SKETCH OF SOLUTIONS (HOMEWORK VI)

1. \( H, \alpha_5 H, \alpha_3 H \) Notice that there must be 3 cosets, and different cosets are disjoint.

7. \(|U(30)| = 8\) and \(|H| = 2\) therefore there must be \(4\) cosets, by inspection we get \(H, 7H, 13H\) and \(19H\)

15. Let \( H \) be a proper subgroup of \( G \) then \(|H| < |G| = pq\) and by Lagrange’s theorem \(|H| \mid pq\), so \(|H|\) equals either 1, \( p \) or \( q \) and by corollary 3 this implies \( H \) is cyclic.

3. \[
5^{2^7+1} \equiv (5^7)^2 \cdot 5 \equiv 5^2 \cdot 5 \equiv 4 \cdot 5 \equiv 6 \mod 7
\]

\[
7^{11+2} \equiv 7 \cdot 7^2 \equiv 7 \cdot 5 \equiv 2 \mod 11
\]

19. \( g^m = e \) therefore \(|g|\) divides \( m \) but since \((m,n) = 1\) if \(|g| \mid n\), then \(|g| = 1\). Using Lagrange’s theorem we get that \(|g| = 1\) therefore \( g = e \)

25. \( 1, 3, 11, 33 \). The number of elements of order 11 must be a multiple of 10 why? Therefore since \(33 \equiv 3 \mod 10\) there must be at least two elements \( g \) and \( h \) of order not equal to 11 or 1. If the order of one of these elements (say \( g \)) is \(33\) then \(|g^{11}| = 3\) and we are done.

31. \[
\begin{align*}
stab(1) &= \{(1), (24)(56)\} \quad \text{orb}(1) = \{1, 2, 3, 4\} \\
stab(3) &= \{(1), (24)(56)\} \quad \text{orb}(3) = \{1, 2, 3, 4\} \\
stab(1) &= \{(1), (12)(34), (13)(24), (14)(23)\} \quad \text{orb}(5) = \{5, 6\}
\end{align*}
\]

42. Label the points in the diagrams in the following way:

\[
\begin{array}{ccc}
a & b & c \\
d & e & f
\end{array}
\]

\[
\begin{align*}
\text{stab}(a) &= \{R_0, H\} \quad \text{stab}(d) = \{R_0\} \\
\text{stab}(b) &= \{R_0, D^f\} \quad \text{stab}(e) = \{R_0\} \\
\text{stab}(c) &= \{R_0, H\} \quad \text{stab}(f) = \{R_0\}
\end{align*}
\]

45. 50. By Lagrange’s theorem \(10 \mid |G|\), \(25 \mid |G|\), therefore \( \text{lcm}(10, 25) \mid |G| \)

therefore \(50 \mid |G|\). Since \(|G| < 100\) we get \(50 = |G|\)