f(\lambda) \, d\mathcal{P}_\lambda , \]
\( f \in \mathcal{B}(\mathbb{R}) \) (bounded Borel)
linear by polarization:

\[ (x, y) = \frac{1}{4} \left[ (x+y, x+y) - (x-y, x-y) + i(x+iy, x+iy) - i(x-iy, x-iy) \right]. \]

\[ \|Tx\|^2 = \int f(t) d\left( P_t x, Tx \right), \]

but

\[ (P_t x, Tx) = \overline{(Tx, P_t x)} \]

\[ = \int \overline{f(t)} \overline{d(\bar{P}_t x, P_t x)} \]

\[ = \int \overline{f(t')} d(P_{t'}, x, x), \]

thus

\[ \|Tx\|^2 = \int |f(t)|^2 d(P_t x, x) \]

\[ \leq \sup |f(t)| \|x\|^2. \]

Also,

\[ (T^* x, y) = \int \overline{f(t)} \overline{d(\bar{P}_t x, y)}. \]
1) \[ l(y) = \int f(t) d(P_t x, y) \]

is bounded anti-linear functional (\( x \in H \) is fixed).

Indeed, it is sufficient to prove for \( f = a x \omega \), but

\[ l(y) = a (P_\omega x, y) \]

\[ \| l \| \leq |a| \| x \| \]

so that in general,

\[ \| l \| \leq \| f \| \infty \| x \| . \]

2) If \( f \in C([-a,a]) \), then

\[ T = \int f(t) dP_t \text{ uniformly.} \]

Proof

Since \( f \) is uniformly continuous, \( \forall \varepsilon > 0 \)
\( \exists \ n \ s.t. \) for \( \| x \| = 1 \),

\[ \left| \left( T - \sum_{i=1}^{n} f(t_{i-1}) P_{\Delta_i} \right) x, x \right| \]

\( \left( \Delta_i = [t_{i-1}, t_i] \right) \)

\[ \leq \left| \left( \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (f(t) - f(t_i)) dP_t \cdot x, x \right) \right| \]
\[ \varepsilon \parallel x \parallel^2 = \varepsilon. \text{ But it follows from the polarization identity that} \]

\[ |(Ax, x)| \leq C \quad \forall \parallel x \parallel = 1 \]

\[ \Rightarrow \parallel A \parallel \leq 2C. \]

(Indeed, for \( \parallel y \parallel = 1 \), polarization identity gives:

\[ |(Ax, y)| \leq 2C, \text{ take } y = \frac{Ax}{\parallel Ax \parallel} \]

if \( Ax \neq 0 \), so that

\[ \parallel Ax \parallel \leq 2C \]

3) If

\[ T_1 = \int f(t) \, dP_t \]

\[ T_2 = \int g(t) \, dP_t \]

then

\[ T_1 T_2 = \int f(t) g(t) \, dP_t. \]

**Proof 1.** Sufficient to verify for

\( f = \chi_{\Omega_1}, \ g = \chi_{\Omega_2} \), then it is obvious.

**Proof 2.**
\[ (T_1 T_2 x, y) = (T_2 x, T_1^* y) \]
\[ = \int f(t) d (P_t x, T_1^* y) . \]

But, as before,
\[ (P_t x, T_1^* y) = (T_1^* y, P_t x) \]
\[ = \int g(t') d (P_{t'} y, P_t x) \]
\[ = \int g(t') d (x, P_{t'} y) \]
\[ = \int g(t') d (P_{t'} x, y) , \]

so that
\[ (T_1 T_2 x, y) = \int f(t) g(t) d (P_t x, y) . \]

4) This refers to Problem 14 in Ch. VII
\[ \text{If } A_n \geq 0, A_n \uparrow A \Rightarrow \sqrt{A_n} \uparrow \sqrt{A} \]
(Could be stated for any fractional power).

Will use Problem 1 in Ch. VII:
\[ f(A) = \frac{1}{2\pi i} \int f(z) R_z(A) dz, \]
\[ C \]
where $C$ contains $\sigma(A)$ inside.

Can assume $\|I-A\| = q' < q < 1$, so that

$$\|I-A_n\| < q, \ n \geq N.$$  

Then $\sigma(A), \sigma(A_n) \subseteq (1-q, 1-q)$

and we use Cauchy integral formula with $f(z) = \sqrt{z}$.

Now

$$R_z(A_n) - R_z(A) = (A - A_n) R_z(A) R_z(A_n),$$

so that on $C$

$$\|R_z(A_n) - R_z(A)\| \leq C^2 \|A - A_n\|,$$

$$C = \max_{z \in C} \|R_z(A_n)\| < \infty.$$  

$\mathbf{III.5.3. \ Spectral \ theorem \ for \ unitary}$

operators

$U$ - unitary, $UU^* = U^*U = I$.

First, for every trigonometric polynomial

$$P(e^{i ht}) = \sum_{k=\text{-m}}^{\text{m}} c_k e^{ikt},$$

set
\[ P(U) = \sum_{k=-m}^{n-1} c_k U^k. \]

Clearly, \( P(U)^* = \overline{P(U^{-1})} \). Also if \( P(e^{it}) \geq 0 \), then \( P(U) \geq 0 \).

Indeed, if "polynomial" \( P(z) \) is real on \( |z| = 1 \), then

\[ \overline{P(z)} = P \left( \frac{1}{\overline{z}} \right) \]

Since \( P(z) = \frac{1}{z^m} F(z) \), all roots of the polynomial \( F(z) \) are symmetric with respect to \( |z| = 1 \) and \( \exists \ Q(z) \) — a polynomial in \( z \), s.t.

\[ P(z) = \overline{Q} \left( \frac{1}{z} \right) Q(z), \]

so that \( P(e^{it}) = \left| Q(e^{it}) \right|^2 \), and

\[ P(U) = T^* T \geq 0, \]

\[ T = Q(U) \]

**Def:** Let \( \mathcal{H} = C(S^1, \mathbb{R}) \) \( U \) \( \overline{\text{PCMC}}(S^1, \mathbb{R}), \)

where \( \text{PCMC}(S^1, \mathbb{R}) \) are real-valued piece-wise continuous functions which are point-wise limits of monotone functions in \( C(S^1, \mathbb{R}) \). \( \overline{\text{Sequences of}} \)
Lemma 1 \( \psi(e^{it}) \in \mathcal{H} \Rightarrow \exists \) monotone system of quasipolynomials \[
\{ P_n(e^{it}) \} \quad \text{s.t.} \quad \lim_{n \to \infty} P_n(e^{it}) = \psi(e^{it}) \quad \text{everywhere on } S^1.
\]

**Proof.** By definition, \( \psi(e^{it}) \) is a point-wise limit of monotone continuous functions \( \{ \varphi_n(e^{it}) \} \),

\[
\varphi_1(e^{it}) > \varphi_2(e^{it}) \geq \ldots \geq \varphi_n(e^{it}) \geq \ldots \to \psi(e^{it}).
\]

By Weierstrass theorem, \( \forall n \exists P_n \) s.t. everywhere on \( S^1 \),

\[
|P_n(e^{it}) - (\varphi_n(e^{it}) + \frac{3}{2^{n+2}})| \leq \frac{1}{2^{n+2}},
\]

i.e.

\[
\frac{1}{2^{n+1}} \leq P_n(e^{it}) - \varphi_n(e^{it}) \leq \frac{1}{2^n}
\]

everywhere on \( S^1 \), so that

\[
\lim_{n \to \infty} P_n(e^{it}) = \psi(e^{it}).
\]

Also,

\[
P_{n+1}(t) \leq \varphi_{n+1}(t) + \frac{1}{2^{n+1}} \leq \varphi_n(t) + \frac{1}{2^{n+1}}
\]

\[
\leq P_n(t).
\]

Want to define \( \psi^*(U) \) as \( \text{s-lim}_{n \to \infty} P_n(U) \).
Lemma 2. Let $A_1 \geq A_2 \geq \ldots \geq 0$. Then $\exists A \in L(H)$ s.t.

$$s\text{-lim}_{n \to \infty} A_n = A.$$ 

Proof. $A_m - A_n \geq 0$ for $m \geq n$, so

$$\sqrt{((A_m - A_n) x, y)} \leq \sqrt{((A_m - A_n) x, x)}.$$ 

Because $(b, y)$, $b \geq 0$, satisfies Cauchy-Schwarz inequality! Setting $y = (A_m - A_n) x$, we get

$$\sqrt{(A_m - A_n) x, y) \leq \sqrt{\|A_1\| \|y\| \sqrt{((A_m - A_n) x, x)}}.$$ 

Since $(A_m x, x) \downarrow$,

$$\lim_{n \to \infty} \sqrt{(A_m x, x) - (A_n x, x)} = 0,$$

that $\{A_n x\}$ is Cauchy sequence, i.e.

$$\lim_{n \to \infty} A_n x = y := A x.$$ 

Clearly, $A \in L(H)$.

Now, since $P_n (e^{it}) - \inf \psi (e^{it}) > 0$, by Lemma 2 we have

$$\psi (U) = s\text{-lim}_{n \to \infty} P_n(U).$$
Lemma 3 \( \Psi^*(U) \) does not depend on the choice of the sequence \( \{P_n(e^{it})\} \).

Proof. Let \( \{Q_n(e^{it})\} \) be s.t.
\[
\lim_{n \to \infty} Q_n(e^{it}) = \Psi(e^{it}) \quad \text{pointwise.}
\]

\( \forall m \) and given \( t \exists N \) s.t.
\[
Q_N(e^{it}) < P_m(e^{it}) + \frac{1}{m}
\]
- true on same interval, but \( S^1 \) is compact, so that \( \exists N \) s.t. for all \( n \geq N \) and everywhere on \( S^1 \),
\[
Q_n(e^{it}) < P_m(e^{it}) + \frac{1}{m}
\]
Setting \( S = \lim_{n \to \infty} Q_n(e^{it}) \), we get
\[
S \leq P_m(U) + \frac{1}{m}.
\]
Setting \( T = \lim_{n \to \infty} P_n(e^{it}) \), we get
\[
S \leq T.
\]
By symmetry, \( T \leq S \), so \( S = T \).

Let \( \hat{K} \) be \( C \)-span of \( K \); set for
\[
\Psi = \sum_{k=1}^{m'} c_k \Psi_k \in \hat{K}
\]
\[
\Psi^*(U) = \sum_{k=1}^{m'} c_k \Psi_k(U).
\]
Claim \( \Psi(U) \) is well-defined. Indeed, if
\[ \Psi_1 - \Psi_2 = \Psi_3 - \Psi_4, \quad \Psi_i \in \mathcal{H}, \]
then
\[ \Psi_1 + \Psi_4 = \Psi_2 + \Psi_3, \]
so
\[ \Psi_1(U) + \Psi_4(U) = \Psi_2(U) + \Psi_3(U). \]

Also, \( U \Psi(U) = \Psi(U) U \)
\( \forall \Psi \in \mathcal{H} \) — a corollary of \( Z \Psi(z) = \Psi(z) Z \).

**Theorem (spectral decomposition of unitary operator)** Let \( U \in \mathcal{L}(\mathcal{H}) \) be a unitary operator. Then \( \exists \) projection-valued measure \( P_\Omega \) s.t.

\[ U^\lambda = \int_0^{2\pi} e^{ikt} dP_t, \quad \lambda \in \mathbb{Z}, \]

in a uniform topology (as a limit of Riemann sums). In general, for \( \Psi \in \mathcal{B}(S^1), \]
\[ \Psi(U) = \int \Psi(e^{ikt}) dP_t \]
(in a weak-sense) — operator Lebesgue-Stieltjes integral.

**Proof** Set
\[ \psi_\lambda(e^{it}) = \begin{cases} 1, & 0 \leq t < \lambda \\ 0, & \lambda \leq t < 2\pi \\ 1, & t = 2\pi \end{cases}, \quad \psi_\lambda(0) = 0 \]

and \[ P_\lambda = \psi_\lambda(\mathcal{U}), \quad \lambda \in [0, 2\pi] \]

Then

(i) \[ P_0 = 0, \quad P = I \]

(ii) \[ P_\lambda = P_\lambda^*, \quad P_\lambda^2 = P_\lambda \]

(iii) \[ P_\lambda \leq P_\mu \quad \text{for} \quad 0 \leq \lambda < \mu \leq 2\pi \]

(iv) \[ s-lim_{\lambda' \rightarrow \lambda-0} P_{\lambda'} = P_\lambda \]

This is because

\[ \lim_{\epsilon \rightarrow 0^+} \psi_{\lambda-\epsilon}(e^{it}) = \psi_{\lambda}(e^{it}) \quad \forall t. \]

This properties define projection-value measure through

\[ P_\Omega = \int \chi_\Omega(e^{it}) \, dP_t. \]

To prove the spectral decomposition, let \( 0 = t_0 < t_1 < \ldots < t_n = 2\pi \).

Using the inequality

\[ |e^{ikt} - e^{ikt'}| \leq |k| |t-t'|, \]

we get
\[ e^{-ikt} - \sum_{j=1}^{n} e^{ikt_{j-1}} (w_{t_{j}}(e^{it}) - w_{t_{j-1}}(e^{it})) \leq |k| \Delta t, \]

\[ \Delta t = \max_{j} \{|t_{j} - t_{j-1}|\} = \frac{\varepsilon}{|k|}. \]

Then

\[ 0 \leq \left[ (U^{K} - \sum_{j=1}^{n} e^{ikt_{j-1}} P_{\Delta j})^* \right] \cdot \left[ (U^{K} - \sum_{j=1}^{n} e^{ikt_{j-1}} P_{\Delta j}) \right] \leq \varepsilon^{2} I, \]

i.e.,

\[ \| U^{K} - \sum_{j=1}^{n} e^{ikt_{j-1}} P_{\Delta j} \| \leq \varepsilon. \]

**Corollary** Let \( s\text{-lim} U_{n} = U \) (\( U \) is obviously unitary), and \( \lambda = 1 \) is not an eigenvalue for \( U \). Then for every \( 0 < t < 2\pi \) s.t. \( e^{it} \) is not an eigenvalue for \( U \),

\[ s\text{-lim} P_{t}^{(n)} = P_{t}. \]
Proof let
\[ \psi(e^{i\xi}) = \psi_t(e^{i\xi})(e^{i\xi} - e^{it})(e^{i\xi} - 1), \]
\[ \psi \in C(S^1). \]
Set
\[ B_n = (U_n - e^{it}I)(U_n - I), \]
\[ B = (U - e^{it}I)(U - I). \]
Since \( \psi \in C(S^1), \) \( \psi(U_n) \rightarrow \psi(U), \)
\[ B_n \rightarrow B. \]
But
\[ \psi(U_n) = P_t^{(n)} B_n \rightarrow P_t B. \]
But for all \( x \in \mathcal{H}, \)
\[ \| (P_t^{(n)} - P_t) B x \| \leq \| P_t^{(n)} (B - B_n) x \| \]
\[ + \| P_t^{(n)} B_n x - P_t B x \| \rightarrow 0, \]
so
\[ s-lim_{n \rightarrow \infty} P_t^{(n)} B \rightarrow P_t B. \]
By assumption, \( \text{Ker} B = 0 \) and \( \overline{\text{Im} B} = \mathcal{H}, \)
so for \( x \in \text{Im} B, \)
\[ \lim_{n \rightarrow \infty} P_t^{(n)} x = \lim_{n \rightarrow \infty} P_t^{(n)} B y \]
\[ = P_t B y = P_t x. \]
Cayley Transform

Let $A \in L(\mathcal{H})$, $A = A^*$. Set

$$U = (A - iI)(A + iI)^{-1} \in L(\mathcal{H}).$$

(Note that $\text{Im}(A \pm iI) = \mathcal{H}$.)

Claim $U$ is unitary and $1 \notin \sigma(U)$.

Proof $U^* = (A - iI)^{-1}(A + iI)

= (A + iI)(A - iI)^{-1} = U^{-1}.$

Also, $Ux = x \iff (A - iI)x = (A + iI)x$,
i.e. $x = 0$; and $\text{Im}(U - I) = \text{Im}(A - iI)^{-1}\mathcal{H}$
since $\text{D}(A) = \mathcal{H}$!

$U$ is called Cayley transform of $A$.

Theorem 1 (Spectral decomposition of bounded self-adjoint operator).

Let $A \in L(\mathcal{H})$, $A = A^*$ and let

$$P_s, \ 0 \leq s \leq 2\pi,$$

be the projection-valued measure for $U = \frac{A - iI}{A + iI}$.

Then

$$A = \int t \, dE_t,$$

where $E_t = P_s, \ t = -\cot \frac{s}{2}$.

Proof
Lemma 1 \( e^{it} \in \sigma(U) \iff \forall \varepsilon > 0 \text{ s.t.} \ P(t-\varepsilon, t+\varepsilon)(U) \neq 0 \).

Proof (a) Suppose that \( \exists \varepsilon > 0 \text{ s.t.} \ P(t-\varepsilon, t+\varepsilon) = 0 \). Since for \( \forall x \in X \),

\[
\| (U - e^{it}I)x \|^2 \\
= \int_{0}^{2\pi} |e^{ix} - e^{it}|^2 \, d(P_{\frac{x}{2}}x, x) \\
= 4 \int_{0}^{2\pi} \sin^2 \frac{x-t}{2} \, d(P_{\frac{x}{2}}x, x) \\
\geq 4 \int_{t-\varepsilon}^{t+\varepsilon} + 4 \int_{0}^{t-\varepsilon} \geq \varepsilon^2 \|x\|^2.
\]

Thus \( e^{it} \in \sigma(U) \) (see next page).

(b) Let \( e^{it} \in \sigma(U) \). Then \( \exists \varepsilon > 0 \) s.t.

\[
\| (U - e^{it})x \|^2 \geq \varepsilon^2 \|x\|^2, \forall x \in X
\]

Suppose that \( \exists \eta > 0 \text{ s.t.} \ P(t-\eta, t+\eta)(U) \neq 0; \ \eta < \varepsilon. \)
Remark. Indeed, if $\forall x \in \mathcal{H}$

$$\| (U - e^{it} I) x \| \geq c \| x \|, \quad (\ast)$$

then $e^{it} \in \sigma(U)$. First, $\text{Im} (U - e^{it} I) = \mathcal{H}_0$ is closed, since if $\mathcal{H}_0 \ni y_n = (U - e^{it} I) x_n \to y \in \mathcal{H}$, then \{ $y_n$ \} is Cauchy and by (\ast) \{ $x_n$ \} is also Cauchy. Thus $\exists x \in \mathcal{H}$ s.t. $x_n \to x$ and $y \in \mathcal{H}_0$.

$$(U - e^{it} I) x$$

Second, suppose $y \perp \mathcal{H}_0$, i.e.

$$(y, (U - e^{it} I) x) = 0 \quad \forall x \in \mathcal{H},$$

$$(U^{-1} - e^{-it} I) y, x = 0 \quad \| \to$$

so that

$$(U^{-1} - e^{-it}) y = 0,$$

i.e.

$$U y = e^{it} y,$$

which contradicts $\text{Ker} (U - e^{it} I) = \{ 0 \}$, as follows from (\ast).
so that \( \exists y \in \mathcal{Y} \) s.t.
\[
(P_{t+\eta} - P_{t-\eta}) y = x \neq 0.
\]

For this \( x \) we get
\[
\| (U - e^{it}) x \|_2^2
\]
\[
= \int_0^{2\pi} 4 \sin^2 \frac{\xi - t}{2} d \left( P_{\xi} (P_{t+\eta} - P_{t-\eta}) y, y \right)
\]
\[
= \int_{t-\eta}^{t+\eta} 4 \sin^2 \frac{\xi - t}{2} d (P_{\xi} y, y)
\]
\[
\geq \varepsilon^2 \| x \|_2^2 = \varepsilon^2 \int_{t-\eta}^{t+\eta} d (P_{\xi} y, y).
\]

But
\[
\int_{t-\eta}^{t+\eta} 4 \sin^2 \frac{\xi - t}{2} d (P_{\xi} y, y)
\]
\[
\leq \eta^2 \int_{t-\eta}^{t+\eta} d (P_{\xi} y, y), \eta < \varepsilon - \text{ contradiction}.
\]
$A = i \frac{I+U}{I-U}$ and $1 \notin \sigma(U)$, so $P_S$ is supported on $[a, b] \subseteq [0, 2\pi]$. 

Thus, 

$$A = \int_{\sigma(U)} i \frac{1+e^{is}}{1-e^{is}} \, dP_S$$

Using Lemma 1, integration goes over $\sigma(U)$ only. 

$$= \int_{\sigma(U)} -\cot \frac{s}{2} \, dP_S,$$

$$\cot \frac{s}{2} \in C(\sigma(U)).$$

Setting $t = -\cot \frac{s}{2}$, $E_t = P_S$, we get 

$$A = \int t \, dE_t.$$

Corollary: Let $s\lim A_n = A$, where all $A_n$ are bounded & self-adjoint. Then for every $t \in \mathbb{R}$ s.t. $t$ is not an eigenvalue for $A$, 

$$s\lim E_t^{(n)} = E_t.$$ 

$n \to \infty$ $t \in \mathbb{R}$.

Follows from similar fact about unitary operators.

Theorem 2: $SA = AS$ if and only if $SE_t = E_t S$ for all $t \in \mathbb{R}$. 
Direct proof of spectral decomposition
for \( A \) (i.e. without
\[
A = i \frac{I + U}{I - U} \Rightarrow A = \psi(U)\]

\( Ax = i (I + U)y \), where
\[ y = (I - U^{-1})x. \]
We have
\[
Ax = i \int_0^{2\pi} (1 + e^{is}) \, dP_s \, y
\]
and for \( x = (I - U)y \),
\[
P_s x = (I - U)P_s y
\]
\[
= \int_0^{2\pi} (1 - e^{i\tau}) \, dP_{\tau} \, P_s y
\]
\[
= \int_0^{2\pi} (1 - e^{i\tau}) \, dP_{\tau} \, y, \text{ so that}
\]
\[
dP_s x = (1 - e^{is}) \, dP_s \, y
\]
and
\[
Ax = \int_0^{2\pi} i \frac{1 + e^{is}}{1 - e^{is}} \, dP_s \, x.
\]
Ex. \( A = A^* \), \( A \) compact \( \Rightarrow \)
\[
A = \sum_{n=1}^{l} \lambda_n P_n
\]
\( \lambda_n \neq 0 \)

\( \lambda_n \to 0 \); \( \lambda = 0 \to P_0 \) - projector on \( \text{Ker} A \). Set
\[
E_\lambda = \sum_{n=1}^{l} P_n \quad (\text{always convergent})
\]

\( \lambda_n < \lambda \)

\[
E_{\lambda-\epsilon} = E_\lambda \quad \text{and}
E_{\lambda_n+\epsilon} - E_{\lambda_n} = P_n,
\]
so that
\[
A = \int \lambda \, dE_\lambda.
\]
Proof. $SE_t = E_t S$ for all $t \in \mathbb{R}$

$\Rightarrow$ $SA = AS$ - trivial.

Converse follows that $SA = AS \Rightarrow SU = US \Rightarrow \psi_\lambda^\prime(U) S = S \psi_\lambda(U)$.

Lemma 1' $\lambda \in \sigma(A) \iff$

$\forall \varepsilon > 0 \quad E_{(\lambda - \varepsilon, \lambda + \varepsilon)}(A) \neq 0$.

Proof. As for lemma 1, replacing $4 \sin^2 \frac{\pi - t}{2}$ by $(\lambda - t)^2$.

Lemma 2

(a) $\lambda \in \sigma(A)$ is an eigenvalue $\iff$

$E_{\lambda+0} \neq E_{\lambda}$.

(b) $e^{it} \in \sigma(U)$ is an eigenvalue $\iff$

$P_{t + 0} \neq P_t$.

Proof. It is sufficient to prove (a).

Let $A x = \lambda x$. Then

$0 = \| (A - \lambda) x \|^2 = \int (t - \lambda)^2 d(E_t x, x)$

$\Rightarrow d(E_t x, x)$ is supported at $t = \lambda$. 


Since \( \|x\|^2 = \int d(E_t x, x) \),

\( (E_t x, x) \) has a jump at \( t = \lambda \) and is constant everywhere else. Thus

\[ (E_{\lambda+0} x, x) \neq (E_\lambda x, x), \]

so \( E_{\lambda+0} \neq E_\lambda \).

Conversely, let \( E_{\lambda+0} \neq E_\lambda \): 

\[ \exists y \in \mathcal{H} \text{ s.t.} \]

\[ x = (E_{\lambda+0} - E_\lambda)y \neq 0. \]

Now

\[ \| (A - \lambda I)x \|^2 = \int (t - \lambda)^2 d(E_t x, x) \]

But

\[ (E_t x, x) = (E_t (E_{\lambda+0} - E_\lambda)y, y) \]

\[ = \begin{cases} 0, & t < \lambda \\ \text{const}, & t > \lambda \end{cases}, \]

so \( d(E_t x, x) \) is supported at \( t = \lambda \) and

\[ \| (A - \lambda) x \| = 0. \]

Corollary: \( P_\lambda = E_{\lambda+0} - E_\lambda \)

is the projector on the corresponding eigenspace.
Another proof of the spectral theorem (a sketch)

\[ A = A^*, \quad R_z = R_z^*(A) = (A - zI)^{-1}, \quad \text{Im} \, z \neq 0. \]

Fix \( f \in H \) & put

\[ \varphi(z) = (R_z f, f). \]

• Hilbert identity \( \Rightarrow \varphi(z) \) is holomorphic for \( \text{Im} \, z > 0 \)

• \( R_z^* = R_z \Rightarrow \text{Im} \, \varphi(z) = y \| R_z f \|^2 > 0 \)
  for \( y = \text{Im} \, z > 0. \)

• \( \| R_z f \| \leq \frac{1}{y} \| f \| \Rightarrow \sup \ y \left| \varphi(iy) \right| < \infty \)
  for \( y > 0 \)

Key fact: \( \exists \) 1 function of bounded variation \( \omega(t) = \omega(t; f) \),

\[ \omega(-\infty) = 0, \quad \omega(t - 0) = \omega(t) \quad \forall \, t \]

s.t.

\[ \varphi(z) = \frac{d}{dt} \omega(t), \quad \text{Im} \, z > 0 \]

Defining \( \omega(t; f, g) \) by polarization...
we get \( (R_{z} f, g) = \int_{-\infty}^{\infty} \frac{d\omega(t; f, g)}{t-z} \)

\( \omega(t; f, g) = \omega(t; g, f) \) and is linear in \( f \); \( \omega(t; f, f) \leq \|f\|^2 \)

By Riesz representation theorem, \( \exists E_{t} \) s.t.

\( \omega(t; f, g) = \langle E_{t} f, g \rangle \),

\( (R_{z} f, g) = \int_{-\infty}^{\infty} \frac{d\langle E_{t} f, g \rangle}{t-z} \)

Hilbert identity \( \Rightarrow E_{t}, \ t \in \mathbb{R} \), is the distribution function for projection-valued measure associated with \( A \)!
III.5.4. Simple spectrum

(Multiplicity-free operators)

Def. A is multiplicity-free (or ω(A) is simple) if ∃ x ∈ H s.t.

(*) C(E(Δ)x) = H.

(Similar for unitary operators)

Lemma. U has simple spectrum

⇔ ∃ x ∈ H s.t. C{U^nx} = H.

Proof. Suppose U has simple spectrum, and suppose that ∃ y ∈ H s.t.

(y, U^n x) = 0 ∀ n ∈ Z,

i.e.

\[ \int e^{i \lambda t} d(P_t x, y) = 0 \]

(+) ⇒ (P_t x, y) = 0 ⇒ y \perp P(Δ)x

-contradicting (*). Hence, \{U^n x\} = H, then clearly (*) holds.

(+) Uniqueness of Fourier series.
Example: let $\sigma(t)$ be a distribution function: non-decreasing and satisfying $\sigma(t-0) = \sigma(t) \; \forall \; t \in \mathbb{R}$.

$$H = L^2_0(\mathbb{R}) = L^2(\mathbb{R}, d\mu),$$

$$\mu(E) = \int E \, d\sigma \quad \text{(Lebesgue-Stieltjes integral)}$$

$\mu$ has compact Q-multiplication by $t$ operator; support finite, $\mu(R) < \infty$; then $Q \in L(H)$.

$$E(\Delta)f(t) = \chi_{\Delta}(t)f(t)$$

$g(t) = \begin{cases} \alpha_k, & k-1 \leq t \leq k \\ \end{cases}$

s.t. $\sum_{n \in \mathbb{Z}} |\alpha_n|^2 \left( \sigma(n) - \sigma(n-1) \right) < \infty$

$g$ is a cyclic vector, since $C\{E(\Delta)g\}$ finite piece-wise constant functions - dense in $H$.

Exercise: any $g \neq 0$ except $\sigma$-measure 0 set is a cyclic vector.
\[ dE_t f = \delta(x-t) \delta(\lambda) \]
so that
\[ \int dE_t f = f \]
and \[ (\int t dE_t f)(\lambda) = \lambda f(\lambda); \]
\[ d(E_t f, f) = |f(t)|^2 \int d\sigma(t). \]
(Here \( f \in L^2_0(\mathbb{R}) \))

- We'll use these formulas on next three pages.
Theorem 1 Let \( A \) be multiplicity-free, \( x \in \mathcal{H} \) a cyclic vector, 
\( \sigma(t) = (E_t x, x) \). Then

\[ L^2_\sigma(\mathbb{R}) \ni f(t) \mapsto f = \int f(t) dE_t x \in \mathcal{H} \]

is an isometry s.t.

\[ A f \leftrightarrow t f(t). \]

Proof

Let \( G = \left\{ f \in \mathcal{H} \mid \exists f(t) \in L^2_\sigma(\mathbb{R}), f = \int f(t) dE_t x \right\} \).

First, for \( f(t) = \chi_\Delta(t) \),

\[ f = E(\Delta) x, \text{ so that} \]

\( G = \mathcal{H} \).

Moreover,

\[ (E_t x, f) = \int_t \overline{f(s)} d_s (E_t x, E_s x) = \int_t \overline{f(s)} d (E_s x, x), \]
so that
\[
(f, f) = \int f(t) \, d(E_t x, f)
\]
\[
= \int |f(t)|^2 \, d\sigma(t),
\]
so the map
\[
L^2_\sigma(\mathbb{R}) \to G
\]
is an isometry. Thus \( G = \mathcal{H} \).

Next,
\[
Af = \int \lambda \, dE_\lambda f
\]
and
\[
(Af, h) = \int \lambda \, d(E_\lambda f, h)
\]
\[
= \int \lambda \, d(f, E_\lambda h)
\]
\[
= \int \lambda \, d \left\{ \int f(t) \, d(E_t x, E_\lambda h) \right\}
\]
\[
= \int \lambda \, \int f(t) \, d(E_t x, h)
\]
\[
= \int \lambda \, f(\lambda) \, d(E_\lambda x, h)
\]
so that
\[ Af = \int \lambda f(\lambda) \, dE \lambda \, x \]
for
\[ f = \int f(\lambda) \, dE \lambda \, x, \]
which is a canonical form of multiplicity-free operator.

**Theorem 2.** \( A = A^* \), bounded is multiplicity-free \( \iff \exists \ h \in \mathcal{H} \) s.t.
\[ C \left\{ A^n h \right\} = \mathcal{H}. \]  
This \( h \) is also a cyclic vector for \( A \).

**Proof.** If part is easy. Suppose that such \( h \) exists and that \( \exists \ y \) s.t.
\[ y \perp C \left\{ E(\lambda) h \right\}, \quad \text{so} \]
y \perp E_t h \ for \( \forall \ t. \) Then
\[ (A^n h, y) = \int t^n d \left( E_t h, y \right) = 0 \]
\( \forall n \geq 0 \) - contradictory.

For only if part, set
\[ h = \int e^{-t^2} \, dE_t \, x, \]
so that
\[ A^n h = \int t^n e^{-t^2} dE_t x. \]

If \( \exists f \in \mathcal{H} \) s.t. \( (A^n h, f) = 0, \forall n > 0 \), then
\[ 0 = (A^n h, f) = \int e^{-t^2} t^n f(t) d\sigma(t), \]

where \( f \leftrightarrow f(t) \) and
\[ ||f||^2 = \int |f(t)|^2 d\sigma(t) \neq 0. \]

Set
\[ w(t) = \int_0^t f(s) d\sigma(s) + C, \]

\[ |w(t)| \leq \sqrt{1 + a}, \quad t \to \infty. \]

Integrating by parts,
\[ 0 = \int_{-\infty}^{\infty} e^{-t^2} t^{n+1} w(t) dt \]
\[ = \int_{-\infty}^{\infty} e^{-t^2} t^{n-1} w(t) dt, \quad n = 0, 1, 2, \ldots \]
implies
\[ \int_{-\infty}^{\infty} e^{-t^2} t^n w(t) dt = 0 \]
for \( n > 0 \) (\( n = 0 \) - choice of \( C \)).

By completeness of Hermite polynomials,
\[ w(t) = 0, \]
and we get
\[ 0 \neq \int |f(t)|^2 \, d\sigma(t) = \int \overline{f(t)} \, d\omega(t) = 0 \]
- a contradiction.

**Characterization of \( \sigma(A) \)**

*Pure point spectrum*: \( \lambda \in \sigma_{pp}(A) \),
if \( \text{Im}(A - \lambda I) \neq \mathbb{N} \iff \lambda \) is an eigenvalue.

*Continuous spectrum*: \( \lambda \in \sigma_{cont}(A) \),
if \( \text{Im}(A - \lambda I) \) is not closed, or \( \lambda \) is an eigenvalue of infinite multiplicity.

**Remark**: It may happen that finite multiplicity eigenvalue \( \in \sigma_{pp}(A) \cap \sigma_{cont}(A) \).

*Essential spectrum*: \( \lambda \in \sigma_{ess}(A) \),
if \( \lambda \) is non-isolated point with the property that \( E^{\lambda+\varepsilon} - E^{\lambda-\varepsilon} \neq 0 \)
for some \( \varepsilon > 0 \), or \( \lambda \) is an eigenvalue of an infinite multiplicity.

*Discrete spectrum*: \( \lambda \in \sigma_{disc}(A) \),
if \( \lambda \) is an isolated point of \( \sigma(A) \)
but not an eigenvalue of infinite multiplicity.

\( \lambda \in \sigma_{ess}(A) \iff \text{Im} \ E_{(\lambda-\varepsilon, \lambda+\varepsilon)} \) is infinite-dimensional for all \( \varepsilon > 0 \).