SKETCH OF SOLUTIONS (HOMEWORK IV)

15. Suppose $g \in H$ and $12 = k|g|$. Then $g^{12} = g^{k|g|} = e$ Therefore g satisfies the equation $x^{12} = e$. If $g^{12} = e$ then $12 = k|g|$. Therefore $H = \{x \in G \mid x^{12} = e\}$ and this is a subgroup by exercise 23 of chapter III
21.
    a $12 = k|a|$ for some $k \in \mathbb{Z}$
    b $m = k|a|$ for some $k \in \mathbb{Z}$
    c Suppose $< a > \neq G$ then $|a| < 24$ and $|a|$ divides 24. Therefore $|a|$ is either $2, 3, 4, 6, 8$ or 12. But $|a| \neq 2, 4, 8$ because $a^8 \neq e$ also $|a| \neq 2, 3, 4, 6, 12$ because $a^{12} \neq e$!
23. Yes, by the funtamental theorem of cyclic groups. The subgroups of $\mathbb{Z}$ are of the form $< n >$ for $n \in \mathbb{Z}$
24. $aa^n = a^{1+n} = a^{n+1} = a^n a$.
31. Let $a_1, \ldots, a_m$ be all the elements of G. Define $n := |a_1| \cdot |a_2| \cdots |a_m|$, then $a_i^n = e$ *why?*
40. $\text{lcm}(m, n)$
53. Notice that for any $x \in < a >$, $x^{10} = e$, also for any $y \in < b >$ $y^{21} = e$. Take $z \in < a > \cap < b >$, since $1 = 21 - 2(10)$, we get $z = z^1 = z^{21-2(10)} = z^{21} z^{-2(10)} = e$
61. Let a be a generator of the group of order $p^n - 1$, since $(p, p^n - 1) = 1$ we get by corollary 2 pg. 77 that $< a > = < a^p >$. Now take any $b \in < a >$ then $b \in < a^p >$ i.e. $b = (a^p)^k = (a^k)^p$